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RATE OF CONVERGENCE TO THE SEMI-CIRCULAR LAW FOR THE GAUSSIAN UNITARY ENSEMBLE¹⁾

Показано, что расстояние Колмогорова между усредненным значением спектральной функции распределения матрицы Вигнера размера $n \times n$ с гауссовскими элементами и функцией распределения полукругового закона имеет порядок $O(n^{-2/3})$.

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Ключевые слова и фразы: независимые случайные величины, спектральное распределение, случайная матрица.

1. Introduction and results. Let W_n be a Hermitean matrix of order n whose elements $W_{lj} = X_{lj} + iY_{lj}$ ($l, j = 1, \dots, n$) are independent (apart from the conditions of being Hermitean). Assume that the real and imaginary parts of matrix elements are independent as well. Furthermore let $\mathbf{E}X_{lj} = 0$ and $\mathbf{E}Y_{kj} = 0$, $1 \leq l \leq j$ and $1 \leq k < j$. Finally we may assume that $\mathbf{E}X_{ll}^2 = 2\sigma^2$, $l = 1, 2, \dots$, and $\mathbf{E}X_{kj}^2 = \mathbf{E}Y_{kj}^2 = \sigma^2$, $1 \leq k < j$. Let $Y_{ll} = 0$ for $l = 1, 2, \dots$.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote the eigenvalues of the random matrix $n^{-1/2}W$ and define the spectral distribution function of this matrix by

$$F_n(x) = \frac{1}{n} \sum_{j=1}^n \mathbf{I}_{(-\infty, x]}(\lambda_j),$$

where $\mathbf{I}_{(a,b]}(u)$ denotes the indicator function of the interval $(a, b]$. For simplicity we shall assume that $\sigma^2 = \frac{1}{4}$. Denote by $G(x)$ the distribution function of the semi-circular law with density $g(x) = G'(x) = \pi^{-1} \sqrt{2 - x^2} \mathbf{I}_{[-\sqrt{2}, \sqrt{2}]}(x)$. Wigner [13], [14] showed that

$$\Delta_n := \sup_x |\mathbf{E}F_n(x) - G(x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (1.1)$$

under assumption that the random variables X_{jk} and Y_{jk} have finite moments. Pastur [12] has proven that $F_n(x)$ tends to $G(x)$ in probability when Lindeberg's condition is fulfilled for every row vector of matrix W_n . Girko [5] proved that if $\lim_{n \rightarrow \infty} n^{-2} \sum_{l,j=1}^n \mathbf{E}|W_{lj}|^2 \mathbf{I}_{\{|W_{lj}| > \tau \sqrt{n}\}} = 0$, for some $\tau > 0$, then $F_n(x) \rightarrow G(x)$ a.s. as $n \rightarrow \infty$. The actual rate of convergence in limit theorems of type (1.1) is unknown.

Girko [7] obtained the bound $\Delta_n = O(n^{-1/14})$. Bai [1] has proven that $\Delta_n = O(n^{-1/4})$ assuming that $M := \sup_{l,j} \mathbf{E}|W_{lj}|^4 < \infty$ and in [3] improved this result to $\Delta_n = O(n^{-1/3})$. Götze and Tikhomirov [8] proved that $\Delta_n \leq CM^{1/2}n^{-1/2}$. Bai [2] announced the bound $\Delta_n = O(n^{-1/2})$ and Girko [6] stated this bound, but it seems that in his proof some critical steps concerning the Gaussian unitary ensemble (GUE) are not correct. A closer analysis of that GUE case motivated our paper. We investigate the rate of convergence of the expected normalized spectral distribution function to the semi-circular law for *Gaussian* random variables X_{lj} and Y_{kj} . We shall consider nonuniform bounds as well as uniform ones. Throughout this note we shall assume that X_{lj} , $1 \leq l \leq j$, and Y_{kj} , $1 \leq k < j$, are independent *Gaussian* random variables with zero mean and $\sigma^2 = \frac{1}{4}$. The main results are the following.

