Math-Net.Ru

Общероссийский математический портал

M. G. Temido, L. Canto E. Castro, Max-semistable laws in extremes of stationary random sequences, *Теория вероятн. и ее примен.*, 2002, том 47, выпуск 2, 402–410

DOI: <https://doi.org/10.4213/tvp3673>

Использование Общероссийского математического портала Math-Net.Ru подразумевает, что вы прочитали и согласны с пользовательским соглашением

<http://www.mathnet.ru/rus/agreement>

Параметры загрузки:

IP: 34.229.108.108

26 апреля 2023 г., 11:58:53

13. Roos B. Metric multivariate Poisson approximation of the generalized multinomial distribution. — Теория вероятн. и ее примен., 1998, т. 43, в. 2, с. 404–413.
14. Roos B. On the rate of multivariate Poisson convergence. — J. Multivariate Anal., 1999, v. 69, № 1, p. 120–134.
15. Witte H.-J. A unification of some approaches to Poisson approximation. — J. Appl. Probab., 1990, v. 27, № 3, p. 611–621.

Поступила в редакцию
18.IX.2000

© 2002 г.

TEMIDO M. G.*, CANTO E CASTRO L.**

MAX-SEMISTABLE LAWS IN EXTREMES OF STATIONARY RANDOM SEQUENCES¹⁾

Рассматриваются стационарные последовательности, подчиняющиеся некоторому обобщению условия Лидбеттера $D(u_n)$. Для таких последовательностей доказывается, что если $\{k_n\}$ — неубывающая последовательность целых чисел и $\lim_{n \rightarrow +\infty} k_{n+1}/k_n = r \geq 1$, то предельным законом для максимума первых k_n величин является мах-полуустойчивый закон. Это утверждение обобщает соответствующий результат для последовательностей независимых одинаково распределенных случайных величин [3] и теорему Лидбеттера об экстремальных типах [4]. Мы доказываем также, что предельное поведение этого максимума можно вывести из предельного поведения соответствующего максимума ассоциированной независимой последовательности, и обобщаем хорошо известное понятие экстремального индекса.

Ключевые слова и фразы: максимум, слабая сходимость, стационарность, мах-полуустойчивые законы.

1. Introduction. Max-semistable laws arise in the theory of extremes as limit in distribution of the maximum, suitably normalized, of k_n independent and identically distributed (i.i.d.) random variables, where $\{k_n\}$ is a nondecreasing integer sequence satisfying

$$\lim_{n \rightarrow +\infty} \frac{k_{n+1}}{k_n} = r, \quad r \in [1, +\infty[. \quad (1)$$

Following [2] we will say that a distribution function (d.f.) G on $\mathbf{R}$ is ℓ -max-semistable if there are reals $r > 1$, $\gamma > 0$ and β such that

$$G(x) = G^r\left(\frac{x}{\gamma} + \beta\right), \quad x \in \mathbf{R}, \quad (2)$$

or equivalently, if there are a sequence of i.i.d. random variables with d.f. F and two real sequences $\{a_n > 0\}$ and $\{b_n\}$ for which

$$\lim_{n \rightarrow +\infty} F^{k_n}\left(\frac{x}{a_n} + b_n\right) = G(x), \quad (3)$$

for each continuity point of G , with $\{k_n\}$ satisfying (1) — where ℓ stands for *linear* normalization. The general definition of max-semistable law is given in [7].

* Departamento de Matemática, Universidade de Coimbra, Portugal; e-mail: mgtm@mat.uc.pt

** D.E.I.O., Universidade de Lisboa, Portugal.

¹⁾ Partially supported by JNICT/PRAXIS XXI/FEDER.

If the limit (3) holds we shall say that F belongs to the domain of attraction of G .

In [3] we can find an important lemma which describes the relation between the support of the solution G to (2) and $u(x) = x/\gamma + \beta$ and also necessary and sufficient conditions for a d.f. F to belong to the domain of attraction of an ℓ -max-semistable law.

Lemma 1 ([3]). *If the support of G is $]-\infty, t[$, then $\gamma > 1$ and $\beta = t(1 - \gamma^{-1})$ and if it is $]t, +\infty[$, then $0 < \gamma < 1$ and $\beta = t(1 - \gamma^{-1})$. Otherwise if the support of G is $\mathbf{R}$, then $\gamma = 1$ and $\beta > 0$.*

The analytical expressions of the three classes of ℓ -max-semistable laws are given in the following theorem.

Theorem 1 ([2]). 1. *For $u(x) = x\gamma^{-1} + t(1 - \gamma^{-1})$, $t \in \mathbf{R}$, the solutions to (2) are the following distributions:*

$$\begin{aligned} G_1(x) &= \exp \left\{ -(t-x)^\alpha \nu(\ln(t-x)) \right\} \quad \text{if } x \in]-\infty, t[, \\ G_2(x) &= \exp \left\{ -(x-t)^{-\alpha} \nu(\ln(x-t)) \right\} \quad \text{if } x \in]t, +\infty[, \end{aligned}$$

where $\alpha = |\ln r / \ln \gamma|$ and ν are periodic positive and bounded functions with period $p = |\ln \gamma|$. Furthermore, $x^\alpha \nu(\ln x)$ and $x^{-\alpha} \nu(\ln x)$ do not increase on the half line $x > 0$.

2. *For $u(x) = x + \beta$ the solutions to (2) are the following distributions:*

$$G_3(x) = \exp \left\{ -\nu(x) \exp(-\alpha x) \right\} \quad \text{if } x \in \mathbf{R},$$

where $\alpha = (\ln r)/\beta$ and ν are periodic positive and bounded functions with period $p = \beta$, and such that $\exp(-\alpha x) \nu(x)$ does not increase on $\mathbf{R}$.

It should be remarked that ℓ -max-stable laws are ℓ -max-semistable.

