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The fact that $p^{(m)}(x)$ does not vanish in $[-b, 0]$ implies by Rolle's theorem that $p(x)$ does not have more than m zeros in that interval. The m zeros of the polynomial $q(x) = \sum_{j=0}^m a_j(\omega)x^j$ lie in $(-b, 0)$. Consequently, if $x < -b$, then $(-1)^m q(x) > 0$, and so, by (22),

$$\begin{aligned} (-1)^m p(x) &= (-1)^m q(x) + \sum_{j=1}^{\infty} |x|^{m+2j-1} \{a_{m+2j}(\omega)|x| - a_{m+2j-1}(\omega)\} \\ &> \sum_{j=1}^{\infty} |x|^{m+2j-1} (\alpha_{m+2j}b - \beta_{m+2j-1}) > 0. \end{aligned}$$

Hence for all $\omega \in A$, $\sum_{j=0}^{\infty} a_j(\omega)x^j$ has exactly m zeros.

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JANVRESSE E.*

BOUNDS ON SEMIGROUPS OF RANDOM ROTATIONS ON $SO(n)$

В [3] с целью получения случайных ортогональных матриц рассматривалась марковская цепь на ортогональной группе $SO(n)$, порожденной случайными поворотами в выбираемых случайным образом координатных плоскостях. Мы изучаем различные способы измерить приближение блуждания к равновесию. С этой целью мы доказываем, что, с точностью до умножения на константу, спектральная щель этого блуждания ограничена снизу величиной $1/n^2$, а отклонение энтропии к диссипации энтропии ограничено сверху величиной n^3 .

Ключевые слова и фразы: приближение к равновесию, спектральная щель.

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1. Introduction and main result. We consider a random walk on the orthogonal group $SO(n)$ based on random rotations on randomly selected coordinate planes i, j with $1 \leq i < j \leq n$. This dynamics on $SO(n)$ was used by W. Hastings [3] to generate an approximate random element in $SO(n)$ and by M. Kac [6] as a model for a Boltzmann equation.

More precisely, let $R_{i,j}^\theta$ be the clockwise rotation by θ in the coordinate plane i, j . Explicitly, $R_{i,j}^\theta$ is given by

$$R_{i,j}^\theta = \begin{pmatrix} & i & j & & \\ & & & & \\ & & \ddots & & \\ & & & c & s \\ & & & -s & c \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \begin{matrix} i \\ j \end{matrix}$$

with $c = \cos(\theta)$ and $s = \sin(\theta)$. Then the generator of the dynamics is defined, for $M \in SO(n)$, by

$$\mathcal{L}_n F(M) = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \int_0^{2\pi} \left(F(R_{i,j}^\theta M) - F(M) \right) \frac{d\theta}{2\pi}.$$

Let P_t denote the corresponding semigroup and let $\mu_n^{\mathcal{H}}$ denote the uniform distribution on $SO(n)$ (now called Haar measure). One can check that $\mu_n^{\mathcal{H}}$ is invariant and reversible with respect to the generator $\mathcal{L}_n$.

A critical issue for users of Monte Carlo Markov chain (MCMC) methods in applications is how to determine when it is safe to stop sampling and use the samples to estimate characteristics of the distribution of interest.

By the spectral theorem, we know that for any initial condition ϕ , the rate of approach to equilibrium in L^2 norm is given by

$$\|\phi(t) - \mathbf{E}_{\mu_n^{\mathcal{H}}}(\phi)\|_2 \leq e^{-\lambda t} \|\phi(0) - \mathbf{E}_{\mu_n^{\mathcal{H}}}(\phi)\|_2,$$

where $\mathbf{E}_{\mu_n^{\mathcal{H}}}(\phi)$ is the expectation of ϕ and λ is the smallest nonzero eigenvalue of $-\mathcal{L}_n$, which corresponds to the spectral gap of $\mathcal{L}_n$.

This means the speed of convergence to equilibrium in L^2 norm is exponential and depends on λ . Thus, we are very much interested in estimating the order in n of the spectral gap.