Theorem 1.1. *There exist positive absolute constants C_1, C such that for any $x \in [-\sqrt{2} + C_1 n^{-1/3}, \sqrt{2} - C_1 n^{-1/3}]$ the following inequality holds for the GUE ensemble:*

$$|\mathbf{E}F_n(x) - G(x)| \leq \frac{C}{n(2 - x^2)}.$$

Theorem 1.2. *There exists a positive absolute constant C such that for the GUE ensemble*

$$\Delta_n \leq Cn^{-2/3}.$$

2. Asymptotics of Hermitean orthogonal polynomials. Let $\varphi_n(x)$ denote the n -th Hermite function, i.e.,

$$\varphi_n(x) = \frac{1}{\sqrt{n! 2^n \sqrt{\pi}}} e^{-x^2/2} H_n(x),$$

where $H_n(x)$ the n -th Hermite polynomial, $H_n(x) = e^{x^2} (-\frac{d}{dx})^n e^{-x^2}$. In what follows we shall denote by θ some quantity, possible depending on n and x , such that $|\theta| \leq 1$. In this section we recall the asymptotics of $\varphi_n(x)$. We have the following asymptotic property of Hermitean functions. Let $N = 2n + 1$. Then

$$\varphi_n(x) = \left(\frac{2}{\pi}\right)^{1/2} (N - x^2)^{-1/4} \cos\left(\frac{N(2\beta - \sin 2\beta) - \pi}{4}\right) + O\left(N^{1/2}(N - x^2)^{-7/4}\right), \quad (2.1)$$

where $0 \leq x \leq N^{1/2}$ and $\beta := \beta_n(x) = \arccos(xN^{-1/2})$. For the proof of this equality see [4] and [11].

We rewrite this result as follows.

Corollary 2.1. *There exists some absolute positive constant C such that, for any $0 \leq x \leq \sqrt{2} - Cn^{-1/3}$, the following equality holds:*

$$n^{1/4} \varphi_n(x\sqrt{n}) = \left(\frac{2}{\pi}\right)^{1/2} (2-x^2)^{-1/4} \cos \left\{ \frac{n}{2} (2\beta_0 - \sin(2\beta_0)) + \frac{\beta_0}{2} - \frac{\pi}{4} \right\} + \theta \frac{C}{n(2-x^2)^{7/4}}, \quad (2.2)$$

where $\beta_0 := \beta_0(x) = \arccos(x/\sqrt{2})$.

P r o o f. At first we note that

$$(N - x^2n)^{-1/4} = n^{-1/4} (2 - x^2)^{-1/4} \left(1 + O\left(\frac{1}{n(2-x^2)}\right) \right). \quad (2.3)$$

Furthermore, using Taylor's expansion for the function $\arccos y$ we obtain

$$\beta(x\sqrt{n}) = \arccos \left(x \sqrt{\frac{n}{2n+1}} \right) = \arccos \left(\frac{x}{\sqrt{2}} \right) + \frac{1}{4n} \frac{\cos \beta_0}{\sin \beta_0} + O\left(\frac{1}{n^2(2-x^2)^{3/2}}\right). \quad (2.4)$$

Analogously we obtain

$$\sin(2\beta(x\sqrt{n})) = \sin(2\beta_0) + \frac{1}{2n} \frac{\cos \beta_0}{\sin \beta_0} \cos(2\beta_0) + O\left(\frac{1}{n^2(2-x^2)^{3/2}}\right). \quad (2.5)$$

The equalities (2.4) and (2.5) together imply

$$\frac{2n+1}{4} (2\beta - \sin(2\beta)) = \frac{n}{2} (2\beta_0 - \sin(2\beta_0)) + \frac{\beta_0}{2} + O\left(\frac{1}{n(2-x^2)^{3/2}}\right). \quad (2.6)$$

Substituting (2.3) and (2.6) in (2.1) we obtain the result of Corollary 2.1.

3. Non-uniform bound. The proof of Theorem 1.1. Let $\sigma_n(x)$ denote the density of the expected normalized spectral distribution function of a Gaussian Hermitean matrix W . It is well known that

$$\begin{aligned} \sigma_n(x) &= \frac{d}{dx} \mathbf{E} F_n(x) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi_k^2(x\sqrt{n}) \\ &= \sqrt{n} \varphi_n^2(x\sqrt{n}) - \sqrt{n+1} \varphi_{n-1}(x\sqrt{n}) \varphi_{n+1}(x\sqrt{n}) \end{aligned} \quad (3.1)$$

(see, for example, [10, Appendix A.8]). Similarly to Girko's [6] approach we write

$$(n+1)^{1/4} \varphi_{n+1}(x\sqrt{n}) = (n+1)^{1/4} \varphi_{n+1} \left(x\sqrt{n+1} \sqrt{\frac{n}{n+1}} \right).$$

