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MOMENT INEQUALITIES FOR SUMS OF CERTAIN
DEPENDENT RANDOM VARIABLES

Представлен ряд примеров зависимых последовательностей со свойством «некоррелированность влечет независимость». Изучаются моментные неравенства для этих последовательностей.

Ключевые слова и фразы: моментные неравенства, неравенства типа Розенталя, отрицательная зависимость, ассоциированность.

1. Introduction, statement of results and discussion. Let $X = (X_1, \dots, X_n)$ be a centered vector of real-valued random variables (r.v.'s) with finite variance. Put $S_n = X_1 + \dots + X_n$, with the convention that $S_0 = 0$ and let $r > 2$ be a fixed real number. The Rosenthal inequality known for independent and identically distributed random variables as

$$\mathbf{E}|S_n|^r \leq C_r \left\{ (\mathbf{D}S_n)^{r/2} + n\mathbf{E}|X_1|^r \right\}, \quad (1)$$

has got a lot of applications in probability and statistics. Its main advantage is that the controls of the higher order moments of the partial sums are reduced to those of order 2. We refer for instance to Rosenthal [21], Petrov [18] for proofs and other versions of moment inequalities and to Stout [26], Bretagnolle and Huber [4], Peskir and Shiryaev [17] for various applications of inequality (1).

Since almost all the models from the statistical theory are not independent and in view of the utility of Rosenthal's inequality in such a field, several authors are interested in the extension of inequality (1) to the depending setting. Those extensions are far from being trivial and depend widely on the dependence structure of the processes.

A very well known measure of dependence in the probabilistic literature is described by the mixing coefficients. Versions of Rosenthal's inequality for mixing sequences were established by several authors. We refer to the works of Doukhan [9], Doukhan and Portal [8], Ibragimov and Linnik [13], Rio ([19], [20]), Shao ([22], [25]), . . .

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Pursuing the objective of Doukhan and Louhichi [10], we would like to prove in this paper a unifying inequality of type (1) for some sequences $(X_i)_{i \in \mathbf{N}}$ obeying the property «*the noncorrelation yields the independence*», which is described by the following covariance inequality:

$$\left| \text{cov}(h(X_i, i \in A), k(X_j, j \in B)) \right| \leq \sum_{i \in A} \sum_{j \in B} \left\| \frac{\partial h}{\partial x_i} \right\|_{\infty} \left\| \frac{\partial k}{\partial x_j} \right\|_{\infty} |\text{cov}(X_i, X_j)|, \quad (2)$$

where A and B are arbitrary finite disjoint subsets of $\mathbf{N}$ and h, k are any real-valued functions having uniformly bounded first derivatives.

Inequality (2) appears as a particular case of the general notion of weak dependence stated in [1] or independently in [10].

We have called in [15], (AG) sequences the processes fulfilling the covariance inequality (2). This terminology is due to the fact that both associated and Gaussian processes fulfill inequality (2). Associated random variables were introduced by Esary, Proschan, and Walkup [11] and independently by Fortuin, Kasteleyn, and Ginibre [12]. Sequences $(X_i)_{i \in \mathbf{N}}$ are associated or fulfilling the FKG inequality (in the statistical mechanics terminology) if for all coordinatewise nondecreasing functions f, g and all $n \in \mathbf{N}$ there holds:

$$\text{cov}(f(X_1, \dots, X_n), g(X_1, \dots, X_n)) \geq 0.$$

The (AG) property for associated processes follows from Lemma 3.1 in [3]. Birkel [2] was the first to establish moment inequalities under association. He proved the moment bound $\mathbf{E}|S_n|^r = O(n^{r/2})$ (i.e., Marcinkiewicz–Zygmund bound) under optimal conditions. Bulinski [6] proved the analog of this bound for random fields under optimal conditions as well, see also [5].

Rosenthal-type inequality for associated processes is recently established by Shao and Yu [23] for functions of associated sequences and by Vronski [27] for associated random fields.

The (AG) property is also satisfied by negatively associated processes. The sequence $(X_i)_{i \in \mathbf{N}}$ is said to be negatively associated if for every pair of disjoint subsets A and B of $\mathbf{N}$

$$\text{cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0,$$

whenever f and g are coordinatewise nondecreasing functions. This definition was introduced by Joag Dev [14]. The (AG) property for negatively associated processes follows from Lemma 1 in [7].

In the negatively dependent setting, moment bounds were established by Matula [16], cf. also [24] and the references therein.

For (AG) sequences, we propose the following moment inequality which will be the main result of this paper.

