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MODERATE DEVIATIONS FOR STUDENT'S STATISTIC¹⁾

Для самонормированных сумм, скажем, S_n/V_n , при условиях симметрии рассматриваются зоны Линника, где отношение $\mathbf{P}\{S_n/V_n \geq x\}/(1 - \Phi(x))$ сходится к 1, и устанавливаются оптимальные оценки для остаточных членов при этой сходимости.

Ключевые слова и фразы: зоны Линника, самонормированная сумма, t -статистика, умеренные отклонения, неравномерные оценки.

1. Introduction and results. Let $X, X_1, X_2, \dots$ denote independent identically distributed (i.i.d.) random variables such that $\mathbf{E}X = 0$ and $0 < \sigma^2 = \mathbf{E}X^2 < \infty$. Write

$$S_n = \sum_{j=1}^n X_j \quad \text{and} \quad V_n^2 = \sum_{j=1}^n X_j^2.$$

Define the self-normalized sum as S_n/V_n , if $V_n \neq 0$, and 0 if $V_n = 0$.

Let $\Lambda(n)$ denote a nondecreasing function satisfying $\Lambda(n) \rightarrow \infty$ as $n \rightarrow \infty$. The interval $[0, \Lambda(n)]$ is called zone of normal convergence if

$$\frac{1 - \mathbf{P}\{S_n < x\sigma\sqrt{n}\}}{1 - \Phi(x)} \rightarrow 1, \quad \frac{\mathbf{P}\{S_n < -x\sigma\sqrt{n}\}}{\Phi(-x)} \rightarrow 1 \quad (1.1)$$

uniformly in $x \in [0, \Lambda(n)]$, where Φ denotes the standard Gaussian distribution function. It is well known, assuming that the condition $\beta = \mathbf{E}|X|^3 < \infty$ holds, that the Berry–Esseen bound implies (1.1) with $\Lambda(n) = (1 - \varepsilon)\sqrt{\ln n}$, for $0 < \varepsilon < 1$. In this case the zone of normal convergence is of logarithmic size and is rather narrow. Consider the condition

$$\mathbf{E} \exp \{|X_1|^{4\alpha/(2\alpha+1)}\} < \infty \quad (1.2)$$

with some α such that $0 < \alpha \leq \frac{1}{6}$. Linnik [13] (see as well [12]) proved the following result. For any function $\rho(n)$ with $\lim_{n \rightarrow \infty} \rho(n) = \infty$ the relation (1.1) holds with $\Lambda(n) = n^\alpha/\rho(n)$. Furthermore, if (1.1) holds with

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$\Lambda(n) = n^\alpha \rho(n)$, then the condition (1.2) is fulfilled. This result of Linnik shows that for deterministically normalized sums the existence of exponential moments is a natural condition for the existence of power type zones of normal convergence.

Giné, Götze, and Mason [8] noticed that the tails of S_n/V_n are uniformly sub-Gaussian in cases when the sequence S_n/V_n is stochastically bounded. This observation suggests that Linnik type results for self-normalized sums have to hold under conditions considerably weaker than (1.2). Shao [19] established that for self-normalized sums Linnik type results hold under the more natural condition of the existence of the third moment $\beta < \infty$. In this paper under symmetry conditions we shall obtain optimal explicit bounds for the remainders in relations (1.1) for the self-normalized sums (see Theorem 1 and Corollary 1 below). The bounds will depend on n, x, σ, β only. In Theorem 2 we extend the results to the non-i.i.d. situation. In this more general setting the bounds depend on x and on Lyapunov's ratio L_n only. Below we shall denote by θ different real valued quantities such that $|\theta| \leq 1$. The notation $c, c_1, c_2, \dots$ will be used throughout for absolute positive constants. By c we denote different constants in different (or even in the same) formulae. The symbols $c_1, c_2, \dots$ are applied for concrete constants. We shall prove the following relations:

$$\frac{\mathbf{P}\{S_n/V_n > x\}}{1 - \Phi(x)} = 1 + c\theta r_n(x), \quad \text{for } x \geq 0, \quad (1.3)$$

$$\frac{\mathbf{P}\{S_n/V_n < x\}}{\Phi(x)} = 1 + c\theta r_n(x), \quad \text{for } x \leq 0, \quad (1.4)$$

where $r_n(x)$ is a function which we have to specify. Notice that for $|x| > \sqrt{n}$ the left-hand sides of (1.3) and (1.4) are equal to zero since $\mathbf{P}\{|S_n|/V_n > |x|\} = 0$.

Theorem 1. *Assume that X is a symmetric random variable such that $\sigma^2 = 1$ and $\beta < \infty$. Then relations (1.3) and (1.4) hold with*

$$r_n(x) = \begin{cases} \frac{(|x|^3 + 1)\beta}{\sqrt{n}}, & \text{for } 0 \leq |x| \leq \frac{n^{1/6}}{\beta^{1/3}}, \\ 1, & \text{for } |x| > \frac{n^{1/6}}{\beta^{1/3}}. \end{cases}$$

Write

$$\begin{aligned} G(x, l) &= \sup \mathbf{P}\left\{\frac{S_n}{V_n} > x\right\}, \quad x \geq 0, \\ G(x, l) &= \sup \mathbf{P}\left\{\frac{S_n}{V_n} < x\right\}, \quad x \leq 0, \end{aligned} \quad (1.5)$$

where the supremum is taken over all symmetric X such that the Lyapunov ratio $L_n := \beta/(\sigma^3\sqrt{n})$ is fixed, that is, $L_n = l$. Let $\rho(l)$ be a function such

that $\rho(l) \geq 1$. Consider the relations

$$\frac{G(x, l)}{1 - \Phi(x)} = 1 + c\theta\left(l + \frac{1}{\rho(l)}\right), \quad \text{for } x \geq 0, \quad (1.6)$$

$$\frac{G(x, l)}{\Phi(x)} = 1 + c\theta\left(l + \frac{1}{\rho(l)}\right), \quad \text{for } x \leq 0. \quad (1.7)$$

Corollary 1. *Under the assumptions of Theorem 1 relations (1.6) and (1.7) hold for $|x| \leq 1/(\rho(l))^{1/3}$.*

In particular, Corollary 1 yields

$$\frac{\mathbf{P}\{S_n/V_n > x\}}{1 - \Phi(x)} = 1 + o(1), \quad (1.8)$$

uniformly for $x = o(n^{1/6})$. To see this it suffices to choose an appropriate function ρ .