Recall the variational definition for the spectral gap of $\mathcal{L}_n$:

$$SG(\mathcal{L}_n) = \inf_F \frac{\mathcal{D}_n(F)}{\mathbf{E}_{\mu_n^{\mathcal{H}}}[F; F]},$$

where $\mathbf{E}_{\mu_n^{\mathcal{H}}}[F; F]$ denotes the variance of F with respect to $\mu_n^{\mathcal{H}}$ and $\mathcal{D}_n(F)$ is the Dirichlet form of F , which is given by

$$\mathcal{D}_n(F) = - \int F \mathcal{L}_n F d\mu_n^{\mathcal{H}} = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} D_{i,j}(F),$$

with

$$D_{i,j}(F) = \frac{1}{2} \int_0^{2\pi} \int \left(F(R_{i,j}^\theta M) - F(M) \right)^2 d\mu_n^{\mathcal{H}}(M) \frac{d\theta}{2\pi}.$$

It is easy to get an upper bound of order $1/n$ for $SG(\mathcal{L}_n)$ by using a test function (e.g. $F(M) = M_{1,1}$). The difficulty lies in finding a lower bound. Using comparisons techniques, Diaconis and Saloff-Coste [1] proved that $SG(\mathcal{L}_n)$ is bounded below by C/n^3 , where C is a constant independent of n . More precisely, they compare the spectral gap of $\mathcal{L}_n$ to the one of the walk generated by random rotations on *any* planes. The spectral behaviour of the latter walk can be determined by character theory on $SO(n)$. In [1], it was also announced that Maslen managed to compute the exact spectral gap and found it to be

the same as the one for the walk acting on the $(n-1)$ -dimensional sphere S^{n-1} , that is, $(n+2)/(2n(n-1))$. The following theorem, which is proved by a completely different method, provides us with a lower bound of order $1/n^2$.

Theorem 1.1. *There exists a constant $C > 0$ such that for any smooth function F on $SO(n)$,*

$$\mathbf{E}_{\mu_n^{\mathcal{H}}}[F; F] \leq Cn^2 \mathcal{D}_n(F).$$

Theorem 1.1 is proved by the martingale approach of [7]. In order to apply this method, a factorization of Haar measure by Hurwitz [4] is crucial, as well as the knowledge of the spectral gap for the same walk acting on the sphere.

Let $Q_n = \mathcal{L}_n + I_n$, where I_n denotes the identity and let Q_n^{*k} be the k -th convolution power. A consequence of Theorem 1.1 is that Q_n^{*k} is close to $\mu_n^{\mathcal{H}}$ for k of order $n^4 \ln n$. We refer to [1] for general results (using spectral gap and curvature information) about rate of convergence to stationarity of random walks generated by a probability on a compact group.

Corollary 1.1. *For any bounded Lipschitz function F of norm at most 1, we have*

$$|Q_n^{*k}(F) - \mu_n^{\mathcal{H}}(F)| \leq 7\sqrt{n} \left(1 - \frac{1}{Cn^2}\right)^{2k/\binom{n}{2}+2}.$$

In his paper [3], Hastings used the random walk to estimate $\mu_n^{\mathcal{H}}(F)$ by $\tilde{J}_F = k^{-1} \sum_{i=1}^k F(M_i)$. When $F(M) = \sum_{i=1}^n M_{i,i}^2$, we know $\mu_n^{\mathcal{H}}(F) = 1$. Using $n = 50$, $k = 1000$ and starting the walk at the identity, Hastings found $\tilde{J}_F = 3.5 \pm 1.5$. According to him, this poor estimate was due to the choice of the identity as a starting place. Therefore, instead of increasing k to get greater precision, he chose to start the walk more «centrally» and obtained satisfactory estimates.

What about other ways to measure convergence to equilibrium? Gross [2] showed that logarithmic Sobolev inequality (in short, L.S.I.) is equivalent to the hypercontractivity of the semigroup, so one may wonder whether Hastings' walk satisfy a L.S.I. Can one find a constant K_n such that for any function F

$$H_n(F) = \int F \ln \frac{F}{\int F d\mu_n^{\mathcal{H}}} d\mu_n^{\mathcal{H}} \leq K_n \mathcal{D}_n(\sqrt{F})? \quad (1.1)$$

Notice that (1.1) is stronger than Spectral Gap Inequality. In fact, choosing $F = 1 + \varepsilon g$ with $\int g d\mu_n^{\mathcal{H}} = 0$ and letting ε go to 0, we recover a Spectral Gap Inequality.