Theorem 1. *Let r be a fixed real number, $r > 2$. Let (X_n) be a strictly stationary sequence of centered and (AG) r.v.'s. Suppose moreover that this*

sequence is bounded by M . Then there exists a positive constant C_r depending only on r , such that

$$\mathbf{E}|S_n|^r \leq C_r \left[s_n^r + \sum_{k=1}^n \sum_{i=0}^{k-1} M^{r-2} (i+1)^{r-2} |\operatorname{cov}(X_1, X_{1+i})| \right], \quad (3)$$

where $s_n^2 := n \sum_{i=0}^n |\operatorname{cov}(X_1, X_{1+i})|$.

Theorem 1 gives then a unifying Rosenthal-type inequality for at least two models: associated or negatively associated processes.

An immediate consequence of Theorem 1 is the following corollary.

Corollary 1. *Let r be a fixed real number, $r > 2$. Let (X_n) be a strictly stationary sequence of centered, (AG) r.v.'s bounded by M . Suppose that*

$$v_r := M^{-2} \sum_{i=0}^{+\infty} (i+1)^{r-2} |\operatorname{cov}(X_1, X_{1+i})| < +\infty, \quad (4)$$

then there exists a positive constant depending only on r and on v_r such that

$$\mathbf{E}|S_n|^r \leq C_r [n^{r/2} M^r + n M^r]. \quad (5)$$

Inequality (3) is close to (1). The linear term (the second term in the right-hand side (RHS) of (3)) is of order n under the summability condition $\sum_{i=0}^{+\infty} (i+1)^{r-2} |\operatorname{cov}(X_1, X_{1+i})| < +\infty$, in contrast with the corresponding term in the Shao and Yu inequality (see [23]) which is of order $n^{1+\varepsilon}$ (for some $\varepsilon > 0$). Their inequality is however available for nonbounded functions of associated random variables. In this latter context, Vronski [27] proves versions of Rosenthal inequality for random fields. For bounded associated sequences Vronski's inequalities apply, and they lead to inequality (5) under the summability condition $\sum_{i=0}^{+\infty} (i+1)^{r-2} U(r) < +\infty$, where $U(r)$ is the Cox–Grimmett coefficient defined by

$$U(r) = \sup_{i \in \mathbf{N}} \sum_{j: |i-j| \geq r} \operatorname{cov}(X_i, X_j).$$

An immediate consequence of Theorem 1 is the following Marcinkiewicz–Zygmund bound.

Corollary 2. *Let r be a fixed real number, $r > 2$. Let (X_n) be a strictly stationary sequence of centered, (AG) r.v.'s bounded by M . Suppose that*

$$|\operatorname{cov}(X_1, X_{1+i})| = O(i^{-r/2}), \quad \text{as } i \rightarrow +\infty. \quad (6)$$

Then

$$\mathbf{E}|S_n|^r = O(n^{r/2}). \quad (7)$$

For bounded associated sequences, condition (6) is shown to be optimal for the Marcinkiewicz–Zygmund bound (7) (cf. [2]).

In the next section we prove Theorem 1. The method for its proof is a generalization of the Lindeberg decomposition to an order $r > 2$. This method was first developed by Rio [20] for mixing sequences and for $r \in]2, 3]$.

Some auxiliary lemmas, needed for the proofs, are proved in the Appendix.

2. Proofs. In the sequel p is a fixed integer not less than 2 and Φ_p is the class of functions defined by:

$$\Phi_p := \left\{ \phi: \mathbf{R}^+ \rightarrow \mathbf{R}^+; \phi(0) = \phi'(0) = \dots = \phi^{(p-1)}(0) = 0, \right. \\ \left. \phi^{(p)} \text{ is nondecreasing, concave} \right\}.$$

Let ϕ be a function of the set Φ_p . Theorem 1 is proved if we suitably control $\mathbf{E}\phi(|S_n|)$ (since the function $x \rightarrow x^r$, for $r \in]p, p + 1]$ is one of those functions ϕ).

Such a control will be done into four steps.

2.1. Step 1. The purpose of Step 1 is to reduce the control of $\mathbf{E}\phi(|S_n|)$ to that of a suitable *polynomial* function of $|S_n|$, as follows.

Define the function $g_p: \mathbf{R}^+ \times \mathbf{R} \rightarrow \mathbf{R}^+$ by

$$g_p(t, x) := \frac{1}{(p + 1)!} \left[x^{p+1} \mathbf{1}_{0 \leq x \leq t} + (x^{p+1} - (x - t)^{p+1}) \mathbf{1}_{t < x} \right], \tag{8}$$

for any $x \geq 0$ and $g_p(t, x) = g_p(t, -x)$.

The following lemma is a generalization of equality (4.3) in [20], which was written for $p = 2$.