According to Corollary 2.1 we get

$$\begin{aligned} (n+1)^{1/4} \varphi_{n+1}(x\sqrt{n}) &= \left(\frac{2}{\pi}\right)^{1/2} (2-x^2)^{-1/4} \cos \left\{ \frac{n}{2} (2\beta_0 - \sin(2\beta_0)) + \frac{3\beta_0}{2} - \frac{\pi}{4} \right\} \\ &\quad + \theta \frac{C}{n(2-x^2)^{7/4}}. \end{aligned} \quad (3.2)$$

Analogously we obtain

$$\begin{aligned} (n+1)^{1/4} \varphi_{n-1}(x\sqrt{n}) &= \left(\frac{2}{\pi}\right)^{1/2} (2-x^2)^{-1/4} \cos \left\{ \frac{n}{2} (2\beta_0 - \sin(2\beta_0)) - \frac{\beta_0}{2} - \frac{\pi}{4} \right\} \\ &\quad + \theta \frac{C}{n(2-x^2)^{7/4}}. \end{aligned} \quad (3.3)$$

The equalities (3.1)–(3.3) and (2.2) together imply that

$$\begin{aligned}\sigma_n(x) &= \frac{2}{\pi\sqrt{2-x^2}} \left[\cos^2 \left\{ \frac{n}{2} \left(2\beta_0 - \sin(2\beta_0) \right) - \frac{\pi}{4} + \frac{\beta_0}{2} \right\} \right. \\ &\quad \left. - \cos \left\{ \frac{n}{2} \left(2\beta_0 - \sin(2\beta_0) \right) - \frac{\pi}{4} - \frac{\beta_0}{2} \right\} \right. \\ &\quad \left. \times \cos \left\{ \frac{n}{2} \left(2\beta_0 - \sin(2\beta_0) \right) - \frac{\pi}{4} + \frac{3\beta_0}{2} \right\} \right] + \theta \frac{C}{n(2-x^2)^2} \\ &= \frac{2}{\pi\sqrt{2-x^2}} \sin^2 \beta_0 + \theta \frac{C}{n(2-x^2)^2} = \frac{\sqrt{2-x^2}}{\pi} + \theta \frac{C}{n(2-x^2)^2}.\end{aligned}\quad (3.4)$$

This implies that, uniformly for $x \in [-\sqrt{2} + C_1 n^{-1/3}, \sqrt{2} - C_1 n^{-1/3}]$,

$$|\sigma_n(x) - g(x)| \leq \frac{C}{n(2-x^2)^2}.$$

Here $g(x)$ denote the density of the semi-circular law, $g(x) = \pi^{-1} \sqrt{2-x^2} \mathbf{I}_{[-\sqrt{2}, \sqrt{2}]}(x)$. Since $\mathbf{E}F_n(0) = G(0)$ we get

$$|\mathbf{E}F_n(x) - G(x)| \leq \frac{C}{n} \int_0^{|x|} \frac{du}{(2-u^2)^2} \leq \frac{C}{n(2-x^2)}.$$

This implies that, for $x \in [-\sqrt{2} + C_1 n^{-1/3}, \sqrt{2} - C_1 n^{-1/3}]$,

$$|\mathbf{E}F_n(x) - G(x)| \leq \frac{C}{n(2-x^2)} \leq \frac{C}{n^{2/3}}$$

and completes the proof.

4. The uniform bound and proof of Theorem 1.2. To obtain the uniform bound we consider now the intervals

$$\begin{aligned}\mathbf{J}_\eta^{(l)} &= [-\sqrt{2}, -\sqrt{2} + \eta], & \mathbf{J}_\eta^{(r)} &= [\sqrt{2} - \eta, \sqrt{2}], \\ \mathbf{J}_\varepsilon^{(l)} &= [-\sqrt{2} + \varepsilon, -\sqrt{2} + \eta], & \mathbf{J}_\varepsilon^{(r)} &= [\sqrt{2} - \eta, \sqrt{2} - \varepsilon], \\ \mathbf{J}_\varepsilon'^{(l)} &= [-\sqrt{2}, -\sqrt{2} + \eta + \varepsilon], & \mathbf{J}_\varepsilon'^{(r)} &= [\sqrt{2} - \eta - \varepsilon, \sqrt{2}].\end{aligned}$$