For $|x| \leq L_n^{-1/3}$ estimates (1.3) and (1.4) with $r_n(x)$ from Theorem 1 are exact in the following sense.

Proposition. *Let $\{b_n\}_{n=1}^\infty$ be a sequence of real numbers such that $b_n \geq 1$ and $b_n/\sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$. Then for every fixed $x \in (0, n^{1/6}/b_n^{1/3}]$ there exist symmetric random variables $X = X(x, n)$ and absolute positive constants c, c_1, c_2 such that $\mathbf{E}X^2 = 1$, $c_1b_n \leq \mathbf{E}|X|^3 \leq c_2b_n$, and*

$$\frac{\mathbf{P}\{S_n/V_n > x\}}{1 - \Phi(x)} \leq 1 - c(x^3 - c(x^2 + 1)) \frac{b_n}{\sqrt{n}}.$$

This inequality holds for sufficiently large $\sqrt{n}/b_n \geq c$.

Let us discuss some results for self-normalized sums related to the central limit theorem (CLT) and their refinements. Self-normalized sums satisfy the CLT provided that X belongs to the domain of attraction of a normal law [15], [4]). Griffin and Mason [9] proved that the condition is necessary in the symmetric case. Giné, Götze, and Mason [8] proved that the condition is necessary for nonsymmetric X as well. Egorov [6] established necessary and sufficient conditions in the non-i.i.d. symmetric case. These results indicate that the self-normalization may improve the limiting behaviour of sums.

Berry-Esseen bounds of increasing generality and accuracy were obtained in [10], [22], [21], [1], [2]. The paper [1] contains bounds which imply, in particular, sufficient conditions for the CLT. The bounds extend to the non-i.i.d. case, see [2]. The bounds of the last two papers imply our results for $|x| \leq c$.

Sub-Gaussian type behaviour of tails related to self-normalized sums was observed in [14]. Giné, Götze, and Mason [8] showed that the tails of S_n/V_n are uniformly sub-Gaussian in the case when a sequence S_n/V_n , $n \geq 1$, is stochastically bounded. Shao ([18]–[20]) considered large and moderate deviations. In particular, he proved (1.8) in the zone $x = o(n^{1/6})$.

Wang and Jing [23] proved a bound for symmetric independent nonidentically distributed random variables. Under the conditions of Theorem 1 it implies (1.3) and (1.4) in the i.i.d. case with

$$r_n(x) = \begin{cases} L_n, & \text{for } |x| \leq 1, \\ |x|^4 L_n, & \text{for } 1 \leq |x| \leq 5^{-1} L_n^{-1/3}, \\ |x|, & \text{for } |x| > 5^{-1} L_n^{-1/3}. \end{cases} \quad (1.9)$$

Notice that the function $r_n(x)$ in (1.9) contains for $|x| \geq 1$ an extra factor $|x|$, compared to $r_n(x)$ from Theorem 1. Wang and Jing [23] proved as well a bound of $r_n(x)$ in (1.3) and (1.4) for nonsymmetric i.i.d. random variables. Chistyakov and Götze [3] improved their results and considered even more general situation of non-i.i.d. random variables (see Theorems 2, 3, and 4 in [3]). In particular, the Wang and Jing bound for nonsymmetric i.i.d. random variables differs from the bound of Theorem 2 (ibid.) by an extra factor $\exp\{\varepsilon x^2\}$, $\varepsilon > 0$.

Now let us consider the case of non-i.i.d. random variables. Henceforth $X_1, \dots, X_n$ denote random variables such that

$$\mathbf{E}X_j = 0, \quad 0 < \sigma_j^2 = \mathbf{E}X_j^2 < \infty, \quad \text{and} \quad \beta_j = \mathbf{E}|X_j|^3$$

for all j . Write

$$B_n^2 = \sum_{j=1}^n \sigma_j^2, \quad A_n = \sum_{j=1}^n \beta_j, \quad L_n = \frac{A_n}{B_n^3}.$$

Theorem 2. *Let $X_1, \dots, X_n$ be symmetric independent random variables such that $L_n < \infty$. Then relations (1.3) and (1.4) hold with*

$$r_n(x) = \begin{cases} (|x|^3 + 1)L_n, & \text{for } 0 \leq |x| \leq L_n^{-1/3}, \\ 1, & \text{for } |x| > L_n^{-1/3}. \end{cases}$$

Corollary 2. *Let $G(x, l)$ be defined by (1.5) with supremum taken over all symmetric independent nonidentically distributed random variables $X_1, \dots, X_n$ such that $L_n = l$. Then under the assumptions of Theorem 2 relations (1.6) and (1.7) are valid for $|x| \leq 1/(l\rho(l))^{1/3}$.*

Note that Theorem 1 is an obvious consequence of Theorem 2. We omit the evident proofs of Corollaries 1 and 2. Consider Student's t -statistic T_n defined by

$$T_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n X_j / \left(\frac{1}{n-1} \sum_{j=1}^n (X_j - \bar{X}_n)^2 \right)^{1/2},$$

where $\bar{X}_n = S_n/n$. It is well known (see [5]) that, for $x \geq 0$,

$$\mathbf{P}\{T_n > x\} = \mathbf{P}\left\{ \frac{S_n}{V_n} > \left(\frac{nx^2}{n+x^2-1} \right)^{1/2} \right\}. \quad (1.10)$$

Denote $\Delta_{T_n}(x) = |\mathbf{P}\{T_n < x\} - \Phi(x)|$. With the help of (1.10) we obtain the following nonuniform bounds and Cramér large (moderate) deviations for T_n as an immediate consequence of Theorem 2.