Lemma 1. *Let p be a fixed integer, $p \geq 2$. Let $\phi \in \Phi_p$. Suppose that $\phi^{(p+1)}$ exists and that $\lim_{x \rightarrow \infty} \phi^{(p+1)}(x) = 0$. Then*

$$\phi(x) = \int_0^{+\infty} g_p(t, x) \nu_p(dt),$$

where ν_p is the Stieltjes measure of $-\phi_p^{(p+1)}$ defined by $\nu_p(dt) = -d\phi^{(p+1)}(t)$.

Lemma 1 reduces then the estimation of $\mathbf{E}\phi(|S_n|)$ to that of $\mathbf{E}g_p(t, S_n)$.

2.2. Step 2: Main terms. The purpose of Step 2 is then to give bounds of $\mathbf{E}f(S_n)$, for real-valued functions f belonging to a suitable set containing the functions $x \rightarrow g_p(t, x)$. For this, let $\mathcal{C}_p$ be the class of real-valued, p times continuously differentiable functions f such that $f(0) = \dots = f^{(p)}(0) = 0$.

Let $\mathcal{F}_p(b_1, b_2)$ be the subclass of $\mathcal{C}_{p+1}$ such that $\|f^{(p)}\|_\infty \leq b_1$ and $\|f^{(p+1)}\|_\infty \leq b_2$, where $\|f^{(i)}\|_\infty = \sup_{x \in \mathbf{R}} |f^{(i)}(x)|$ and $f^{(i)}$ denotes the differential of order i of the function f .

In this step, we give an estimation of $\mathbf{E}f(S_n)$, for $f \in \mathcal{F}_p(b_1, b_2)$. Let us note that the function g_p as defined by (8) belongs to the set $\mathcal{F}_p(t, 1)$.

We first exhibit the mains terms of our calculations.

N o t a t i o n . We denote by $\sum_{(p-2)}$ the sum over $i_1, \dots, i_{p-2}$ such that $0 := i_0 \leq i_1 \leq \dots \leq i_{p-2} \leq k - 1$, that is, $\sum_{0 \leq i_1 \leq \dots \leq i_{p-2} \leq k-1}$. We define

$$\mathbf{E}_{p-2, k}(\Delta f) = \sup_{0 \leq u \leq 1} \sum_{(p-2)} |\mathbf{E}X_k X_{k-i_1} \dots X_{k-i_{p-2}} \Delta_{p-2, k}(f, u)|,$$

where

$$\begin{aligned}\Delta_{p-2,k}(f) &:= \Delta_{p-2,k}(f, u) = [f(S_{k-i_{p-2}-1} + uX_{k-i_{p-2}}) - f(S_{k-i_{p-2}-1})] \\ &= uX_{k-i_{p-2}} \int_0^1 f'(S_{k-i_{p-2}-1} + uvX_{k-i_{p-2}}) dv.\end{aligned}$$

Finally, we set for $p \geq 2$,

$$\mathbf{E}_{p-2,k}(f) = \sum_{(p-2)} |\mathbf{E}X_k X_{k-i_1} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})|.$$

Let us note that $i_0 = 0$,

$$\mathbf{E}_{0,k}(f) = |\mathbf{E}X_k f(S_{k-1})| \quad \text{and that} \quad \mathbf{E}_{0,k}(\Delta f) = \sup_{0 \leq u \leq 1} |\mathbf{E}X_k \Delta_{0,k}(f, u)|.$$

For any real-valued function f of the set $\mathcal{F}_p(b_1, b_2)$, the quantity $|\mathbf{E}(f(S_n))|$ is evaluated by means of the main terms $\mathbf{E}_{p-2,k}(f^{(p-1)})$ and $\mathbf{E}_{p-2,k}(\Delta f^{(p-1)})$ as shows the following lemma.

Lemma 2. *Let p be a fixed integer, $p \geq 2$. Let (X_n) be a sequence of (AG) r.v.'s, centered and bounded by M . There exists a positive constant C_p depending only on p , such that for any $f \in \mathcal{F}_p(b_1, b_2)$*

$$\begin{aligned}|\mathbf{E}(f(S_n))| &\leq C_p \left\{ s_n^p(b_1 \wedge b_2 s_n) + (b_1 \wedge b_2 M) M^{p-2} \sum_{k=1}^n \sum_{i=0}^{k-1} |\text{cov}(X_k, X_{k-i})| \right. \\ &\quad \left. + \sum_{k=1}^n \mathbf{E}_{p-2,k}(f^{(p-1)}) + \sum_{k=1}^n \mathbf{E}_{p-2,k}(\Delta f^{(p-1)}) \right\}.\end{aligned}$$

From now C_p denotes a positive constant depending only on p and that will be different from line to line.