Consider the distance $\Delta_\eta := \sup_{x \in \mathbf{J}_\eta^{(r)} \cup \mathbf{J}_\eta^{(l)}} |F(x) - G(x)|$. In what follows we shall assume that

$$\Delta := \sup_x |F(x) - G(x)| = \Delta_\eta. \quad (4.1)$$

Lemma 4.1. *Let F be a distribution function and let G denote the semi-circular distribution function. Denote their Stieltjes transforms by $f(z)$ and $g(z)$ respectively, where $z = u + iv$. Assume that $\int |F(x) - G(x)| dx < \infty$ and the condition (4.1) holds. Let $v > 0$ and let a and η be positive numbers such that*

$$\gamma = \frac{1}{\pi} \int_{|u| \leq a} \frac{1}{u^2 + 1} du > \frac{3}{4}, \quad (4.2)$$

$$\eta \geq 2\varepsilon \geq 2va. \quad (4.3)$$

Then there exist some positive constants $C_1(\gamma)$, $C_2(\gamma)$, $C_3(\gamma)$, depending on γ such that

$$\Delta_\eta \leq C_1(\gamma) \sup_{x \in \mathbf{J}_\varepsilon'^{(r)} \cup \mathbf{J}_\varepsilon'^{(l)}} \left| \operatorname{Im} \left(\int_{-\infty}^x (f(z) - g(z)) du \right) \right| + C_2(\gamma) v \sqrt{\eta} + C_3(\gamma) \varepsilon^{3/2}.$$

P r o o f. Introduce the following notation $\Delta_\varepsilon := \sup_{x \in \mathbf{J}_\varepsilon^{(r)} \cup \mathbf{J}_\varepsilon^{(l)}} |F(x) - G(x)|$. Note that $\Delta_\eta \leq \Delta_\varepsilon + C\varepsilon^{2/3}$. We have the following relations:

$$\begin{aligned} \frac{1}{\pi} \operatorname{Im} \left(\int_{-\infty}^x (f(z) - g(z)) du \right) &= \frac{1}{\pi} \int_{-\infty}^x \left[\int_{-\infty}^{\infty} \frac{v d(F(y) - G(y))}{(y-u)^2 + v^2} \right] du \\ &= \frac{1}{\pi} \int_{-\infty}^x \left[\int_{-\infty}^{\infty} \frac{2v(y-u)(F(y) - G(y)) dy}{((y-u)^2 + v^2)^2} \right] du \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} (F(y) - G(y)) \left[\int_{-\infty}^x \frac{2v(y-u) du}{((y-u)^2 + v^2)^2} \right] dy \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(F(x-vy) - G(x-vy)) dy}{y^2 + 1}. \end{aligned}$$

Since F is nonincreasing we obtain

$$\begin{aligned} \frac{1}{\pi} \int_{|y| < a} \frac{(F(x-vy) - G(x-vy)) dy}{y^2 + 1} &\geq \gamma(F(x-va) - G(x-va)) \\ &\quad - \frac{1}{\pi} \int_{|y| < a} |G(x-vy) - G(x-va)| dy \\ &\geq \gamma(F(x-va) - G(x-va)) - \frac{1}{v\pi} \int_{|y| < va} |G(x-y) - G(x-va)| dy. \end{aligned}$$

Replacing $x-va$ by x we get

$$\begin{aligned} \frac{1}{\pi} \int_{|y| < a} \frac{(F(x+va-vy) - G(x+va-vy)) dy}{y^2 + 1} \\ &\geq \gamma(F(x) - G(x)) - \frac{1}{\pi} \int_{|y| < a} |G(x+va-vy) - G(x+va-va)| dy \\ &\geq \gamma(F(x) - G(x)) - \frac{1}{v\pi} \int_{|y| < va} |G(x+va-y) - G(x+va-va)| dy. \end{aligned}$$

If (4.2) and (4.3) hold and $x \in \mathbf{J}_\varepsilon^{(r)} \cup \mathbf{J}_\varepsilon^{(l)}$ then $x \pm va \in \mathbf{J}_\varepsilon^{(r)} \cup \mathbf{J}_\varepsilon^{(l)}$.