Grinevich [3] notice also that, due to Khinchin's Theorem (see [6]) and Lemma 1, the existence of a nondegenerate limit G in (3) implies that the normalizing constants $a_n > 0$ and b_n satisfy $\lim_{n \rightarrow +\infty} a_{n+1}/a_n = \gamma$ and $\lim_{n \rightarrow +\infty} a_n(b_{n+1} - b_n) = t(1 - \gamma^{-1})$ if G belongs to the classes G_1 or G_2 , and, if G belongs to the class G_3 the corresponding conditions are $\lim_{n \rightarrow +\infty} a_{n+1}/a_n = 1$ and $\lim_{n \rightarrow +\infty} a_n(b_{n+1} - b_n) = \beta$.

Let us consider now a strictly stationary (stationary) sequence $\{X_n\}$ with marginal d.f. F . Let $M_n = \max\{X_1, X_2, \dots, X_n\}$.

Leadbetter [4] introduced the dependence restriction $D(u_n)$, weaker than strong mixing, and has shown that, under this restriction, the limit laws are precisely the same as the classical ones. That is, if there are normalizing constants $a_n > 0$ and b_n such that $\mathbf{P}\{a_n(M_n - b_n) \leq x\}$ converges to $G(x)$, then G , whenever nondegenerate, is a ℓ -max-stable distribution. Moreover (see [6]), if the underlying stationary sequence satisfies $D(u_n)$, then the limiting behaviour of M_n can be inferred from the limiting behaviour of the maximum $\bar{M}_n$ of n i.i.d. random variables with the same d.f., through what is known as the extremal index. Specifically if, for each $\tau > 0$, there is a real sequence $\{u_n\}$ such that $\lim_{n \rightarrow +\infty} n(1 - F(u_n)) = \tau > 0$ then $\{X_n\}$ has extremal index θ if $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_n \leq u_n\} = e^{-\theta\tau}$.

In this paper we consider that $\{X_n\}$ is a stationary sequence satisfying a new dependence restriction, $D_{k_n}(u_n)$, which generalizes Leadbetter's condition $D(u_n)$. We will prove that when $\{X_n\}$ satisfies $D_{k_n}(u_n)$ and $\{k_n\}$ satisfies (1) if, for the same normalizing constants $a_n > 0$ and b_n , $\mathbf{P}\{a_n(M_{k_n} - b_n) \leq x\}$ has a nondegenerate limit, then the limit law is ℓ -max-semistable. Thus, the corresponding classical result stated in (3) and the Leadbetter's Extremal Types Theorem (Theorem 2.1 in [4]) are generalized. We also extend the notion of extremal index.

2. Limiting distribution for the maximum. As we have already said before $\{X_n\}$ will be, throughout this section, a stationary sequence with common d.f. F . We start by defining our new dependence condition $D_{k_n}(u_n)$. Leadbetter's condition $D(u_n)$ is obtained from it with $k_n = n$.

Definition 1. Let $\{k_n\}$ be a sequence of positive integers. The random sequence $\{X_n\}$ satisfies condition $D_{k_n}(u_n)$ if for any integers $1 \leq i_1 < \dots < i_p < j_1 < \dots <$

$j_q \leq k_n$, for which $j_1 - i_p > \ell_{k_n}$, we have

$$\left| \mathbf{P}\{X_{i_1} \leq u_n, \dots, X_{i_p} \leq u_n, X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n\} \right. \\ \left. - \mathbf{P}\{X_{i_1} \leq u_n, \dots, X_{i_p} \leq u_n\} \mathbf{P}\{X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n\} \right| \leq \alpha_{n, \ell_{k_n}},$$

where $\lim_{n \rightarrow +\infty} \alpha_{n, \ell_{k_n}} = 0$ for some sequence $\ell_{k_n} = o_n(k_n)$.

The following lemma shows how the condition $D_{k_n}(u_n)$ gives an asymptotic independence for maxima on separated sets, which is appropriate for the subsequent results. We must remark that stationarity is not an assumption of the lemma. The proof is omitted because it is similar to the one of Lemma 3.2.2 of [6]. If E is any set of integers we will denote $\max\{X_j, j \in E\}$ by $M(E)$ and we will say that E is an interval if the integers are consecutive. Moreover if $E = \{j_1, \dots, j_2\}$ its length will be $j_2 - j_1 + 1$ and, E and $F = \{i_1, \dots, i_2\}$ with $i_1 > j_2$ are separated by $i_1 - j_2$.

Lemma 2. *Let $\{k_n\}$ be a sequence of positive integers. Suppose that $\{X_n\}$ satisfies $D_{k_n}(u_n)$ for some real sequence $\{u_n\}$. Let t_n and ℓ_{k_n} be positive integers and $E_1, E_2, \dots, E_{t_n}$ subintervals of $\{1, 2, \dots, k_n\}$ such that any two are separated by ℓ_{k_n} .*

Then

$$\left| \mathbf{P}\left(\bigcap_{j=1}^{t_n} \{M(E_j) \leq u_n\}\right) - \prod_{j=1}^{t_n} \mathbf{P}\{M(E_j) \leq u_n\} \right| \leq (t_n - 1) \alpha_{n, \ell_{k_n}}.$$

Let $\{s_n\}$ be an integer sequence satisfying

$$\lim_{n \rightarrow +\infty} s_n = +\infty, \quad \lim_{n \rightarrow +\infty} \frac{k_n}{s_n} = +\infty, \quad \lim_{n \rightarrow +\infty} s_n \alpha_{n, \ell_{k_n}} = 0, \quad \lim_{n \rightarrow +\infty} \frac{\ell_{k_n} s_n}{k_n} = 0.$$

For any integer $m \geq 0$ let $r_{n,m} = [k_{n-m}/(r^{-m} s_n)]$ and $\beta_{n,m}$ be integer sequences such that $(\beta_{n,m} - 1) r_{n,m} \leq k_{n-m} \leq \beta_{n,m} r_{n,m}$. Under these assumptions we can obtain the following limits, which will be used in the proof of next lemma. Namely,

$$\begin{aligned} \text{i)} \quad \lim_{n \rightarrow +\infty} \frac{\beta_{n,m} \ell_{k_n}}{k_{n-m}} &= 0, & \text{ii)} \quad \lim_{n \rightarrow +\infty} \frac{\beta_{n,m} r_{n,m}}{k_{n-m}} &= 1, \\ \text{iii)} \quad \lim_{n \rightarrow +\infty} \beta_{n,m} \alpha_{n, \ell_{k_n}} &= 0, & \text{iv)} \quad \lim_{n \rightarrow +\infty} \frac{\beta_{n,m}}{r^{-m} s_n} &= 1, \\ \text{v)} \quad \lim_{n \rightarrow +\infty} \frac{r_{n,0}}{r_{n,m}} &= 1, & \text{vi)} \quad \lim_{n \rightarrow +\infty} \frac{\beta_{n,0}}{\beta_{n,m}} &= r^m. \end{aligned} \quad (4)$$