Theorem 3. *Under the assumptions of Theorem 2, we have, for all positive integers n and $x \geq 0$,*

$$\Delta_{T_n}(x) \leq c \min\{(x^3 + 1) L_n, 1\} \left(1 - \Phi\left(\left(\frac{nx^2}{n + x^2 - 1}\right)^{1/2}\right)\right).$$

Theorem 2 and Theorem 3 are consequences of Theorem 4 which shows that the preceding results hold for wide classes of nonindependent random variables $X_1, \dots, X_n$.

Theorem 4. *Let $X_1, \dots, X_n$ be random variables such that the distributions of the random vectors $(\delta_1 X_1, \dots, \delta_n X_n)$ coincide for all values of δ_j , where $\delta_j = \pm 1$, $j = 1, \dots, n$, and $L_n < \infty$. Then, for all $x \geq 0$,*

$$\begin{aligned} & \left| \mathbf{P}\left\{\frac{S_n}{V_n} < x\right\} - \Phi(x) \right| \\ & \leq c \min \left\{ 1, (x^3 + 1) \left(L_n + \mathbf{P}\left\{V_n^2 < \frac{B_n^2}{2}\right\} \right) \right\} (1 - \Phi(x)), \end{aligned} \quad (1.11)$$

$$\begin{aligned} \Delta_{T_n}(x) & \leq c \min \left\{ 1, (x^3 + 1) \left(L_n + \mathbf{P}\left\{V_n^2 < \frac{B_n^2}{2}\right\} \right) \right\} \\ & \quad \times \left(1 - \Phi\left(\left(\frac{nx^2}{n + x^2 - 1}\right)^{1/2}\right) \right). \end{aligned} \quad (1.12)$$

To prove Theorem 4 we apply the saddle-point method to the inversion formula for $\mathbf{P}\{S_n/V_n < x\}$ and with the help of this method obtain the optimal result in the case of symmetric random variables. Our method of the proof of Theorem 4 essentially differs from the methods used by Shao [18], [20], and Wang and Jing [23].

2. Proof of Theorem 2, Theorem 4, and Proposition. In the first step we shall prove Theorem 4 assuming, without loss of generality, that $x > 0$.

We have, by the inversion formula (see [16, Ch. I, § 3],

$$\begin{aligned} & \mathbf{P}\left\{\frac{S_n}{V_n} < N\right\} - \mathbf{P}\left\{\frac{S_n}{V_n} < x\right\} \\ & = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ixt} - e^{-iNt}}{it} \mathbf{E} e^{itS_n/V_n} dt \end{aligned} \quad (2.1)$$

for all real x, N . Denote by $P_{1,\dots,n}$ the distribution of the random vector $(X_1, \dots, X_n)$ and rewrite the preceding formula, using the well-

known relation

$$\mathbf{E} e^{itS_n/V_n} = \int_{\mathbf{R}^n} \prod_{j=1}^n \cos \frac{tx_j}{v_n} dP_{1,\dots,n}, \quad t \in \mathbf{C}, \quad (2.2)$$

where $v_n = \{x_1^2 + \dots + x_n^2\}^{1/2}$ and $\mathbf{C}$ is the open complex plane. Here and in the sequel we shall assume that $\frac{0}{0} = 0$. Let D denote Borel's set of $\mathbf{R}^n$. Denote

$$I(x, N, D) = \int_D \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ixt} - e^{-iNt}}{it} \prod_{j=1}^n \cos \frac{tx_j}{v_n} dt dP_{1,\dots,n}.$$

Then with the help of (2.2) and Lebesgue's dominated convergence theorem we obtain from (2.1)

$$1 - \mathbf{P} \left\{ \frac{S_n}{V_n} < x \right\} = \lim_{N \rightarrow \infty} I(x, N, \mathbf{R}^n \setminus D) + \lim_{N \rightarrow \infty} I(x, N, D). \quad (2.3)$$

The function under the integral sign on the right-hand side of (2.1) is a regular function in t for all values of $t \in \mathbf{C}$ and its absolute value is bounded in each strip $|\operatorname{Im} t| < a$ ($a > 0$). Using Cauchy's theorem we obtain the formula

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ixt} - e^{-iNt}}{it} \prod_{j=1}^n \cos \frac{tx_j}{v_n} dt \\ &= e^{-x^2} \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ixt}}{x + it} \prod_{j=1}^n \cos \frac{(t - ix)x_j}{v_n} dt \\ & \quad - ye^{-Nx} \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-iNt}}{x + it} \prod_{j=1}^n \cos \frac{(t - ix)x_j}{v_n} dt. \end{aligned}$$

It is easy to see that

$$\sup_{(x_1, \dots, x_n) \in \mathbf{R}^n} \sup_{N \in \mathbf{R}} \left| \lim_{T \rightarrow \infty} \int_{-T}^T e^{-iNt} \frac{1}{x + it} \prod_{j=1}^n \cos \frac{(t - ix)x_j}{v_n} dt \right| < \infty.$$

Indeed, the product of the second and third factor under the integral sign is the Fourier transform of a function f such that $0 \leq f(u; x_1, \dots, x_n) \leq A < \infty$, where A does not depend on $x_1, \dots, x_n$ and N .