2.3. Step 3: Evaluation of the main terms $\mathbf{E}_{p-2,k}(f)$ and $\mathbf{E}_{p-2,k}(\Delta f)$. The object of this step is to evaluate the main terms $\mathbf{E}_{p-2,k}(f)$ and $\mathbf{E}_{p-2,k}(\Delta f)$ of Lemma 2. This evaluation evokes the following covariance quantities:

$$M_{m,n} = M^{m-2} \sum_{k=1}^n \sum_{i=0}^{k-1} (i+1)^{m-2} |\text{cov}(X_1, X_{i+1})|, \quad \text{for } 2 \leq m \leq p, \quad (9)$$

and

$$\begin{aligned}M_{m,n}(b_1, b_2) \\ := \sum_{k=1}^n \sum_{r=0}^{k-1} (b_1 \wedge b_2(r+1)M)(r+1)^{m-2} M^{m-2} |\text{cov}(X_1, X_{r+1})|. \quad (10)\end{aligned}$$

Let us note that $M_{2,n}$ is close to $\mathbf{D}S_n$ and that in the i.i.d. case $M_{2,n} = n\mathbf{D}X_1$.

We now state the basic technical lemma of this paper.

Lemma 3. *Let f be a real-valued function of the set $\mathcal{F}_1(b_1, b_2)$. Let (X_n) be a centered sequence of (AG) r.v.'s. Suppose that (X_n) is uniformly bounded by M . Then, for any integer $p \geq 2$, there exists a positive constant C_p depending only on p , for which*

$$\sum_{k=1}^n \mathbf{E}_{p-2,k}(\Delta f) + \sum_{k=1}^n \mathbf{E}_{p-2,k}(f) \leq C_p \left\{ s_n^p(b_1 \wedge b_2 s_n) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2) + M_{p,n}(b_1, b_2) \right\}, \tag{11}$$

where the sum $\sum_{m=2}^{p-2}$ equals to 0, whenever $p \in \{2, 3\}$.

2.4. Step 4: End of the proof of Theorem 1. Finally, we combine the three previous steps in order to finish the proof of Theorem 1. Let us explain. We first make use of Lemma 1, together with Fubini’s theorem, to obtain:

$$\mathbf{E}\phi(|S_n|) = \int_0^{+\infty} \mathbf{E}g_p(t, S_n) \nu_p(dt). \tag{12}$$

Now, we recall that the functions $x \rightarrow g_p(t, x)$ and $x \rightarrow g_p^{(p-1)}(t, x)$ belong respectively to $\mathcal{F}_p(t, 1)$ and to $\mathcal{F}_1(t, 1)$ (in fact if $f \in \mathcal{F}_p(t, 1)$, then $f^{(p-1)} \in \mathcal{F}_1(t, 1)$). Hence we deduce by applying Lemma 2 to the function $x \rightarrow g_p(t, x)$ and Lemma 3 to $x \rightarrow g_p^{(p-1)}(t, x)$:

$$\mathbf{E}g_p(t, S_n) \leq C_p \left\{ \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(t, 1) + M_{p,n}(t, 1) + s_n^p(t \wedge s_n) \right\}. \tag{13}$$

Taking into account Lemma 1 and the fact $g^{(p)}(x) = x \wedge t$, we deduce that

$$\phi^{(p)}(x) = \int_0^{+\infty} (t \wedge x) \nu_p(dt). \tag{14}$$

Inequalities (12), (13) and (14) yield:

$$\begin{aligned} \mathbf{E}\phi(|S_n|) \leq C_p \Bigg\{ & \sum_{k=1}^n \sum_{i=0}^{k-1} \phi^{(p)}(M(i+1)) M^{p-2}(i+1)^{p-2} \\ & \times |\operatorname{cov}(X_1, X_{1+i})| + s_n^p \phi^{(p)}(s_n) \\ & + \sum_{m=2}^{p-2} M_{m,n} \left(\sum_{k=1}^n \sum_{i=0}^{k-1} \phi^{(p)}(M(i+1)) M^{p-m-2}(i+1)^{p-m-2} \right. \\ & \left. \times |\operatorname{cov}(X_1, X_{1+i})| \right) \Bigg\}. \end{aligned} \tag{15}$$