The L.S.I. does not hold for this model. For any $\varepsilon > 0$, one can find a set A_ε of probability ε . Choosing $F = 1_{A_\varepsilon}$, the indicator function of A_ε , we get that the entropy of F is equal to

$$\int F \ln \frac{F}{\int F d\mu_n^{\mathcal{H}}} d\mu_n^{\mathcal{H}} = -\varepsilon \ln \varepsilon$$

and that $\mathcal{D}_n(\sqrt{F}) \leq \varepsilon$. Hence, one cannot bound the entropy of F by the Dirichlet form of $\sqrt{F}$ and the L.S.I. is wrong for this model.

Another important question is the rate of convergence of the entropy $H_n(F)$. Notice that the derivative in time of $H_n(P_t F)$ is equal to $\int P_t F \mathcal{L}_n(\ln P_t F) d\mu_n^{\mathcal{H}}$, the entropy dissipation. Using the same method as for Theorem 1.1, we were able to prove the following theorem, which implies the exponential convergence of the entropy.

Theorem 1.2. *There exists a constant $C > 0$ such that for any smooth function F on $SO(n)$,*

$$H_n(F) \leq Cn^3 \mathcal{E}_n(F, \ln F),$$

where $\mathcal{E}_n(F, \ln F) = \int F \mathcal{L}_n(\ln F) d\mu_n^{\mathcal{H}}$ denotes the entropy dissipation of F .

The paper is organized as follows. The next section gives a description of the factorization of Haar measure used in our proofs. Theorem 1.1 and Theorem 1.2 are proved in Sections 3 and 4.

2. Factorization of Haar measure. The aim of this section is to describe the factorization of Haar measure that we use in our proof of Theorem 1.1. For $2 \leq j \leq n$, let $G_j^i(\theta_{ij})$ be the clockwise rotation by θ_{ij} in the coordinate plane $j, j-1$: $G_j^i(\theta_{ij}) = R_{j-1,j}^{\theta_{ij}}$. Choosing θ_{ij} adequately, we can bring a given $M \in SO(n)$ to the diagonal form by multiplying it (on the left side) by a succession of such rotations: The final matrix $G_n^{n-1}(G_n^{n-2}G_{n-1}^{n-2}) \cdots (G_3^2 \cdots G_n^2)(G_2^1 \cdots G_{n-1}^1 G_n^1)M$ is orthogonal and can be taken as the identity by changing the sign of $\cos(\theta_{ij})$ and $\sin(\theta_{ij})$ in some of the rotations. Thus, any element $M \in SO(n)$ can be represented as

$$M = (G_n^1 G_{n-1}^1 \cdots G_2^1)(G_n^2 \cdots G_3^2) \cdots (G_n^{n-2} G_{n-1}^{n-2}) G_n^{n-1}.$$

Hurwitz [4] discovered that a uniform probability distribution on $SO(n)$ (Haar measure) can be derived by giving an appropriate measure to the $(G_j^i)_{i,j}$. The following proposition (proved in [1]) gives a probabilistic version of Hurwitz's construction of Haar measure on $SO(n)$.

Proposition 2.1. *Let $(G_j^i)_{i < j}$ be independent random matrices in $SO(n)$ with $(G_j^i)_{1 \leq i \leq j-1}$ having a common distribution μ_j for each fixed j . The measure μ_j is supported on rotations of the form*

$$g_j(x) = \begin{pmatrix} & & j-1 & & j & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & x & & \sqrt{1-x^2} & \\ & & & -\sqrt{1-x^2} & & x & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \begin{matrix} j-1 \\ j \end{matrix}$$

Under μ_j , x has the distribution of the first coordinate of a point uniformly chosen on the $(n+2-j)$ -sphere. Then

$$(G_n^1 G_{n-1}^1 \cdots G_2^1)(G_n^2 \cdots G_3^2) \cdots (G_n^{n-2} G_{n-1}^{n-2}) G_n^{n-1}$$

is uniformly distributed on $SO(n)$.

Observe that $(G_n^1 G_{n-1}^1 \cdots G_2^1)$ depends on M only through its first column $C_1 = (M_{11}, \dots, M_{n1})^t$. This matrix is of the form

$$P(C_1) = (G_n^1 G_{n-1}^1 \cdots G_2^1) = \begin{pmatrix} M_{11} & * & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ M_{n-2,1} & * & & \cdots & 0 \\ M_{n-1,1} & * & & \cdots & * \\ M_{n1} & * & & \cdots & * \end{pmatrix}.$$

Moreover, $P(C_1)$ has its first column uniformly distributed on the n -sphere $S^{n-1}(1)$ (see the proof of Proposition 2.1).