From the last two relations we deduce the following key formula

$$\lim_{N \rightarrow \infty} I(x, N, D) = e^{-x^2} \int_D \prod_{j=1}^n \operatorname{ch} \frac{xx_j}{v_n} \cdot \lim_{T \rightarrow \infty} I(x, T) dP_{1,\dots,n} \quad (2.4)$$

for all $x > 0$, where

$$I(x, T) = \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ixt}}{x + it} \prod_{j=1}^n \psi_j(t; x) dt$$

and $\psi_j(t; x) = \cos\{(t - ix)x_j/v_n\} / \operatorname{ch}\{xx_j/v_n\}$.

In the sequel we use (2.3) and (2.4) with $D = Q$, where Q is the set of points $(x_1, \dots, x_n) \in \mathbf{R}^n \setminus \{0\}$ such that

$$L = L(x_1, \dots, x_n) = \frac{|x_1|^3 + \dots + |x_n|^3}{v_n^3} \leq \frac{1}{8(x+1)^3}. \quad (2.5)$$

To estimate $I(x, N, Q)$ we need the following lemmas.

Lemma 1. *Let $X_1, \dots, X_n$ be independent random variables such that $\mathbf{E}X_j = 0$, $\sigma_j^2 > 0$, $\beta_j < \infty$, $j = 1, \dots, n$. Then, for any $\varepsilon > 0$,*

$$\mathbf{P}\{|V_n^2 - B_n^2| > \varepsilon B_n^2\} \leq \frac{4}{\varepsilon^{3/2}} L_n. \quad (2.6)$$

P r o o f. To prove (2.6) we use the von Bahr and Esseen result (see [16, Ch. III, § 5]). Let $Y_1, \dots, Y_n$ be independent random variables with zero means and finite absolute moments of order p ($1 \leq p \leq 2$). Then

$$\mathbf{E} \left| \sum_{k=1}^n Y_k \right|^p \leq \left(2 - \frac{1}{n} \right) \sum_{k=1}^n \mathbf{E} |Y_k|^p.$$

We obtain with the help of Chebyshev's inequality and the previous bound

$$\begin{aligned} \mathbf{P}\{|V_n^2 - B_n^2| > \varepsilon B_n^2\} &\leq (\varepsilon B_n^2)^{-3/2} \mathbf{E} \left| \sum_{j=1}^n (X_j^2 - \sigma_j^2) \right|^{3/2} \\ &\leq 2 \sum_{j=1}^n (\varepsilon B_n^2)^{-3/2} \mathbf{E} |X_j^2 - \sigma_j^2|^{3/2} \leq 4\varepsilon^{-3/2} L_n \end{aligned}$$

and (2.6) is proved.

Lemma 2. *Let $X_1, \dots, X_n$ denote independent random variables under the assumptions of Lemma 1. Denote by $f_n(t)$ the characteristic function of the random variable $B_n^{-1} \sum_{j=1}^n X_j$. Then*

$$|f_n(t) - e^{-t^2/2}| \leq 16L_n |t|^3 e^{-t^2/3}, \quad |t| \leq \frac{1}{4L_n}.$$

The proof of this lemma see in [16, Ch. V, § 2].

Lemma 3. *Let X be a random variable such that $\mathbf{E}e^{yX} < \infty$ for $0 \leq y \leq a$. Then, for $x > 0$ and $0 \leq y \leq a$,*

$$1 - \mathbf{P}\{X < x\} = e^{-yx} \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ixt}}{y + it} \mathbf{E}e^{(y+it)X} dt.$$

The proof of this lemma is similar to the proof of formula (2.4) and hence will be omitted.

Corollary 3. *The following formula holds, for $x > 0$,*

$$1 - \Phi(x) = e^{-x^2/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2/2}}{x + it} dt.$$

For the proof it suffices to apply Lemma 3 for a standard normal random variable X .

Lemma 4. *The following inequality holds*

$$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{x} - \frac{1}{x^3} \right) e^{-x^2/2} \leq 1 - \Phi(x) \leq \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}, \quad x > 0.$$

The proof of this well-known inequality can be found, for instance, in [7, Ch. VII, § 1].

The following result is due to Pinelis [17, p. 359].

Lemma 5. *Let $X_1, \dots, X_n$ be i.i.d. random variables such that $\mathbf{P}\{X_1 = \pm 1\} = \frac{1}{2}$ and let $y_1, \dots, y_n$ be real numbers such that $y_1^2 + \dots + y_n^2 = 1$. Then, for any $u > 0$,*

$$\mathbf{P}\{|y_1 X_1 + \dots + y_n X_n| > u\} < 2c(1 - \Phi(u)),$$

where $c = 2e^3/9$.

Lemma 6. *Let $X_1, \dots, X_n$ denote independent random variables under assumptions of Lemma 1. Then*

$$\sup_x \left| \mathbf{P} \left\{ B_n^{-1/2} \sum_{j=1}^n X_j < x \right\} - \Phi(x) \right| \leq cL_n.$$

The proof of Lemma 6 (the Berry–Esseen inequality) see in [16, Ch. V, § 2].