Let $x_n \in \mathbf{J}_\varepsilon^{(r)} \cup \mathbf{J}_\varepsilon^{(l)}$ be such that $F(x_n) - G(x_n) \rightarrow \Delta_\eta$. Then we have

$$\begin{aligned} \sup_{x \in \mathbf{J}_\varepsilon^{(r)} \cup \mathbf{J}_\varepsilon^{(l)}} \left| \frac{1}{\pi} \operatorname{Im} \left(\int_{-\infty}^x (f(z) - g(z)) du \right) \right| &\geq \gamma \lim_{n \rightarrow \infty} (F(x_n) - G(x_n)) \\ &\quad - \frac{1}{v\pi} \sup_{x \in \mathbf{J}_\varepsilon^{(r)} \cup \mathbf{J}_\varepsilon^{(l)}} \int_{|y| < 2va} |G(x+y) - G(x)| dy - (1-\gamma) \Delta_\eta \\ &\geq \gamma \Delta_\eta - Cav\sqrt{\eta} - (1-\gamma) \Delta_\eta - C\varepsilon^{2/3} \geq (2\gamma-1) \Delta_\eta - Cav\sqrt{\eta} - C\varepsilon^{2/3}. \end{aligned}$$

Similar arguments may be used for the sequence $x_n \in \mathbf{J}_\varepsilon^{(r)} \cup \mathbf{J}_\varepsilon^{(l)}$ such that $(F(x_n) - G(x_n)) \rightarrow -\Delta_\eta$. This completes the proof.

Lemma 4.2. For any $V > v_0 > 0$ the following inequality holds:

$$\begin{aligned} \sup_{x \in \mathbf{J}_\varepsilon^{(r)} \cup \mathbf{J}_\varepsilon^{(l)}} \left| \int_{-\infty}^x \operatorname{Im} (f(z) - g(z)) du \right| &\leq \int_{-\infty}^{\infty} |f(u+iV) - g(u+iV)| du \\ &\quad + \sup_{x \in \mathbf{J}_\varepsilon^{(r)} \cup \mathbf{J}_\varepsilon^{(l)}} \left| \operatorname{Im} \left\{ \int_{v_0}^V (f(x+iv) - g(x+iv)) dv \right\} \right|. \end{aligned}$$

For the proof of this lemma see [8, Proof of Lemma 2.2].

Lemmas 4.1 and 4.2 together imply

Corollary 4.1. *There exist some absolute positive constants C_1, C_2 and C_3 such that for any $V > v_0 > 0$ and $\varepsilon > 0$ such that $\eta \geq 2\varepsilon \geq 2va$ the following inequality holds:*

$$\begin{aligned} \Delta_\eta \leq & C_1 \left(\int_{-\infty}^\infty \left| f(u + iV) - g(u + iV) \right| du \right. \\ & + \sup_{x \in \mathbf{J}'_\varepsilon(r) \cup \mathbf{J}'_\varepsilon(l)} \left| \operatorname{Im} \left\{ \int_{v_0}^V \left(f(x + iu) - g(x + iu) \right) du \right\} \right| \Big) \\ & + C_2 v_0 \sqrt{\eta} + C_3 \varepsilon^{2/3}. \end{aligned} \tag{4.4}$$

Let $W(k)$ be the matrix obtained from W by deleting the k -th row and k -th column, and let $\alpha'(k) = (X_{1k} + iY_{1k}, \dots, X_{(k-1)k} + iY_{(k-1)k}, X_{(k+1)k} + iY_{(k+1)k}, \dots, X_{nk} + iY_{nk})$. Set

$$\varepsilon_k = \frac{1}{\sqrt{n}} X_{kk} - \frac{1}{n} \overline{\alpha'(k)} \left(W(k) - z \mathbf{I}_{n-1} \right)^{-1} \alpha_k + \frac{s_n(z)}{2}, \tag{4.5}$$

where $\mathbf{I}_{n-1}$ denotes the $(n-1) \times (n-1)$ identity matrix and $\bar{a}$ denotes the complex conjugate to a complex number a . Introduce

$$\delta_n(z) = -\frac{1}{n} \sum_{k=1}^n \mathbf{E} \varepsilon_k \frac{1}{(z + s_n(z)/2)(z + s_n(z)/2 - \varepsilon_k)}. \tag{4.6}$$