On the other hand, $(G_n^2 \cdots G_3^2) \cdots (G_n^{n-2} G_{n-1}^{n-2}) G_n^{n-1}$ is independent of C_1 and has the following form

$$Q_{n-1}(M) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & Q \in SO(n-1) & & \end{pmatrix}.$$

Hence, the marginal of Haar measure $\mu_n^{\mathcal{H}}$ on C_1 is the uniform measure S_{n-1} on $S^{n-1}(1)$, the n -sphere of radius 1, and the conditional probability of $\mu_n^{\mathcal{H}}$ with respect to C_1 can be identified with the Haar measure $\mu_{n-1}^{\mathcal{H}}$ on $SO(n-1)$. In other words, the following map is isometric:

$$\begin{aligned} SO(n) &\longrightarrow S^{n-1} \times SO(n-1) \\ M &\longrightarrow (C_1, Q_{n-1}(M)) = (C_1, P(C_1)^{-1}M). \end{aligned}$$

Of course, we can do the same using the first line $L_1 = (M_{11}, \dots, M_{1n})$ of M instead of the first column C_1 . We obtain the following isometric map:

$$\begin{aligned} SO(n) &\longrightarrow S^{n-1} \times SO(n-1) \\ M &\longrightarrow \left(L_1, \tilde{Q}_{n-1}(M) \right) = \left(L_1, M(P(L_1)^t)^{-1} \right). \end{aligned}$$

3. Proof of Theorem 1.1. We now turn to the proof of Theorem 1.1. For all integer $l \geq 2$, define a_l as the smallest constant such that for all function $h \in L^2(\mu_l^{\mathcal{H}})$

$$\mathbf{E}_{\mu_l^{\mathcal{H}}} [h; h] \leq a_l l^2 \mathcal{D}_l(h). \quad (3.1)$$

The goal is to set up a recursive equation for a_n and to prove that a_n is bounded by a constant independent of n .

Decomposition of the variance. Let $F_1(L_1)$ be the expectation of $F(M)$ conditioned on the first line L_1 of M : $F_1(L_1) = \mathbf{E}_{\mu_n^{\mathcal{H}}} [F(M) | L_1] = \int F(\tilde{Q}_{n-1} \cdot P(L_1)^t) d\mu_{n-1}^{\mathcal{H}}(\tilde{Q}_{n-1})$, and denote by $\mathbf{E}_{\mu_n^{\mathcal{H}}} [f; g | \mathcal{F}]$ the covariance of f and g with respect to the measure $\mu_n^{\mathcal{H}}$ conditioned on $\mathcal{F}$:

$$\mathbf{E}_{\mu_n^{\mathcal{H}}} [f; g | \mathcal{F}] = \mathbf{E}_{\mu_n^{\mathcal{H}}} [fg | \mathcal{F}] - \mathbf{E}_{\mu_n^{\mathcal{H}}} [f | \mathcal{F}] \mathbf{E}_{\mu_n^{\mathcal{H}}} [g | \mathcal{F}].$$

We can decompose the variance of F in the following way:

$$\mathbf{E}_{\mu_n^{\mathcal{H}}} [F; F] = \mathbf{E}_{\mu_n^{\mathcal{H}}} \left[\mathbf{E}_{\mu_n^{\mathcal{H}}} [F; F | L_1] \right] + \mathbf{E}_{\mu_n^{\mathcal{H}}} [F_1; F_1]. \quad (3.2)$$

Induction. We have seen in Section 2 that $\mu_n^{\mathcal{H}}$ conditioned on L_1 can be identified with $\mu_{n-1}^{\mathcal{H}}$, the Haar measure on $SO(n-1)$. Hence, we can use (3.1) to bound the first term on the right-hand side of (3.2) by

$$\begin{aligned} &a_{n-1}(n-1)^2 \int_0^{2\pi} d\theta \operatorname{Av}_{i,j \neq 1} \mathbf{E}_{\mu_n^{\mathcal{H}}} \left[\mathbf{E}_{\mu_n^{\mathcal{H}}} [(F(R_{i,j}^\theta M) - F(M))^2 | L_1] \right] \\ &= a_{n-1}(n-1)^2 \operatorname{Av}_{i,j \neq 1} D_{i,j}(F), \end{aligned}$$

where $\operatorname{Av}_{i,j \neq 1}$ denotes the average over $i, j \neq 1$.