Lemma 3. *Let $\{k_n\}$ be an integer sequence satisfying (1) and let $\{u_n\}$ be a real sequence such that $\limsup_{n \rightarrow +\infty} k_n(1 - F(u_n)) < +\infty$. Suppose that the stationary sequence $\{X_n\}$ satisfies $D_{k_n}(u_n)$. Hence, for all integers $m \geq 0$,*

$$\lim_{n \rightarrow +\infty} \left(\mathbf{P}\{M_{k_{n-m}} \leq u_n\} - \mathbf{P}^{\beta_{n,m}}\{M_{r_{n,m}} \leq u_n\} \right) = 0, \quad (5)$$

$$\lim_{n \rightarrow +\infty} \left(\mathbf{P}\{M_{k_{n-m}} \leq u_n\} - \mathbf{P}^{r^{-m} s_n}\{M_{r_{n,m}} \leq u_n\} \right) = 0, \quad (6)$$

$$\lim_{n \rightarrow +\infty} \left(\mathbf{P}\{M_{k_n} \leq u_n\} - \mathbf{P}^{r^m}\{M_{k_{n-m}} \leq u_n\} \right) = 0, \quad (7)$$

$$\lim_{n \rightarrow +\infty} \left(\mathbf{P}\{M_{k_n} \leq u_n\} - \mathbf{P}^m\{M_{[k_n/m]} \leq u_n\} \right) = 0 \quad (m \neq 0). \quad (8)$$

P r o o f. Let $m \geq 0$ be a fixed integer and, for large n , divide the first $\beta_{n,m} r_{n,m}$ integers into $2\beta_{n,m}$ consecutive intervals as follows

$$I_1 = \{1, \dots, r_{n,m} - \ell_{k_n}\},$$

$$I_1^* = \{r_{n,m} - \ell_{k_n} + 1, \dots, r_{n,m}\},$$

...

$$I_{\beta_{n,m}} = \{(\beta_{n,m} - 1) r_{n,m} + 1, \dots, \beta_{n,m} r_{n,m} - \ell_{k_n}\},$$

$$I_{\beta_{n,m}}^* = \{\beta_{n,m} r_{n,m} - \ell_{k_n} + 1, \dots, \beta_{n,m} r_{n,m}\},$$

where $I_1, \dots, I_{\beta_{n,m}}$ are separated by ℓ_{k_n} and have all the same length.

If k_{n-m} belongs to $I_{\beta_{n,m}}^*$ we have

$$\begin{aligned} 0 &\leq \mathbf{P}\left(\bigcap_{j=1}^{\beta_{n,m}} \{M(I_j) \leq u_n\}\right) - \mathbf{P}\{M_{k_{n-m}} \leq u_n\} \\ &\leq \mathbf{P}\left(\bigcup_{j=1}^{\beta_{n,m}} \{M(I_j) \leq u_n < M(I_j^*)\}\right) \leq \frac{\beta_{n,m} \ell_{k_n}}{k_{n-m}} k_{n-m} (1 - F(u_n)) = o_n(1), \end{aligned}$$

because (4) i) holds.

On the other hand, if k_{n-m} belongs to $I_{\beta_{n,m}}$, then

$$\begin{aligned} 0 &\leq \mathbf{P}\left(\bigcap_{j=1}^{\beta_{n,m}} \{M(I_j) \leq u_n\}\right) - \mathbf{P}\{M_{\beta_{n,m} r_{n,m} - \ell_{k_n}} \leq u_n\} \\ &\leq \mathbf{P}\left(\bigcup_{j=1}^{\beta_{n,m}-1} \{M(I_j) \leq u_n < M(I_j^*)\}\right) \\ &\leq \frac{\beta_{n,m} \ell_{k_n}}{k_{n-m}} k_{n-m} (1 - F(u_n)) - \frac{\ell_{k_n}}{k_n} k_n (1 - F(u_n)) = o_n(1) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \mathbf{P}\{M_{k_{n-m}} \leq u_n\} - \mathbf{P}\{M_{\beta_{n,m} r_{n,m} - \ell_{k_n}} \leq u_n\} \leq \mathbf{P}\left(\bigcup_{j=k_{n-m}+1}^{\beta_{n,m} r_{n,m} - \ell_{k_n}} \{X_j > u_n\}\right) \\ &\leq \frac{\beta_{n,m} r_{n,m} - \ell_{k_n} - k_{n-m}}{k_{n-m}} k_{n-m} (1 - F(u_n)) = o_n(1), \end{aligned} \quad (9)$$

by (4) ii). Hence, we conclude that

$$\lim_{n \rightarrow +\infty} \left(\mathbf{P}\left(\bigcap_{j=1}^{\beta_{n,m}} \{M(I_j) \leq u_n\}\right) - \mathbf{P}\{M_{k_{n-m}} \leq u_n\} \right) = 0. \quad (10)$$

Moreover, taking into account that $D_{k_{n-m}}(u_n)$ holds, from Lemma 2 and using stationarity, we have

$$\left| \mathbf{P}\left(\bigcap_{j=1}^{\beta_{n,m}} \{M(I_j) \leq u_n\}\right) - \mathbf{P}^{\beta_{n,m}} \{M(I_1) \leq u_n\} \right| \leq (\beta_{n,m} - 1) \alpha_{n, \ell_{k_n}} = o_n(1), \quad (11)$$

since (4) iii) holds.