Let

$$s(z) = -(z - \sqrt{z^2 - 2}), \quad s_n(z) = \int_{-\infty}^\infty \frac{1}{x - z} d\mathbf{E} F_n(x). \tag{4.7}$$

This implies that

$$s_n(z) = -\frac{2}{2z + s_n(z)} + \delta_n(z). \tag{4.8}$$

In [8] it was proved that there exists some positive constant C_0 such that, for any $v \geq C_0 M^{1/2} n^{-1/2}$, $M = \sup_{l,j} \{ \mathbf{E} |X_{lj}|^4, \mathbf{E} |Y_{lj}|^4 \}$,

$$\operatorname{Im} \left(z + \frac{\delta_n(z)}{2} \right) > 0, \quad z = u + iv.$$

For $z = u + iv$ such that $|u| \leq \sqrt{2}$, $v \geq C_0 M^{1/2} n^{-1/2}$ we obtained the bound

$$\left| \delta_n(z) \right| \leq \frac{CM^{1/2}}{nv}. \tag{4.9}$$

(See [8, Lemmas 5.1, 5.2].) Solving the equation (4.8) and choosing a solution with $\operatorname{Im} s_n(z) > 0$ we get, for $v \geq C_0 M^{1/2} n^{-1/2}$

$$s_n(z) = -\left(z - \frac{\delta_n(z)}{2} \right) + \sqrt{\left(z + \frac{\delta_n(z)}{2} \right)^2 - 2}. \tag{4.10}$$

The equality (4.10) implies that, for $v \geq C_0 M^{1/2} n^{-1/2}$

$$|s_n(z) - s(z)| \leq |\delta_n(z)| \left(1 + \left| \sqrt{\left(z + \frac{\delta_n(z)}{2} \right)^2 - 2} + \sqrt{z^2 - 2} \right|^{-1} \right).$$

Since $\operatorname{Im} \sqrt{(z + \delta_n(z)/2)^2 - 2}$ and $\operatorname{Im} \sqrt{z^2 - 2}$ are positive, we have, for $z = u + iv$ such that $u \in \mathbf{J}'_\varepsilon(r) \cup \mathbf{J}'_\varepsilon(l)$ and $v \geq C_0 M^{1/2} n^{-1/2}$,

$$|s_n(z) - s(z)| \leq |\delta_n(z)| \left(1 + \left| \operatorname{Im} \{ \sqrt{z^2 - 2} \} \right|^{-1} \right). \tag{4.11}$$

Note that

$$\operatorname{Im} \sqrt{z^2 - 2} = \frac{|\operatorname{Im}(z^2 - 2)|}{\sqrt{2(|z^2 - 2| + \operatorname{Re}(z^2 - 2))}}. \tag{4.12}$$

Furthermore, since $|u|^2 \leq 2$ we have

$$\operatorname{Re}(z^2 - 2) \leq 0, \quad (4.13)$$

and

$$|z^2 - 2|^2 = (u^2 - 2)^2 + 4u^2v^2 - 2(u^2 - 2)v^2 + v^4 \leq 9(v^2 + \eta^2). \quad (4.14)$$

In the last inequality we used that $v \leq 1$ and $0 \leq 2 - u^2 \leq 2\eta$. The relations (4.12)–(4.14) together imply that

$$\left| \operatorname{Im} \sqrt{z^2 - 2} \right|^{-1} \leq \frac{C(v^2 + \eta^2)^{1/4}}{v}. \quad (4.15)$$

The inequalities (4.15) and (4.11) together imply that

$$|s_n(z) - s(z)| \leq C|\delta_n(z)| \frac{(\eta^2 + v^2)^{1/4}}{v} \leq \frac{C(\eta^2 + v^2)^{1/4}}{nv^2}.$$

Using now the inequality (4.4) with $\eta = Cn^{-1/3}$, $\varepsilon = Cn^{-1/2}$, $v_0 = Cn^{-1/2}$ we get

$$\Delta \leq \Delta_\eta \leq \frac{C\eta^{1/2}}{nv_0} + \frac{C}{n\sqrt{\eta}} + Cv_0\sqrt{\eta} \leq Cn^{-2/3}.$$

If $\Delta \geq \Delta_\eta$, we use the nonuniform bound and obtain $\Delta \leq Cn^{-2/3}$. These inequalities together imply that $\Delta \leq Cn^{-2/3}$, which proves Theorem 1.2.

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