Variance on one coordinate. The second term on the right-hand side of (3.2) is the variance of F_1 with respect to the marginal of Haar measure $\mu_n^{\mathcal{H}}$ on L_1 , that is, the uniform measure on $S^{n-1}(1)$, the n -sphere of radius 1. From Theorem 2.1 of [5], we can bound this term, up to a multiplicative constant, by

$$n \operatorname{Av}_{1 \leq i, j \leq n} \mathbf{E}_{\mu_n^{\mathcal{H}}} \left[\int_0^{2\pi} d\theta (F_1(R_{i,j}^\theta L_1) - F_1(L_1))^2 \right].$$

The above expression can be rewritten as

$$\begin{aligned} &n \operatorname{Av}_{1 \leq i, j \leq n} \mathbf{E}_{\mu_n^{\mathcal{H}}} \left[\int_0^{2\pi} d\theta \left(\mathbf{E}_{\mu_n^{\mathcal{H}}} \left[F(\tilde{Q}_{n-1}(M) \cdot P(R_{i,j}^\theta L_1)^t) \right. \right. \right. \\ &\quad \left. \left. \left. - F(\tilde{Q}_{n-1}(M) \cdot P(L_1)^t) \mid L_1 \right) \right]^2 \right]. \end{aligned}$$

Since $P(R_{i,j}^\theta L_1) = R_{i,j}^\theta P(L_1)$, we obtain that

$$\begin{aligned} \mathbf{E}_{\mu_n^{\mathcal{H}}} [F_1; F_1] &\leq C n \operatorname{Av}_{1 \leq i, j \leq n} \mathbf{E}_{\mu_n^{\mathcal{H}}} \left[\int_0^{2\pi} d\theta \left(\mathbf{E}_{\mu_n^{\mathcal{H}}} \left[F(\tilde{Q}_{n-1}(M) \cdot P(L_1)^t \cdot R_{i,j}^\theta) \right. \right. \right. \\ &\quad \left. \left. \left. - F(\tilde{Q}_{n-1}(M) \cdot P(L_1)^t) \mid L_1 \right) \right]^2 \right] \\ &= C n \operatorname{Av}_{1 \leq i, j \leq n} \int_0^{2\pi} d\theta \mathbf{E}_{\mu_n^{\mathcal{H}}} \left[\left(F(M R_{i,j}^\theta) - F(M) \right)^2 \right]. \end{aligned}$$

Notice the above term is equal to $C n \operatorname{Av}_{1 \leq i, j \leq n} D_{i,j}(\tilde{F}) = C n \mathcal{D}_n(\tilde{F})$ where $\tilde{F}(M) = F(M^t)$.

Conclusion. Eventually, we proved that

$$\mathbf{E}_{\mu_n^{\mathcal{H}}} [F; F] \leq Cn \mathcal{D}_n(\tilde{F}) + a_{n-1}(n-1)^2 \operatorname{Av}_{k,l \neq 1} D_{k,l}(F). \quad (3.3)$$

Let $P_{1,i}$ be the permutation matrix which permutes the lines 1 and i and denote by H_i the function $H_i(M) = F(P_{1,i}M)$. Note that $\mathbf{E}_{\mu_n^{\mathcal{H}}} [F; F] = \mathbf{E}_{\mu_n^{\mathcal{H}}} [H_i; H_i]$ and $\mathcal{D}_n(\tilde{H}_i) = \mathcal{D}_n(\tilde{F})$. Moreover, we have

$$P_{1,i} R_{k,l}^\theta = \begin{cases} R_{k,l}^\theta P_{1,i}, & i \neq k, \\ R_{1,l}^\theta P_{1,i}, & i = k, \end{cases}$$

which implies that $\mathcal{D}_n(H_i) = \mathcal{D}_n(F)$. We now apply (3.3) to the function H_i and average over $i \in \{1, \dots, n\}$ to get

$$\mathbf{E}_{\mu_n^{\mathcal{H}}} [F; F] \leq Cn \mathcal{D}_n(\tilde{F}) + a_{n-1}(n-1)^2 \mathcal{D}_n(F).$$