Due to $|\mathbf{P}\{M(I_1) \leq u_n\} - \mathbf{P}\{M_{r_{n,m}} \leq u_n\}| \leq \ell_{k_n} (1 - F(u_n))$, we may use the inequality

$$\left| \prod_{i=1}^k a_i - \prod_{i=1}^k b_i \right| \leq \sum_{i=1}^k |a_i - b_i|, \quad (12)$$

with $a_1, \dots, a_k, b_1, \dots, b_k$ in $[0, 1]$, to obtain

$$\begin{aligned} &\left| \mathbf{P}^{\beta_{n,m}} \{M(I_1) \leq u_n\} - \mathbf{P}^{\beta_{n,m}} \{M_{r_{n,m}} \leq u_n\} \right| \\ &\leq \frac{\beta_{n,m} \ell_{k_n}}{k_{n-m}} k_{n-m} (1 - F(u_n)) = o_n(1). \end{aligned} \quad (13)$$

Therefore, from (10), (11) and (13), follows (5).

To prove (6), observe that for each m

$$\begin{aligned} &\left| \mathbf{P}\{M_{k_{n-m}} \leq u_n\} - \mathbf{P}^{r^{-m} s_n} \{M_{r_{n,m}} \leq u_n\} \right| \\ &\leq \left| \mathbf{P}\{M_{k_{n-m}} \leq u_n\} - \mathbf{P}^{\beta_{n,m}} \{M_{r_{n,m}} \leq u_n\} \right| \\ &\quad + \left| \mathbf{P}^{\beta_{n,m}} \{M_{r_{n,m}} \leq u_n\} - \mathbf{P}^{r^{-m} s_n} \{M_{r_{n,m}} \leq u_n\} \right|. \end{aligned} \quad (14)$$

It is thus enough to prove that the second summand in the right-hand side of (14) is an $o_n(1)$. The Lagrange Theorem applied to the function $f(x) = u^x$, u in $]0, 1]$, x in $]0, +\infty[$, gives $0 \leq u^x - u^y \leq u^x(\ln u)(x - y)$, $0 < x \leq y$. Then, for n and m such that $\mathbf{P}\{M_{r_{n,m}} \leq u_n\} \neq 0$ and taking into account that $\beta_{n,m} \geq r^{-m}s_n$, we can write

$$\begin{aligned} 0 &\leq \mathbf{P}^{r^{-m}s_n}\{M_{r_{n,m}} \leq u_n\} - \mathbf{P}^{\beta_{n,m}}\{M_{r_{n,m}} \leq u_n\} \\ &\leq \mathbf{P}^{r^{-m}s_n}\{M_{r_{n,m}} \leq u_n\} \ln \mathbf{P}^{r^{-m}s_n}\{M_{r_{n,m}} \leq u_n\} \left(1 - \frac{\beta_{n,m}}{r^{-m}s_n}\right). \end{aligned} \quad (15)$$

Observe now that $g(x) = x \ln x$ is a bounded function on $]0, 1]$ and satisfies $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 1} g(x) = 0$. Thus, with a careful study of all possible behaviours of $\mathbf{P}^{r^{-m}s_n}\{M_{r_{n,m}} \leq u_n\}$, as $n \rightarrow +\infty$, we prove, taking also (4) iv) into account, that the right-hand side of (15) is an $o_n(1)$.

To obtain (7) we must first observe that from (5) we have

$$\lim_{n \rightarrow +\infty} \left(\mathbf{P}^{r^m}\{M_{k_{n-m}} \leq u_n\} - \mathbf{P}^{r^m\beta_{n,m}}\{M_{r_{n,m}} \leq u_n\} \right) = 0. \quad (16)$$

Furthermore, observe that, for $m \neq 0$,

$$\begin{aligned} \left| \mathbf{P}\{M_{k_n} \leq u_n\} - \mathbf{P}^{r^m}\{M_{k_{n-m}} \leq u_n\} \right| &\leq \left| \mathbf{P}\{M_{k_n} \leq u_n\} - \mathbf{P}^{\beta_{n,0}}\{M_{r_{n,0}} \leq u_n\} \right| \\ &+ \left| \mathbf{P}^{\beta_{n,0}}\{M_{r_{n,0}} \leq u_n\} - \mathbf{P}^{\beta_{n,0}}\{M_{r_{n,m}} \leq u_n\} \right| \\ &+ \left| \mathbf{P}^{\beta_{n,0}}\{M_{r_{n,m}} \leq u_n\} - \mathbf{P}^{\beta_{n,m}r^m}\{M_{r_{n,m}} \leq u_n\} \right| \\ &+ \left| \mathbf{P}^{\beta_{n,m}r^m}\{M_{r_{n,m}} \leq u_n\} - \mathbf{P}^{r^m}\{M_{k_{n-m}} \leq u_n\} \right|. \end{aligned}$$

From (5), with $m = 0$, and (16) we conclude that the first and the fourth sums are asymptotically zero. On the other hand, using (12) and following the procedure used in (9), we prove that the second sum does not exceed

$$\left| \frac{\beta_{n,0}r_{n,0}}{k_n} \right| \left| 1 - \frac{r_{n,m}}{r_{n,0}} \right| k_n(1 - F(u_n)) = o_n(1),$$

since (4) ii) and (4) v) hold.

With the arguments used in (15) we conclude also that, for each positive integer m , by (4) vi), the third sum converges to zero.

The proof of (8) follows exactly the steps of the proof of Lemma 3.3.1 in [6]. The construction of the intervals is the same with the obvious modifications. For instance $I_1 = \{1, \dots, [k_n/m] - \ell_{k_n}\}$ and $I_{m+1}^* = \{m[k_n/m] + 1, \dots, m[k_n/m] + \ell_{k_n}\}$.

Taking into account the fact that a d.f. G is ℓ -max-stable if, for each positive integer $m \geq 2$, there exist real constants $a_m > 0$ and b_m such that $G^m(a_mx + b_m) = G(x)$, the Leadbetter's Extremal Types Theorem has been proven using (8) with $k_n = n$. However the parameter r that arises in the definition of ℓ -max-semistable laws is a real positive number. This justifies the need for (7) in the proof of the following theorem, which is the desired generalization of (3) and of the above mentioned Leadbetter's Extremal Types Theorem.