Our next aim is to prove the statement (1.11) of Theorem 4 for $0 < x \leq \varepsilon_n^{-1/3}$, where $\varepsilon_n := L_n + \mathbf{P}\{V_n^2 \leq B_n^2/2\}$. Let us return to formula (2.4) with $D = Q$ and represent the interior integral $I(x, T)$, $T > \frac{1}{10} L^{-1}$, in the form

$$\begin{aligned} I(x, T) &= \frac{1}{2\pi} \left(\int_{-L^{-1/3}/2}^{L^{-1/3}/2} + \int_{L^{-1/3}/2 \leq |t| \leq L^{-1/10}} + \int_{L^{-1/10} < |t| \leq T} \right) \\ &\times \frac{e^{-ixt}}{x + it} \prod_{j=1}^n \psi_j(t; x) dt := I_1(x) + I_2(x) + I_3(x, T). \end{aligned} \quad (2.7)$$

By (2.5), we see that

$$\frac{1}{v_n} \max_j |x_j| \leq L^{1/3} \leq \frac{1}{2(x+1)}. \quad (2.8)$$

Estimate now the integral $I_1(x)$, using (2.8). Note that $|th y| \leq y$ for all real y and $|tg y| \leq 2|y|$ for real y such that $|y| \leq 1$, and, by (2.8),

$$\max \{x, |t|\} \frac{|x_j|}{v_n} \leq \max \left\{ x, \frac{1}{2} L^{-1/3} \right\} L^{1/3} \leq \frac{1}{2}, \quad 2|t|x \frac{x_j^2}{v_n^2} \leq L^{-1/3} x L^{2/3} \leq \frac{1}{2}$$

for $t \in [-\frac{1}{2}L^{-1/3}, \frac{1}{2}L^{-1/3}]$. With the help of the series expansion for the function $\ln(1+z)$, $|z| < 1$, we easily deduce the following relation for the functions $\psi_j(t; x)$

$$\begin{aligned}\psi_j(t; x) &= \cos \frac{tx_j}{v_n} \left(1 + i \operatorname{th} \frac{xx_j}{v_n} \operatorname{tg} \frac{tx_j}{v_n} \right) \\ &= \cos \frac{tx_j}{v_n} \exp \left\{ i \operatorname{th} \frac{xx_j}{v_n} \operatorname{tg} \frac{tx_j}{v_n} + 3\theta \frac{t^2 x^2 x_j^4}{v_n^4} + 4i\theta \frac{t^3 x^3 x_j^6}{v_n^6} \right\}\end{aligned}\quad (2.9)$$

for $t \in [-\frac{1}{2}L^{-1/3}, \frac{1}{2}L^{-1/3}]$. Recall that $\operatorname{th} y = -i \operatorname{tg}(iy)$. By the inequalities $|\operatorname{th} y - y| \leq |y|^3/2$ for all real y and $|\operatorname{tg} y - y| \leq |y|^3$ for real y such that $|y| \leq 1$, we have, for the same t ,

$$\operatorname{th} \frac{xx_j}{v_n} \operatorname{tg} \frac{tx_j}{v_n} = tx \frac{x_j^2}{v_n^2} + \theta tx \frac{x_j^4}{v_n^4} \left(t^2 + \frac{1}{2}x^2 + \frac{1}{2}t^2 x^2 \frac{x_j^2}{v_n^2} \right). \quad (2.10)$$

We obtain from (2.9) and (2.10)

$$\prod_{j=1}^n \psi_j(t; x) = e^{f_1(t; x) + i f_2(t; x)} e^{itx} \prod_{j=1}^n \cos \frac{tx_j}{v_n},$$

where

$$\begin{aligned}f_1(t; x) &= 3\theta t^2 x^2 \frac{1}{v_n^4} \sum_{j=1}^n x_j^4, \\ f_2(t; x) &= \theta tx \left(t^2 + \frac{1}{2}x^2 \right) \frac{1}{v_n^4} \sum_{j=1}^n x_j^4 + \frac{9}{2} \theta t^3 x^3 \frac{1}{v_n^6} \sum_{j=1}^n x_j^6.\end{aligned}$$

The function $\exp\{f_1(t; x) + i f_2(t; x)\}$ admits the following bound

$$\begin{aligned}\left| e^{f_1(t; x) + i f_2(t; x)} - 1 \right| &\leq \left| e^{f_1(t; x)} - 1 \right| + 2 e^{f_1(t; x)} \sin^2 \frac{f_2(t; x)}{2} \\ &\quad + e^{f_1(t; x)} |\sin f_2(t; x)| \\ &\leq \left| e^{f_1(t; x)} - 1 \right| + \frac{1}{2} e^{f_1(t; x)} f_2^2(t; x) + e^{f_1(t; x)} |f_2(t; x)|.\end{aligned}$$

By (2.8), we have

$$\begin{aligned}|f_1(t; x)| &\leq \frac{3}{2} t^2 x L \leq \frac{1}{4} t^2, & \left| e^{f_1(t; x)} - 1 \right| &\leq |f_1(t; x)| e^{|f_1(t; x)|} \leq ct^2 x L e^{t^2/4}, \\ |f_2(t; x)| &\leq c|t|(t^2 + x^2)L, & f_2^2(t; x) &\leq c \frac{t^2(t^2 + x^2)^2}{(1+x)^3} L\end{aligned}$$

for $t \in [-\frac{1}{2}L^{-1/3}, \frac{1}{2}L^{-1/3}]$. Using these bounds we arrive at the inequality

$$\left| e^{f_1(t; x) + i f_2(t; x)} - 1 \right| \leq c \left(t^2 x + |t|(t^2 + x^2) + \frac{t^2(t^2 + x^2)^2}{(1+x)^3} \right) L e^{t^2/4} \quad (2.11)$$

for the same t . By Lemma 2, note that

$$\left| \prod_{j=1}^n \cos \frac{tx_j}{v_n} - e^{-t^2/2} \right| \leq 16L |t|^3 e^{-t^2/3} \quad (2.12)$$

for $|t| \leq 1/(4L^{1/3})$. Since

$$\begin{aligned} e^{-itx} \prod_{j=1}^n \psi_j(t; x) &= e^{-t^2/2} + \left(\prod_{j=1}^n \cos \frac{tx_j}{v_n} - e^{-t^2/2} \right) \\ &\quad + \left(e^{f_1(t; x) + if_2(t; x)} - 1 \right) \prod_{j=1}^n \cos \frac{tx_j}{v_n}, \end{aligned}$$

we can write the integral $I_1(x)$ with the help of (2.11) and (2.12) in the form

$$I_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2/2}}{x + it} dt + \theta(r_1 + r_2 + r_3). \quad (2.13)$$