Since $\mathbf{E}_{\mu_n^{\mathcal{H}}} [\tilde{F}; \tilde{F}] = \mathbf{E}_{\mu_n^{\mathcal{H}}} [F; F]$, this implies that

$$\mathbf{E}_{\mu_n^{\mathcal{H}}} [F; F] \leq \left(Cn + a_{n-1}(n-1)^2 \right) \frac{\mathcal{D}_n(\tilde{F}) + \mathcal{D}_n(F)}{2}, \quad (3.4)$$

and, by definition of a_n , that $a_n \leq C/n + a_{n-1}(n-1)/n^2$, which is enough to conclude the proof of the theorem.

Notice that using this induction method, we lose a factor n by going from the gap on one coordinate (on the sphere $S^{n-1}(1)$) to the one on $SO(n)$.

4. Proof of Theorem 1.2. Since Theorem 1.2 is also proved using the martingale method, we do not give as many details as in Section 3. For all integer $l \geq 2$, define b_l as the smallest constant such that for all function $h \in L^2(\mu_l^{\mathcal{H}})$

$$H_l(h) \leq b_l l^3 \mathcal{E}_l(h, \ln h). \quad (4.1)$$

The goal is to set up a recursive equation for b_n and to prove that b_n is bounded by a constant independent of n .

Decomposition of the entropy. We can rewrite the entropy of F as

$$H_n(F) = \mathbf{E}_{\mu_n^{\mathcal{H}}} \left[\mathbf{E}_{\mu_n^{\mathcal{H}}} \left[F \ln \frac{F}{F_1} \middle| L_1 \right] \right] + \mathbf{E}_{\mu_n^{\mathcal{H}}} [F_1 \ln F_1]. \quad (4.2)$$

Induction. We use (4.1) to bound the first term on the right-hand side of (4.2) by

$$b_{n-1}(n-1)^3 \int_0^{2\pi} d\theta \operatorname{Av}_{i,j \neq 1} \mathbf{E}_{\mu_n^{\mathcal{H}}} \left[\mathbf{E}_{\mu_n^{\mathcal{H}}} \left[F(M) \ln \frac{F(R_{i,j}^\theta M)}{F(M)} \middle| L_1 \right] \right].$$

Entropy dissipation bound on one coordinate. The second term on the right-hand side of (4.2) is the entropy of F_1 with respect to the uniform measure on $S^{n-1}(1)$. In [8] (Annex III, Appendix B), Villani proved an entropy/entropy dissipation bound on the sphere $S^{n-1}(1)$ with a constant of order n^2 . Hence, we bound the second term, up to a multiplicative constant, by

$$n^2 \operatorname{Av}_{1 \leq i,j \leq n} \mathbf{E}_{\mu_n^{\mathcal{H}}} \left[\int_0^{2\pi} d\theta \left(F_1(R_{i,j}^\theta L_1) - F_1(L_1) \right) \ln \frac{F_1(R_{i,j}^\theta L_1)}{F_1(L_1)} \right].$$

Using the remarks made in Section 3 and the convexity of the function $(x-y) \ln(x/y)$, we obtain that $\mathbf{E}_{\mu_n^{\mathcal{H}}} [F_1 \ln F_1]$ is bounded above, up to a multiplicative constant, by $n^2 \mathcal{E}_n(\tilde{F}, \ln \tilde{F})$.

Conclusion. Eventually, we proved that

$$H_n(F) \leq Cn^2 \mathcal{E}_n(\tilde{F}, \ln \tilde{F}) + b_{n-1}(n-1)^3 \operatorname{Av}_{k,l \neq 1} \mathcal{E}_{i,j}(\tilde{F}, \ln \tilde{F}), \quad (4.3)$$

where $\mathcal{E}_{i,j}(\tilde{F}, \ln \tilde{F})$ is the part of $\mathcal{E}_n(\tilde{F}, \ln \tilde{F})$ corresponding to rotation in the plane (i, j) . We conclude in the same way that

$$b_n \leq \frac{C}{n} + b_{n-1} \left(\frac{n-1}{n} \right)^3,$$

which is enough to conclude the proof of the theorem.

The entropy dissipation bound obtained by Villani is the best one we know, but is probably not optimal. Note that improving this bound would improve our result as well.

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