Theorem 2. *Let $\{k_n\}$ be an integer sequence satisfying (1). Let $\{X_n\}$ be a stationary sequence and let $\{a_n > 0\}$ and $\{b_n\}$ be real sequences such that $\limsup_{n \rightarrow +\infty} k_n(1 - F(x/a_n + b_n)) < +\infty$ and $\lim_{n \rightarrow +\infty} \mathbf{P}\{a_n(M_{k_n} - b_n) \leq x\}$ converges to $G(x)$ for each continuity point of the nondegenerate d.f. G . If $\{X_n\}$ satisfies $D_{k_n}(x/a_n + b_n)$, for all x in $\mathbf{R}$, then G belongs to one of the three classes of ℓ -max-semistable laws listed in Theorem 1.*

Proof. Consider $u_n = x/a_n + b_n$. Let us assume that (1) holds, with $r > 1$. By (7), we obtain $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_{n-1}} \leq u_n\} = G^{r^{-1}}(x)$, or, equivalently, $\lim_{n \rightarrow +\infty} \mathbf{P}\{a_{n+1}(M_{k_n} - b_{n+1}) \leq x\} = G^{r^{-1}}(x)$. From Khinchin's Theorem it results $G(x) = G^r(ax + b)$, where $a = \lim_{n \rightarrow +\infty} a_n/a_{n+1}$ and $b = \lim_{n \rightarrow +\infty} a_n(b_{n+1} - b_n)$. Hence G is ℓ -max-semistable.

If (1) holds with $r = 1$, we observe that for all positive integers n and m there exists $p_n = p_n(m)$ such that $\lim_{n \rightarrow +\infty} p_n = +\infty$ and $k_n/m = k_{p_n}(1 + o_n(1))$. Thus

$\mathbf{P}\{M_{k_{p_n}} \leq u_n\} = \mathbf{P}\{M_{[k_n/m]} \leq u_n\}(1 + o_n(1))$. Therefore, taking (8) into account, we have $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_{p_n}} \leq u_n\} = G^{1/m}(x)$. Since $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_{p_n}} \leq u_{p_n}\} = G(x)$, using again Khinchin's Theorem we obtain the stability equation $G^m(ax + b_m) = G(x)$ that defines ℓ -max-stable laws.

Now, we will investigate if the existence of a limiting distribution for the maximum $\bar{M}_n$ implies a limiting distribution in the same class for M_n , with the same normalizing constants. It is well known that if $\{X_n\}$ is a sequence of i.i.d. random variables then $\lim_{n \rightarrow +\infty} k_n(1 - F(u_n)) = \tau$ and $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq u_n\} = e^{-\tau}$ are equivalent. Thus, in what follows we deal with levels u_n satisfying the above first limit, which will be denoted by $u_n(\tau, k_n)$ and put $\Gamma(F, k_n) = \{\tau > 0: \exists \{u_n\}: \lim_{n \rightarrow +\infty} k_n(1 - F(u_n)) = \tau\}$. We observe that if F is a discrete d.f. and $\lim_{n \rightarrow +\infty} k_{n+1}/k_n = r > 1$ then $\Gamma(F, k_n)$ is not necessarily equal to $]0, +\infty[$.

For the subsequent results we need the following lemma, where stationarity is not imposed. The proof is omitted since it is similar to those of Lemma 3.6.1 ii) and Lemma 3.6.2 ii) of [6], with the obvious modifications.

Lemma 4. *Let $\{X_n\}$ be a random sequence with common marginal distribution function F . Let $\{k_n\}$ and $\{k_n^*\}$ be positive integer sequences.*

1. *Suppose that the real sequences $\{v_n\}$ and $\{w_n\}$ satisfy $\lim_{n \rightarrow +\infty} k_n^*(F(v_n) - F(w_n)) = 0$.*

Then

i) *if $k_n^* \leq k_n$ and if $D_{k_n}(v_n)$ holds, then so does $D_{k_n^*}(w_n)$,*

ii) *if $k_n^* = k_n$ then $D_{k_n}(v_n)$ holds if and only if $D_{k_n^*}(w_n)$ holds.*

2. i) *If for some τ_0 , $D_{k_n}(u_n(\tau_0, k_n))$ holds, so does $D_{k_n^*}(u_n(\tau, k_n^*))$ with $0 < \tau \leq \tau_0$ and $k_n^* = [\tau k_n / \tau_0]$.*

ii) *$D_{k_n}(u_n(\tau_0, k_n))$ holds if and only if $D_{k_n}(u_n(\alpha\tau_0, [\alpha k_n]))$ holds for all $\alpha > 0$.*

The following theorem generalizes Theorem 2.2 of Leadbetter [5]. The proof here follows along similar lines to those of Leadbetter.

Theorem 3. *Let $\{X_n\}$ be a stationary sequence and $\{k_n\}$ an integer sequence satisfying (1).*

1. *If $\{X_n\}$ satisfies $D_{k_n}(u_n(\tau_0, k_n))$ for some $\tau_0 \in \Gamma(F, k_n)$, then there exist real constants θ and θ' , with $0 \leq \theta \leq \theta' \leq 1$, such that*

$$e^{-\theta'\tau} = \liminf_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n^*} \leq u_n(\tau, k_n^*)\} \leq \limsup_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n^*} \leq u_n(\tau, k_n^*)\} = e^{-\theta\tau},$$

for τ in $]0, \tau_0]$ and $k_n^ = [\tau k_n / \tau_0]$. Hence if $\mathbf{P}\{M_{k_n^*} \leq u_n(\tau, k_n^*)\}$ converges for some τ in $]0, \tau_0]$ then $\theta = \theta'$ and $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n^*} \leq u_n(\tau, k_n^*)\} = e^{-\theta\tau}$ for all such τ .*

2. *If $\{X_n\}$ satisfies $D_{k_n}(u_n(\tau, k_n))$ for all $\tau \in \Gamma(F, k_n)$, then there exist real constants θ and θ' , with $0 \leq \theta \leq \theta' \leq 1$, such that*

$$e^{-\theta'\tau} = \liminf_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq u_n(\tau, k_n)\} \leq \limsup_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq u_n(\tau, k_n)\} = e^{-\theta\tau}.$$

Hence, if $\mathbf{P}\{M_{k_n} \leq u_n(\tau, k_n)\}$ converges, for some $\tau \in \Gamma(F, k_n)$, then

$$\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq u_n(\tau, k_n)\} = e^{-\theta\tau},$$

for all such τ .