Here the nonnegative quantities r_1, r_2 , and r_3 admit the bounds

$$\begin{aligned} r_1 &\leq cL \int_{-\infty}^{\infty} \frac{|t|^3}{|x + it|} e^{-t^2/3} dt \leq cL, \\ r_2 &\leq cL \int_{-\infty}^{\infty} \left(|t|x + |t|(t^2 + x^2)^{1/2} \right. \\ &\quad \left. + t^2(t^2 + x^2)^{3/2}(1 + x)^{-3} \right) e^{-t^2/12} dt \leq c(x + 1)L, \\ r_3 &\leq \frac{c}{x + 1} \int_{|t| > L^{-1/3/2}} e^{-t^2/2} dt \leq \frac{cL}{x + 1}. \end{aligned} \quad (2.14)$$

Let us estimate the integral $I_2(x)$. First we determine Lyapunov's ratio L_1 for the independent random variables Y_j , $j = 1, \dots, n$, with the characteristic functions $\psi_j(t; x)$, $j = 1, \dots, n$, respectively. Since

$$\begin{aligned} \mathbf{E}Y_j &= \frac{x_j}{v_n} \operatorname{th} \frac{xx_j}{v_n}, \quad \mathbf{D}Y_j = \frac{x_j^2}{v_n^2} \left(1 - \operatorname{th}^2 \frac{xx_j}{v_n} \right), \\ \mathbf{E}|Y_j - \mathbf{E}Y_j|^3 &= \frac{|x_j|^3}{v_n^3} \left(1 - \operatorname{th}^4 \frac{xx_j}{v_n} \right), \end{aligned}$$

we have

$$L_1 = \left(\sum_{j=1}^n \left(1 - \operatorname{th}^2 \frac{xx_j}{v_n} \right) \frac{x_j^2}{v_n^2} \right)^{-3/2} \sum_{j=1}^n \left(1 - \operatorname{th}^4 \frac{xx_j}{v_n} \right) \frac{|x_j|^3}{v_n^3}.$$

By (2.8), we see that

$$\frac{15}{16} L \leq L_1 \leq \left(\frac{4}{3} \right)^{3/2} L. \quad (2.15)$$

In addition we note that

$$1 \geq \sum_{j=1}^n \mathbf{D}Y_j \geq 1 - \frac{x^2}{v_n^4} \sum_{j=1}^n x_j^4 = 1 - \frac{1}{2} |\theta| xL. \quad (2.16)$$

By Lemma 2 and (2.15), (2.16), we then obtain

$$\left| \prod_{j=1}^n \psi_j(t; x) \right| \leq c(1+t^2) e^{-(1-xL/2)t^2/3} \leq c(1+t^2) e^{-t^2/4}$$

for $|t| \leq \frac{1}{10} L^{-1}$. This bound yields the inequality

$$|I_2(x)| \leq c \int_{|t| > L^{-1/3}/2} \frac{t^2 + 1}{|x + it|} e^{-t^2/4} dt \leq \frac{c}{x+1} L. \quad (2.17)$$

Consider now the integral $I_3(x, T)$ for $T > \frac{1}{10} L^{-1}$. Represent it in the form

$$\begin{aligned} I_3(x, T) &= \frac{1}{2\pi} \int_{L^{-1}/10 < |t| \leq T} \frac{e^{-ixt}}{it} \prod_{j=1}^n \psi_j(t; x) dt \\ &\quad - \frac{x}{2\pi} \int_{L^{-1}/10 < |t| \leq T} \frac{e^{-ixt}}{it(x+it)} \prod_{j=1}^n \psi_j(t; x) dt \\ &:= I_{3,1}(x, T) + I_{3,2}(x, T). \end{aligned}$$

It is easy to see that

$$|I_{3,2}(x, T)| \leq \frac{x}{2\pi} \int_{|t| > L^{-1}/10} \frac{dt}{|t| \sqrt{x^2 + t^2}} \leq \frac{x}{2\pi} \int_{|t| > L^{-1}/10} \frac{dt}{t^2} \leq \frac{5}{\pi} xL. \quad (2.18)$$

Rewrite $I_{3,1}(x, T)$ in the form

$$\begin{aligned} &\frac{1}{2\pi} \int_{-T}^T \frac{e^{-ixt}}{it} \prod_{j=1}^n \left(\psi_j(t; x) - e^{it\mathbf{E}Y - t^2/2} \right) dt \\ &\quad - \frac{1}{2\pi} \int_{|t| \leq L^{-1}/10} \frac{e^{-ixt}}{it} \prod_{j=1}^n \left(\psi_j(t; x) - e^{it\mathbf{E}Y - t^2/2} \right) dt \\ &\quad + \frac{1}{2\pi} \int_{L^{-1}/10 < |t| \leq T} \frac{e^{-ixt}}{it} e^{it\mathbf{E}Y - t^2/2} dt \\ &:= I_{3,1,1}(x, T) - I_{3,1,2}(x) + I_{3,1,3}(x, T), \end{aligned} \quad (2.19)$$

where $Y = \sum_{j=1}^n Y_j$. First we note that

$$\lim_{T \rightarrow \infty} I_{3,1,1}(x, T) = \Phi(x - \mathbf{E}Y) - \mathbf{P}\{Y < x\}.$$

By Lemma 6 and (2.15), (2.16),

$$\left| \lim_{T \rightarrow \infty} I_{3,1,1}(x, T) \right| \leq cL_1 + \left| \Phi\left(\frac{x - \mathbf{E}Y}{\sqrt{\mathbf{D}Y}}\right) - \Phi(x - \mathbf{E}Y) \right| \leq cL.$$