Now, we use the concavity property of the function $\phi^{(p)}$:

$$\phi(x) = \frac{x^p}{(p-1)!} \int_0^1 (1-t)^{p-1} \phi^{(p)}(tx) dt \geq x^p \phi^{(p)}(x) \int_0^1 \frac{t(1-t)^{p-1}}{(p-1)!} dt,$$

to deduce that

$$x^p \phi^{(p)}(x) \leq C_p \phi(x). \tag{16}$$

We conclude, combining inequalities (15) and (16):

$$\begin{aligned} \mathbf{E}\phi(|S_n|) \leq C_p \Bigg\{ & \sum_{k=1}^n \sum_{i=0}^{k-1} \phi(M(i+1)) M^{-2}(i+1)^{-2} |\operatorname{cov}(X_1, X_{1+i})| + \phi(s_n) \\ & + \sum_{m=2}^{p-2} M_{m,n} \left(\sum_{k=1}^n \sum_{i=0}^{k-1} \phi(M(i+1)) M^{-m-2}(i+1)^{-m-2} \right. \\ & \left. \times |\operatorname{cov}(X_1, X_{1+i})| \right) \Bigg\}. \end{aligned}$$

The last inequality applied to $\phi(x) = x^r$, for $r \in]p, p+1]$, leads to

$$\begin{aligned} \mathbf{E}|S_n|^r \leq C_r \Bigg\{ & \sum_{k=1}^n \sum_{i=0}^{k-1} M^{r-2}(i+1)^{r-2} |\operatorname{cov}(X_1, X_{1+i})| + s_n^r \\ & + \sum_{m=2}^{p-2} M_{m,n} M_{r-m,n} \Bigg\}. \end{aligned} \quad (17)$$

In order to control the term $M_{m,n}M_{r-m,n}$ of the last inequality, we need the following lemma (that we prove in the Appendix).

Lemma 4 (Analog of Hölder's inequalities). *We have, for any $r > 4$, $2 < m < r$,*

$$M_{m,n}M_{r-m,n} \leq s_n^{2r/(r-2)} M_{r,n}^{(r-4)/(r-2)} \leq s_n^r + M_{r,n}. \quad (18)$$

We also have, for any $b_1, b_2 > 0$, $r > 2$, and $2 \leq m < r$,

$$s_n^{r-m}(b_1 \wedge b_2 s_n) M_{m,n} \leq s_n^r(b_1 \wedge b_2 s_n) + M_{r,n}(b_1, b_2), \quad (19)$$

$$s_n^{r-m} M_{m,n}(b_1, b_2) \leq s_n^r(b_1 \wedge b_2 s_n) + M_{r,n}(b_1, b_2). \quad (20)$$

The proof of Theorem 1 is now complete, using inequality (17), together with (18) and (20) (recall that $M_{2,n} \leq s_n^2$).

3. Appendix. In this section, we prove all the auxiliary lemmas needed for the proof of Theorem 1.

P r o o f o f L e m m a 2. By induction on $p \geq 2$, and using the following decomposition: for $g \in \mathcal{C}_1$,

$$g(S_n) = \sum_{k=1}^n [g(S_k) - g(S_{k-1})] = \sum_{k=1}^n X_k \int_0^1 g'(S_{k-1} + uX_k) du, \quad (21)$$

we show that, for any $f \in \mathcal{C}_{p-1}$,

$$|\mathbf{E}(f(S_n))| \leq C_p \sum_{k=1}^n \left[\mathbf{E}_{p-2,k}(f^{(p-1)}) + \mathbf{E}_{p-2,k}(\Delta f^{(p-1)}) \right] + \sum_{k=1}^n \mathbf{R}_{p,k}(f), \quad (22)$$

where

$$\mathbf{R}_{p,k}(f) := \left| \mathbf{E} \left[f(S_k) - f(S_{k-1}) - \cdots - \frac{X_k^{p-1}}{(p-1)!} f^{(p-1)}(S_{k-1}) \right] \right|.$$

Let now f be a fixed function of the set $\mathcal{F}_p(b_1, b_2)$. In view of (22), Lemma 2 is proved if we suitably evaluate $\mathbf{R}_{p,k}(f)$. Clearly

$$\begin{aligned} \mathbf{R}_{p,k}(f) &\leq \mathbf{R}_{p+1,k}(f) + |\operatorname{cov}(X_k^p, f^{(p)}(S_{k-1}))| + \mathbf{E}|X_k^p| \mathbf{E}|f^{(p)}|(S_{k-1}) \\ &=: I_{1,k} + I_{2,k} + I_{3,k}. \end{aligned}$$