P r o o f. For each τ in $]0, \tau_0]$ let us define $k_n^* = [\tau k_n / \tau_0]$ and $\limsup_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n^*} \leq u_n(\tau, k_n^*)\} = \Psi(\tau)$.

Suppose first that $\lim_{n \rightarrow +\infty} k_{n+1}/k_n = r > 1$. In view of (7) and Lemma 4.1 i), for each positive integer m , we obtain

$$\limsup_{n \rightarrow +\infty} \mathbf{P}\{M_{k_{n-m}^*} \leq u_n(\tau, k_n^*)\} = \Psi^{r^{-m}}(\tau).$$

On the other hand $\mathbf{P}\{M_{k_{n-m}^*} \leq u_n(\tau, k_n^*)\} = \mathbf{P}\{M_{k_{n-m}^*} \leq u_{n-m}(r^{-m}\tau, k_{n-m}^*)\}(1 + o_n(1))$.

Thus $\Psi^{r^{-m}}(\tau) = \limsup_{n \rightarrow +\infty} \mathbf{P}\{M_{k_{n-m}^*} \leq u_{n-m}(r^{-m}\tau, k_{n-m}^*)\} = \Psi(r^{-m}\tau)$ and hence $\Psi(\tau) = e^{-\theta\tau}$, with $\theta \geq 0$.

Similarly we prove that $\liminf_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n^*} \leq u_n(\tau, k_n^*)\} = e^{-\theta' \tau}$, with $\theta' > \theta$. Using again (7) and Boole's inequality we easily prove that $\liminf_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n^*} \leq u_n(\tau, k_n^*)\} \geq e^{-\tau}$, and thus $\theta' \leq 1$.

Now, if $\lim_{n \rightarrow +\infty} k_{n+1}/k_n = 1$, we must observe that, for each n and m , there exists a positive integer $p_n = p_n(m)$ such that $\lim_{n \rightarrow +\infty} p_n = +\infty$ and $[k_n^*/m] = k_{p_n}^* (1 + o_n(1))$. In this case, from (8) and Lemma 4.1 i), we obtain

$$\limsup_{n \rightarrow +\infty} \mathbf{P}\{M_{[k_n^*/m]} \leq u_n(\tau, k_n^*)\} = \Psi^{1/m}(\tau).$$

Hence, since $\mathbf{P}\{M_{[k_n^*/m]} \leq u_n(\tau, k_n^*)\} = \mathbf{P}\{M_{k_{p_n}^*} \leq u_n(\tau, k_n^*)\} (1 + o_n(1))$ and $\mathbf{P}\{M_{k_{p_n}^*} \leq u_n(\tau, k_n^*)\} = \mathbf{P}\{M_{k_{p_n}^*} \leq u_{p_n}(\tau/m, k_{p_n}^*)\} (1 + o_n(1))$, we conclude the proof of point 1 taking into account that

$$\Psi^{1/m}(\tau) = \limsup_{n \rightarrow +\infty} \mathbf{P}\left\{M_{k_{p_n}^*} \leq u_{p_n}\left(\frac{\tau}{m}, k_{p_n}^*\right)\right\} = \Psi\left(\frac{\tau}{m}\right).$$

After the above procedures the proof of point 2 is now obvious.

It will be convenient to give a name for the parameter appeared in the above theorem. Specifically, we shall say that the stationary sequence $\{X_n\}$ has an extremal index θ , with θ in $[0, 1]$, if there exists $\{k_n\}$ satisfying (1) such that, for all $\tau \in \Gamma(F, k_n)$, we have $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq u_n(\tau, k_n)\} = e^{-\theta \tau}$.

The following lemma, a generalization of Lemma 2.4 of Leadbetter [5], enables us to show the relationship between the limiting behaviour of M_{k_n} and $\widehat{M}_{k_n}$ in stationary sequences having an extremal index. The proof is omitted because it is similar to the one of Leadbetter.

Theorem 4. *Suppose that the stationary sequence $\{X_n\}$ has an extremal index θ . Let $\{k_n\}$ be a positive integer sequence, $\{v_n\}$ a real sequence and let $\rho \in [0, 1]$. Hence*

- 1) *for $\theta > 0$, $\lim_{n \rightarrow +\infty} \mathbf{P}\{\widehat{M}_{k_n} \leq v_n\} = \rho$ if and only if $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq v_n\} = \rho^\theta$, and*
- 2) *for $\theta = 0$*
 - i) *if $\liminf_{n \rightarrow +\infty} \mathbf{P}\{\widehat{M}_{k_n} \leq v_n\} > 0$, then $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq v_n\} = 1$,*
 - ii) *if $\limsup_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq v_n\} < 1$, then $\lim_{n \rightarrow +\infty} \mathbf{P}\{\widehat{M}_{k_n} \leq v_n\} = 0$.*

Observing the case $\theta > 0$ in last theorem we conclude that it is also a question of interest to know if the ℓ -max-semistable d.f.'s G and G^θ , with $\theta > 0$, belong to the same type. That is if there are real constants $c_\theta > 0$ and d_θ such that $G^\theta(x) = G(c_\theta x + d_\theta)$. We notice also that if G is ℓ -max-stable then G and G^θ are always in the same type (see Corollary 1.3.2 in [6]). Thus, the following lemma is trivial for ℓ -max-stable laws.