By Lemma 2 and (2.15), (2.16), we get

$$\begin{aligned} |I_{3,1,2}(x)| &\leq \frac{1}{2\pi} \int_{|t| \leq L^{-1/10}} \left| \prod_{j=1}^n \psi_j(t; x) - e^{it\mathbf{E}Y - (\mathbf{D}Y)t^2/2} \right| \frac{dt}{|t|} \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| e^{-t^2/2} - e^{-(\mathbf{D}Y)t^2/2} \right| \frac{dt}{|t|} \\ &\leq c(x+1) L_1 \int_{-\infty}^{\infty} (|t| + t^2) e^{-t^2/4} dt \\ &\leq c(x+1) L_1 \leq c(x+1) L. \end{aligned}$$

In addition we have $|I_{3,1,3}(x, T)| \leq cL$. In view of the last three bounds, we obtain from (2.19) the inequality

$$\left| \lim_{T \rightarrow \infty} I_{3,1}(x, T) \right| \leq c(x+1) L. \quad (2.20)$$

Comparing (2.18) and (2.20) we finally deduce

$$\left| \lim_{T \rightarrow \infty} I_3(x, T) \right| \leq c(x+1) L. \quad (2.21)$$

Now let us return to formula (2.13). By Corollary 1 and (2.13), (2.14), we have

$$I_1(x) = (1 - \Phi(x)) e^{x^2/2} + \theta c(x+1) L. \quad (2.22)$$

Then, in view of (2.17), (2.21), and (2.22), we obtain from (2.7)

$$\lim_{T \rightarrow \infty} I(x, T) = (1 - \Phi(x)) e^{x^2/2} + \theta c(x+1) L. \quad (2.23)$$

Using (2.8) and the series expansion for $\ln(1+z)$, $|z| < 1$, $\ln 1 = 0$, we see that, for $(x_1, \dots, x_n) \in Q$,

$$\begin{aligned} \ln \operatorname{ch} \frac{xx_j}{v_n} &= \ln \left\{ 1 + \frac{1}{2} \left(\frac{xx_j}{v_n} \right)^2 + \frac{1}{4!} \left(\frac{xx_j}{v_n} \right)^4 \left(1 + \frac{\theta}{20} \left(\frac{xx_j}{v_n} \right)^2 \right) \right\} \\ &= \frac{1}{2} \left(\frac{xx_j}{v_n} \right)^2 - \frac{1}{12} \left(\frac{xx_j}{v_n} \right)^4 \left(1 + 2\theta \left(\frac{xx_j}{v_n} \right)^2 \right). \end{aligned}$$

Using again (2.8) we obtain, for $(x_1, \dots, x_n) \in Q$,

$$\begin{aligned} \prod_{j=1}^n \operatorname{ch} \frac{xx_j}{v_n} &= \exp \left\{ \frac{x^2}{2} - \frac{x^4}{12} \sum_{j=1}^n \left(\frac{x_j}{v_n} \right)^4 + \frac{\theta x^6}{6} \sum_{j=1}^n \left(\frac{x_j}{v_n} \right)^6 \right\} \\ &= e^{x^2/2} \left(1 - \frac{1}{2} |\theta| x^3 L \right). \end{aligned} \quad (2.24)$$

Applying (2.23) and (2.24) to (2.4), we deduce with the help of Lemma 4 the relation

$$\lim_{N \rightarrow \infty} I(x, N, Q) = (1 - \Phi(x)) \int_Q dP_{1, \dots, n} + c\theta(x^2 + 1) e^{-x^2/2} \int_Q \theta L dP_{1, \dots, n}.$$

By Jensen's inequality, $|x_1|^3 + \dots + |x_n|^3 \leq v_n^3$, therefore

$$\begin{aligned} \int_Q L dP_{1,\dots,n} &\leq \mathbf{P}\left\{V_n^2 \leq \frac{1}{2} B_n^2\right\} + 2^{3/2} B_n^{-3} \int_{\mathbf{R}^n} \sum_{j=1}^n |x_j|^3 dP_{1,\dots,n} \\ &\leq \mathbf{P}\left\{V_n^2 \leq \frac{1}{2} B_n^2\right\} + 2^{3/2} L_n. \end{aligned}$$

Thus we have

$$\lim_{N \rightarrow \infty} I(x, N, Q) = (1 - \Phi(x)) \int_Q dP_{1,\dots,n} + c\theta(x^2 + 1) e^{-x^2/2} \varepsilon_n. \quad (2.25)$$

Recall that $\varepsilon_n := L_n + \mathbf{P}\{V_n^2 \leq \frac{1}{2} B_n^2\}$. With the help of (2.5) we note as above that

$$\int_{\mathbf{R}^n \setminus Q} dP_{1,\dots,n} \leq 8(1+x)^3 \int_{\mathbf{R}^n \setminus Q} L dP_{1,\dots,n} + \mathbf{P}\{V_n = 0\} \leq c(x^3 + 1) \varepsilon_n. \quad (2.26)$$

Then, applying (2.26) to (2.25), we arrive, in view of Lemma 4, at the relation

$$\begin{aligned} \lim_{N \rightarrow \infty} I(x, N, Q) &= 1 - \Phi(x) + c\theta(x^2 + 1) e^{-x^2/2} \varepsilon_n \\ &= (1 - \Phi(x)) (1 + c\theta(x^3 + 1) \varepsilon_n). \end{aligned} \quad (2.27)$$

To estimate $I(x, N, \mathbf{R}^n \setminus Q)$ we use Lemma 5 with $y_j = x_j/v_n$ and (2.1), (2.26). We have

$$\begin{aligned} \left| \lim_{N \rightarrow \infty} I(x, N, \mathbf{R}^n \setminus Q) \right| &\leq c(1 - \Phi(x)) \int_{\mathbf{R}^n \setminus Q} dP_{1,\dots,n} \\ &\leq c(x^3 + 1) \varepsilon_n (1 - \Phi(x)). \end{aligned} \quad (2.28)$$

The statement (1.11) of Theorem 4 for $0 < x \leq \varepsilon_n^{-1/3}$ follows from (2.3) with $D = Q$ and (2.27), (2.28).