Taylor's formula and the (AG) property yield:

$$\begin{aligned} I_{1,k} &\leq \mathbf{E}(|X_k|^p(b_1 \wedge b_2|X_k|)), \\ I_{2,k} &\leq c_p \left[b_p M^{p-2} \mathbf{E}|X_k|^2 \wedge M^{p-1} b_{p+1} \sum_{i=0}^{k-1} |\operatorname{cov}(X_k, X_{k-i})| \right] \\ &\leq c_p (b_p \wedge b_{p+1} M) M^{p-2} \sum_{i=0}^{k-1} |\operatorname{cov}(X_k, X_{k-i})|, \quad \text{finally,} \\ I_{3,k} &\leq \mathbf{E}|X_k^p|(b_1 \wedge b_2 s_n) \\ &\leq \mathbf{E}|X_k^p|(b_1 \wedge b_2 s_n) \mathbf{1}_{|X_k| \leq s_n} + \mathbf{E}|X_k^p|(b_1 \wedge b_2 s_n) \mathbf{1}_{|X_k| > s_n} \\ &\leq s_n^{p-2} (b_1 \wedge b_2 s_n) \mathbf{E}(X_k^2) + \mathbf{E}(|X_k|^p(b_1 \wedge b_2|X_k|)). \end{aligned}$$

The proof of Lemma 2 is complete by noting that $\sum_{k=1}^n \mathbf{E}(X_k^2) \leq s_n^2$, and

$$\sum_{k=1}^n \mathbf{E}(|X_k|^p(b_1 \wedge b_2|X_k|)) \leq (b_1 \wedge b_2 M) M^{p-2} \sum_{k=1}^n |\operatorname{cov}(X_k, X_k)|.$$

P r o o f o f L e m m a 4. For all positive integer m , and all positive real numbers α and β that are conjugate (i.e., $1/\alpha + 1/\beta = 1$) there holds, thanks to Hölder's inequality

$$\begin{aligned} \sum_{i=0}^{k-1} M^{m-2} (i+1)^{m-2} |\operatorname{cov}(X_1, X_{i+1})| &\leq \left(\sum_{i=0}^{k-1} |\operatorname{cov}(X_1, X_{i+1})| \right)^{1/\alpha} \\ &\times \left(\sum_{i=0}^{k-1} M^{(m-2)\beta} (i+1)^{(m-2)\beta} |\operatorname{cov}(X_1, X_{i+1})| \right)^{1/\beta}. \end{aligned}$$

Taking the sum over $k \in \{1, \dots, n\}$ in the last inequality and using again Hölder's inequality, we obtain

$$M_{m,n} \leq s_n^{2/\alpha} \left(\sum_{k=0}^n \sum_{i=0}^{k-1} M^{(m-2)\beta} (i+1)^{(m-2)\beta} |\operatorname{cov}(X_1, X_{i+1})| \right)^{1/\beta},$$

hence, for $\alpha = (r-2)/(r-m)$ and $\beta = (r-2)/(m-2)$, we obtain

$$M_{m,n} \leq s_n^{2(r-m)/(r-2)} M_{r,n}^{(m-2)/(r-2)} \quad \text{and} \quad M_{r-m,n} \leq s_n^{2m/(r-2)} M_{r,n}^{(r-m-2)/(r-2)}.$$

The two last inequalities proves the first inequality in (18), the second one is trivial.

Now, we only prove (19) ((20) will hold with the same method).

Clearly

$$\begin{aligned} s_n^{r-m}(b_1 \wedge b_2 s_n) & \left(\sum_{k=1}^n \sum_{i: M(i+1) \leq s_n}^{k-1} M^{m-2}(i+1)^{m-2} |\operatorname{cov}(X_1, X_{i+1})| \right) \\ & \leq s_n^{r-m}(b_1 \wedge b_2 s_n) s_n^{m-2} \left(\sum_{k=1}^n \sum_{i=0}^{k-1} |\operatorname{cov}(X_1, X_{i+1})| \right) \leq s_n^r(b_1 \wedge b_2 s_n) \quad (23) \end{aligned}$$

and

$$\sum_{k=1}^n \sum_{i: M(i+1) \geq s_n}^{k-1} (s_n^{r-m}(b_1 \wedge b_2 s_n) M^{m-2}(i+1)^{m-2} |\operatorname{cov}(X_1, X_{i+1})|) \leq M_{r,n}. \quad (24)$$

Collecting inequalities (23) and (24) we obtain (19).

Before going to the proof of the basic Lemma 3, we will state an intermediate result needed for its proof. Define

$$\mathbf{E}_{p-1,k} = \sum_{(p-1)} |\mathbf{E} X_k X_{k-i_1} \cdots X_{k-i_{p-1}}|.$$

Lemma 5 ([10]). *Let (X_n) be a centered sequence of (AG) r.v.'s. Suppose that (X_n) is uniformly bounded by M . Then, for any integer $p \geq 2$, there exists a positive constant C_p for which*

$$\sum_{k=1}^n \mathbf{E}_{p-1,k} \leq C_p \left\{ s_n^p + \sum_{k=1}^n \sum_{r=0}^{k-1} r^{p-2} M^{p-2} |\operatorname{cov}(X_1, X_{r+1})| \right\}.$$

Lemma 5 was established by Doukhan and Louhichi [10], in order to give moment inequalities with integer order p (cf. Theorem 4 in [10]).