Lemma 5. *Let G be a nondegenerate ℓ -max-semistable distribution function. Then $G^\theta(x) = G(c_\theta x + d_\theta)$ for some real $\theta > 0$, $c_\theta > 0$ and d_θ if and only if there is an integer m such that $\theta = r^{-m}$. Furthermore, if G is in the class G_1 or G_2 , then $c_\theta = \gamma^{-m}$ and $d_\theta = t(1 - \gamma^{-m})$ and if G is in the class G_3 , then $c_\theta = 1$ and $d_\theta = m\beta$.*

P r o o f. If G is ℓ -max-semistable then $G(x) = G^r(x/\gamma + \beta)$ for some $r > 1$, $\gamma > 0$ and β real.

We start by considering a positive integer m and $\theta = r^{-m}$. By induction we deduce $G(u^{\circ m}(x)) = G^{r^{-m}}(x)$ with $u(x) = x/\gamma + \beta$, where $u^{\circ m}(x)$ denotes the composition $u \circ u \circ \dots \circ u$ (m times). Writing $u^{\circ m}(x) = c_\theta x + d_\theta$ we obtain the mentioned values for c_θ and d_θ .

Consider now a negative integer m and $\theta = r^{-m}$. Putting $v(x) = \gamma(x - \beta)$, we deduce, again by induction, that $G^{1/r^{-m}}(v^{\circ(-m)}(x)) = G(x)$ for each real x , that is, $G(v^{\circ(-m)}(x)) = G^\theta(x)$. Thus, with $v^{\circ(-m)}(x) = c_\theta x + d_\theta$ we obtain again the desired values for c_θ and d_θ .

Suppose now that G is ℓ -max-semistable and that there are real constants $\theta > 0$, $c_\theta > 0$ and d_θ such that $G^\theta(x) = G(c_\theta x + d_\theta)$.

Observing that $G^{\theta/r}(x) = G((c_\theta x + d_\theta)/\gamma + \beta) = G(c_\theta(x/\gamma + \beta) + d_\theta)$ we obtain, using Corollary 1.2.2 of [6], $d_\theta/\gamma + \beta = c_\theta \beta + d_\theta$. Therefore $d_\theta = t(1 - c_\theta)$ if G is in G_1 or G_2 , and $c_\theta = 1$ if G is in G_3 .

Taking into account the analytical form of each d.f. in G_1 we deduce

$$\theta \nu(\ln(t-x)) = c_\theta^\alpha \nu(\ln c_\theta + \ln(t-x)). \quad (17)$$

For each integer k , let us consider the real values $x_k = t - c_\theta^k$ for which equation (17) is transformed in $\nu(k \ln c_\theta) = \psi \nu((k+1) \ln c_\theta)$, with $\psi = c_\theta^\alpha / \theta$. Consequently, for each positive integer k ,

$$\dots = \psi^{-k} \nu(-k \ln c_\theta) = \dots = \nu(0) = \dots = \psi^k \nu(k \ln c_\theta) = \dots$$

Hence, if $\psi < 1$, we have

$$\dots < \nu(-k \ln c_\theta) < \dots < \nu(0) < \nu(\ln c_\theta) < \dots < \nu(k \ln c_\theta) < \dots$$

which is a contradiction because ν is bounded and periodic. Similarly we prove that ψ is not less than 1 and thus $\psi = 1$.

Since ν is periodic with period $\ln \gamma$, from (17) with $\theta = c_\theta^\alpha$, we can conclude that there is an integer m satisfying $\ln c_\theta = -m \ln \gamma$. The constant m is, clearly, positive or negative if c_θ is less or greater than 1, respectively. Taking into account the fact that α is positive and $r = \gamma^\alpha$, it follows $\ln \theta = \alpha \ln c_\theta = -m \alpha \ln \gamma$, and then $\theta = r^{-m}$. Thus, $c_\theta = \gamma^{-m}$ and $d_\theta = t(1 - \gamma^{-m})$.

For the class G_2 the proof is similar.

For d.f.'s in the class G_3 we have

$$\theta \nu(x) = \nu(x + d_\theta) \exp(-\alpha d_\theta). \quad (18)$$

Considering, for each integer k , $x_k = kd_\theta$ and $\psi = \exp(-\alpha d_\theta)/\theta$, we obtain $\nu(kd_\theta) = \psi \nu((k+1)d_\theta)$. With the above arguments we prove that $\psi = 1$ and, therefore, from (18) we deduce that there exists an integer m such that $d_\theta = m\beta$. Thus, since $\theta = \exp(-\alpha d_\theta)$ and $\ln r = \alpha\beta$ we have $\theta = r^{-m}$, which finalizes the proof.

Corollary. Let the stationary sequence $\{X_n\}$ have an extremal index θ and suppose that $\{k_n\}$ is an integer sequence satisfying (1).

1. If $\theta > 0$ then, with $a_n > 0$ and b_n real, $\lim_{n \rightarrow +\infty} \mathbf{P}\{a_n(\widehat{M}_{k_n} - b_n) \leq x\} = G(x)$ if and only if $\lim_{n \rightarrow +\infty} \mathbf{P}\{a_n(M_{k_n} - b_n) \leq x\} = G^\theta(x)$, and

i) if there exist a positive integer m such that $\theta = r^{-m}$, then G and G^θ are of the same type,

ii) if there does not exist a positive integer m such that $\theta = r^{-m}$, then G and G^θ are not of the same type but they belong to the same class of ℓ -max-semistable laws.

2. Otherwise, if $\theta = 0$ and $\{X_n\}$ satisfies $D_{k_n}(u_n(\tau, k_n))$ for each $\tau \in \Gamma(F, k_n)$, then M_{k_n} and $\widehat{M}_{k_n}$ cannot both have nondegenerate limiting distributions with the same normalizing constants.

P r o o f. The proof of point 1 follows directly from Theorem 4 and from Lemma 5. In order to prove ii) we observe that if G is a ℓ -max-semistable law then G^θ , with $\theta > 0$, belongs to the same class as G with ν replaced by $\theta\nu$.

We prove 2 following the steps of Corollary 3.7.4 in [6]. Suppose that

$$\lim_{n \rightarrow +\infty} \mathbf{P}\{a_n(\widehat{M}_{k_n} - b_n) \leq x\} = G(x), \quad \lim_{n \rightarrow +\infty} \mathbf{P}\{a_n(M_{k_n} - b_n) \leq x\} = H(x)$$

for nondegenerate G and H .