The statement (1.11) of Theorem 4 for $x > \varepsilon_n^{-1/3}$ immediately follows from Lemma 5 and formulas (2.1) and (2.2).

The statement (1.12) of Theorem 4 is an evident sequence of (1.11) and (1.10).

R e m a r k. Denote by Q_1 the set of points $(x_1, \dots, x_n) \in \mathbf{R}^n \setminus \{\mathbf{0}\}$, satisfying the inequality

$$\frac{1}{v_n} \max_j |x_j| \leq \frac{1}{2(x+1)}.$$

Repeating the argument which we employed to deduce (2.25) and (2.28), we obtain, for $0 < x \leq \varepsilon_n^{-1/3}$, the following formula

$$\begin{aligned} 1 - \mathbf{P}\left\{\frac{S_n}{V_n} < x\right\} &= (1 - \Phi(x)) \int_{Q_1} dP_{1,\dots,n} \\ &\quad - (1 - \Phi(x)) \int_{Q_1} \left(1 - \exp\left\{-\frac{x^4}{12} \left(1 + \frac{\theta}{2}\right) \sum_{j=1}^n \left(\frac{x_j}{v_n}\right)^4\right\}\right) dP_{1,\dots,n} \\ &\quad + c\theta(1 - \Phi(x)) \int_{\mathbf{R}^n \setminus Q_1} dP_{1,\dots,n} + c(x^2 + 1) \varepsilon_n (1 - \Phi(x)). \end{aligned} \quad (2.29)$$

P r o o f o f T h e o r e m 2. Note that the random variables under assumptions of Theorem 2 satisfy the assumptions of Theorem 4. Therefore (1.11) holds. In view of Lemma 1, $\mathbf{P}\{V_n^2 \leq \frac{1}{2} B_n^2\} \leq cL_n$ and we obtain the statement of the theorem from (1.11).

P r o o f o f P r o p o s i t i o n. In the proof of Proposition we continue to use the notation of Section 2. Let $x \in (0, n^{1/6}/b_n^{1/3}]$ and $\sqrt{n}/b_n \geq c$, where c is sufficiently large. Consider symmetric i.i.d. random variables $X_1, \dots, X_n$ such that

$$\mathbf{P}\{X_1 = a\} = \frac{1}{2} - \frac{b_n(x+1)^3}{4n^{3/2}}, \quad \mathbf{P}\left\{X_1 = \frac{a\sqrt{n}}{2(x+1)}\right\} = \frac{b_n(x+1)^3}{4n^{3/2}},$$

where

$$a = a(n, x) := \left(1 - \frac{b_n}{2\sqrt{n}} \left(\frac{(x+1)^3}{n} - \frac{(x+1)}{4}\right)\right)^{-1/2}.$$

Denote the distribution of these random variables by P . It is easy to see that $\mathbf{E}X_1^2 = 1$ and $c_1b_n \leq \mathbf{E}|X_1|^3 \leq c_2b_n$.

Consider vectors $(x_1, \dots, x_n) \in \mathbf{R}^n$ such that $x_j = \pm a, \pm a\sqrt{n}/(2(x+1))$ for $j = 1, \dots, n$. In the proof of proposition we will consider such vectors only. By the simple estimate $v_n^2 \geq na^2$, we see that $v_n^{-1} \max_j |x_j| \leq (2(x+1))^{-1}$ for all considered $(x_1, \dots, x_n)$. Thus in our case $Q_1 = \mathbf{R}^n$ and we obtain from (2.29), taking into account Lemma 1,

$$\begin{aligned} \frac{1 - \mathbf{P}\{S_n/V_n < x\}}{1 - \Phi(x)} &= 1 + c\theta(x^2 + 1) \frac{b_n}{\sqrt{n}} \\ &- \int_{\mathbf{R}^n} \left(1 - \exp\left\{-\frac{x^4}{12} \left(1 + \frac{\theta}{2}\right) \sum_{j=1}^n \left(\frac{x_j}{v_n}\right)^4\right\}\right) dP \times \dots \times P. \end{aligned} \quad (2.30)$$

Denote by Q_2 the set of vectors $(x_1, \dots, x_n)$ such that only one of x_j , $j = 1, \dots, n$, takes the values $\pm a\sqrt{n}/(2(x+1))$. It is easy to see that

$$\int_{Q_2} dP \times \dots \times P = \left(1 - \frac{b_n(x+1)^3}{2n^{3/2}}\right)^{n-1} \frac{b_n(x+1)^3}{2n^{1/2}} \geq \frac{b_n(x+1)^3}{8n^{1/2}}. \quad (2.31)$$

Since $v_n^2 \leq 5na^2/4$, for $(x_1, \dots, x_n) \in Q_2$, we have

$$x^4 \sum_{j=1}^n \left(\frac{x_j}{v_n}\right)^4 \geq \frac{1}{32}, \quad (x_1, \dots, x_n) \in Q_2. \quad (2.32)$$

Using (2.31) and (2.32), we obtain from (2.30)

$$\frac{1 - \mathbf{P}\{S_n/V_n < x\}}{1 - \Phi(x)} \leq 1 - c_3((x+1)^3 - c(x^2 + 1)) \frac{b_n}{\sqrt{n}},$$

as was to be proved.

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