P r o o f o f L e m m a 3. We will prove, using an induction on $p \geq 2$, that each of the terms $\sum_{k=1}^n \mathbf{E}_{p-2,k}(\Delta f)$ and $\sum_{k=1}^n \mathbf{E}_{p-2,k}(f)$ are bounded by the RHS of (11).

For $p = 2$, we refer to [15]. The calculations for the case $p = 2$ are also deduced from the general case that we state just below.

Suppose now that (11) holds for the order $(p-1)$, we will prove it for p . Our purpose is then to evaluate the following sums:

$$\sum_{k=1}^n \sum_{(p-2)} |\mathbf{E} X_k X_{k-i_1} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})|, \quad (25)$$

$$\sum_{k=1}^n \sum_{(p-2)} |\mathbf{E} X_k X_{k-i_1} \cdots X_{k-i_{p-2}} \Delta_{p-2,k}(f)|. \quad (26)$$

We argue as Doukhan and Portal [8]: let $0 =: i_0 \leq i_1 \leq \cdots \leq i_{p-2} \leq k-1$ be a fixed sequence of increasing integers, let m be the smallest integer for which $r_m := i_{m+1} - i_m = \max_{1 \leq q \leq p-2} (i_q - i_{q-1})$. Finally let $\sum_{(p-2)}^{(m)}$ denotes the sum over the subdivisions $i_1 \leq \cdots \leq i_{p-2} \leq k-1$ for which the big hole $\max_{1 \leq q \leq p-2} (i_q - i_{q-1})$ is reached at the index m .

We also define $\sum_{(p-2)}^{(m),r_m}$ to be the sum over the subdivisions $i_1 \leq \dots \leq i_{p-2}$ such that the big hole $i_{m+1} - i_m = \max_{1 \leq q \leq p-2} (i_q - i_{q-1})$ equals to r_m .

Hence, if B denotes a subset of $\mathbf{N}^{p-2}$ and g is a positive function on $\mathbf{N}^{p-2}$, then

$$\sum_{(p-2),B} g(i_1, \dots, i_{p-2}) \leq \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-2),B}^{(m),r_m} g(i_1, \dots, i_{p-2}), \quad (27)$$

we mean by $\sum_{(p-2),B}$ the sum over the subdivision $(i_1, \dots, i_{p-2})$ such that $(i_1, \dots, i_{p-2}) \in B$ and $0 =: i_0 \leq i_1 \leq \dots \leq i_{p-2}$.

We will decompose the sum in (25) (respectively in (26)) into two sums, one over the subdivisions $0 =: i_0 \leq i_1 \leq \dots \leq i_{p-2}$ for which $i_1 > A$ and the second over the subdivisions $0 =: i_0 \leq i_1 \leq \dots \leq i_{p-2}$ for which $i_1 \leq A$, where the truncate term A is defined by:

$$A = \frac{b_1}{Mb_2} \wedge (k-1). \quad (28)$$

(A) *Bounds over the subdivisions* $0 =: i_0 \leq i_1 \leq \dots \leq i_{p-2}$ such that $i_1 > A$. We first need the following technical lemma that we check at the end of the proof of Lemma 3, as soon as all the evoked lemmas.

Lemma 6. *Let (X_n) be a centered sequence of (AG) r.v.'s. Suppose that (X_n) is uniformly bounded by M . Then, for any integer $p \geq 2$, there exists a positive constant C_p such that for any $f \in \mathcal{F}_1(b_1, b_2)$:*

$$\begin{aligned} & \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-2)}^{(m),r_m} |\text{cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} \\ & \quad \times f(S_{k-i_{p-2}-1}))| \mathbf{1}_{i_1 > A} \\ & \leq C_p \{s_n^p(b_1 \wedge b_2 s_n) + M_{p,n}(b_1, b_2)\}. \end{aligned} \quad (29)$$

We obtain, using the trivial fact

$$|\mathbf{E}(XY)| \leq |\text{cov}(X, Y)| + |\mathbf{E}X| |\mathbf{E}Y|,$$

inequalities (27) and (29):

$$\begin{aligned} & \sum_{k=1}^n \sum_{(p-2)} |\mathbf{E}(X_k \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1}))| \mathbf{1}_{i_1 > A} \\ & \leq \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{(p-2)}^{(m)} |\mathbf{E}X_k X_{k-i_1} \cdots X_{k-i_m}| |\mathbf{E}X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})| \\ & \quad + C_p \{s_n^p(b_1 \wedge b_2 s_n) + M_{p,n}(b_1, b_2)\}. \end{aligned} \quad (30)$$

We now evaluate the first term in the last inequality. For this, let $A_{p,n}$ (respectively $A_{p,n}(f)$) denote the sum $\sum_{k=1}^n \mathbf{E}_{p,k}$ (respectively $\sum_{k=1}^n \mathbf{E}_{p,k}(f)$).