Thus, with $v_n = x/a_n + b_n$, it follows from Theorem 4 that $H(x) = 1$ for $x \geq t > -\infty$, where $t = \inf\{x: G(x) > 0\}$. So G belongs to the class G_2 .

By Theorem 2.1 in [3], if $\{\varepsilon_n\}$ is such that $\lim_{n \rightarrow +\infty} k_n(1 - F(\varepsilon_n)) = \nu(0)$ we obtain $\lim_{n \rightarrow +\infty} \mathbf{P}\{\widehat{M}_{k_n} \leq x\varepsilon_n\} = G(x+t)$. Using Khinchin's theorem twice we conclude that $\lim_{n \rightarrow +\infty} \varepsilon_n a_n = 1$, $\lim_{n \rightarrow +\infty} a_n b_n = -t$ and that $\lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq x\varepsilon_n\} = H(x+t)$.

On the other hand, taking into account that $\lim_{n \rightarrow +\infty} k_n(1 - F(x/a_n + b_n)) = -\ln G(x)$, we have that $D_{k_n}(x/a_n + b_n)$ holds. Therefore, H belongs to the class G_1 and then $\lim_{n \rightarrow +\infty} \varepsilon_n < +\infty$ which is a contradiction.

E x a m p l e. Let us consider the max-autoregressive model $X_n = k \max\{X_{n-1}, Y_n\}$, where $\{Y_n\}$ is a sequence of i.i.d. random variables with common d.f. G and k is a constant in $]0, 1[$.

Alpuim [1] has proven that $\{X_n\}$ is strong mixing and has a stationary d.f. H if and only if there exists some $x > 0$ such that $G(x/k) > 0$ and $0 < \sum_{j=1}^{+\infty} (-\ln G(x/k^j)) < +\infty$; in this case, $H(x) = \prod_{j=1}^{+\infty} G(x/k^j)$.

In this example we consider $\{Y_n\}$ with d.f. $G(x) = \exp\{-x^{-\alpha}\nu(\ln x)\}$, $\alpha > 0$, $x > 0$, and $k = r^{-m/\alpha}$, where m is a positive integer. We recall that r and α are such that $\alpha = \ln r / (-\ln \gamma)$ and $G^r(x/\gamma) = G(x)$ which implies that

$$\lim_{n \rightarrow +\infty} G^{[r^n]}(\gamma^{-n}x) = G(x). \quad (19)$$

Since, for $x > 0$, $-\ln G(x/k^j) = (x/k^j)^{-\alpha}\nu(\ln(x/k^j))$ and $0 < \sum_{j=1}^{+\infty} (k^\alpha)^j = \sum_{j=1}^{+\infty} (r^{-m})^j < +\infty$ we have $0 < \sum_{j=1}^{+\infty} (-\ln G(x/k^j)) < +\infty$ since ν is positive and bounded.

Moreover, taking into consideration the fact that, for each positive integer j , $\ln k^j = jm \ln \gamma$ and ν has period $p = -\ln \gamma$ we obtain $\nu(\ln(x/k^j)) = \nu(\ln x)$.

Consequently,

$$H(x) = \prod_{j=1}^{+\infty} \exp\left(-\left(\frac{x}{k^j}\right)^{-\alpha} \nu\left(\ln\left(\frac{x}{k^j}\right)\right)\right) = \exp\left(-x^{-\alpha}\nu(\ln x) \sum_{j=1}^{+\infty} k^{j\alpha}\right) = G^\beta(x),$$

with $\beta = k^\alpha / (1 - k^\alpha)$.

Now, due to (19), with $k_n = [r^n]$, we have $\lim_{n \rightarrow +\infty} H^{k_n}(\gamma^{-n}x) = G^\beta(x) = H(x)$. On the other hand, since $\mathbf{P}\{M_{k_n} \leq \gamma^{-n}x\} = H(\gamma^{-n}x) G^{k_n}(\gamma^{-n}x/k) \rightarrow G(x/k)$, $n \rightarrow +\infty$, we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbf{P}\{M_{k_n} \leq \gamma^{-n}x\} &= \exp\left(-\frac{1}{\beta} \left(\frac{x}{k}\right)^{-\alpha} \beta \nu\left(\ln \frac{x}{k}\right)\right) \\ &= \exp(-(1 - k^\alpha)(x^{-\alpha}) \beta \nu(\ln x)) = H^{1-k^\alpha}(x). \end{aligned}$$

Thus $\{X_n\}$ has an extremal index $\theta = 1 - k^\alpha = 1 - r^{-m}$. Furthermore, H and H^θ are of the same type if and only if there exists a positive integer p such that $1 - r^{-m} = r^{-p}$, for instance if $r = 2$ and $m = 1$.

REFERENCES

1. Alpuim M. T. An extremal Markovian sequence. — J. Appl. Probab., 1989, v. 26, № 2, p. 219–232.
2. Гриневич И. В. Макс-полуустойчивые предельные распределения, отвечающие линейной и степенной нормировке. — Теория вероятн. и ее примен., 1992, т. 37, в. 4, с. 774–776.
3. Гриневич И. В. Области притяжения макс-полуустойчивых законов при линейной и степенной нормировках. — Теория вероятн. и ее примен., 1993, т. 38, в. 4, с. 787–799.
4. Leadbetter M. R. On extreme values in stationary sequences. — Z. Wahrscheinlichkeitstheor. verw. Geb., 1974, v. 28, p. 289–303.
5. Leadbetter M. R. Extremes and local dependence in stationary sequences. — Z. Wahrscheinlichkeitstheor. verw. Geb., 1983, v. 65, p. 291–306.
6. Leadbetter M. R., Lindgren G., Rootzén H. Extremes and Related Properties of Random Sequences and Processes. Berlin–Heidelberg: Springer-Verlag, 1983, 336 p.
7. Pancheva E. Multivariate max-semistable distributions. — Теория вероятн. и ее примен., 1992, т. 37, в. 4, с. 794–795.

Поступила в редакцию
13.V.1999