The inductive assumption and Lemma 5, together with Lemma 4, yield

$$\begin{aligned}
& \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{(p-2)}^{(m)} |\mathbf{E} X_k X_{k-i_1} \cdots X_{k-i_m}| |\mathbf{E} X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})| \\
& \leq \sum_{m=0}^{p-3} A_{m,n} A_{p-m-3,n}(f) \\
& \leq C_p \sum_{m=0}^{p-3} (s_n^{m+1} + M_{m+1,n}) \left(s_n^{p-m-1} (b_1 \wedge b_2 s_n) \right. \\
& \quad \left. + \sum_{m'=2}^{p-m-3} M_{m',n} M_{p-m-1-m',n}(b_1, b_2) + M_{p-m-1,n}(b_1, b_2) \right) \\
& \leq C_p \left\{ s_n^p (b_1 \wedge b_2 s_n) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2) + M_{p,n}(b_1, b_2) \right\}. \quad (31)
\end{aligned}$$

Now, collecting inequalities (30) and (31), we obtain

$$\begin{aligned}
& \sum_{k=1}^n \sum_{(p-2)} \mathbf{1}_{i_1 > A} |\mathbf{E} X_k X_{k-i_1} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})| \\
& \leq C_p \left\{ s_n^p (b_1 \wedge b_2 s_n) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2) + M_{p,n}(b_1, b_2) \right\}. \quad (32)
\end{aligned}$$

In order to prove that

$$\begin{aligned}
& \sum_{k=1}^n \sum_{(p-2)} \mathbf{1}_{i_1 > A} |\mathbf{E} X_k X_{k-i_1} \cdots X_{k-i_{p-2}} \Delta_{p-2,k}(f)| \\
& \leq C_p \left\{ s_n^p (b_1 \wedge b_2 s_n) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2) + M_{p,n}(b_1, b_2) \right\}, \quad (33)
\end{aligned}$$

we argue as for the proof of (32), we just take Δf instead of f , and we use the following inequality given by the lemma below, instead of (29).

Lemma 7. *Let (X_n) be a centered sequence of (AG) r.v.'s. Suppose that (X_n) is uniformly bounded by M . Then, for any integer $p \geq 2$, there exists a positive constant C_p such that for any $f \in \mathcal{F}_1(b_1, b_2)$:*

$$\begin{aligned}
& \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-2)}^{(m), r_m} |\text{cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} \Delta_{p-2,k}(f))| \mathbf{1}_{i_1 > A} \\
& \leq C_p M_{p,n}(b_1, b_2). \quad (34)
\end{aligned}$$

(B) *Bounds over the subdivisions $0 =: i_0 \leq i_1 \leq \cdots \leq i_{p-2}$ such that $i_1 \leq A$.* Here we use analogous calculations as for (A). Those calculations are applied to the function f' instead of f , whose expression appears thanks to Taylor's formula (cf. equality (21)). Let us explain more.

First, we write

$$f(S_{k-i_{p-2}-1}) = \sum_{i_{p-1}=i_{p-2}+1}^{i_{p-2}+r_m} [f(S_{k-i_{p-1}}) - f(S_{k-i_{p-1}-1})] + f(S_{k-i_{p-2}-r_m-1}).$$

The last decomposition yields:

$$\begin{aligned} & \left| \mathbf{E}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})) \right| \\ & \leq \left| \sum_{i_{p-1}=i_{p-2}+1}^{i_{p-2}+r_m} \mathbf{E}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}} [f(S_{k-i_{p-1}}) - f(S_{k-i_{p-1}-1})]) \right| \\ & \quad + \left| \mathbf{E}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-r_m-1})) \right| \\ & =: J_1(i_1, \dots, i_{p-2}) + J_2(i_1, \dots, i_{p-2}). \end{aligned} \quad (35)$$

Evaluation of $\sum_{(p-2)} J_2(i_1, \dots, i_{p-2}) \mathbf{1}_{i_1 \leq A}$. We deduce, since $\mathbf{E}(XY) = \text{cov}(X, Y) + \mathbf{E}(X)\mathbf{E}(Y)$, that

$$\begin{aligned} & \mathbf{E}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-r_m-1})) \\ & = \text{cov}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1})) \\ & \quad + \mathbf{E}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}}) \mathbf{E}(f(S_{k-i_{p-2}-r_m-1})). \end{aligned} \quad (36)$$

The function f belongs to the set $\mathcal{F}_1(b_1, b_2)$, hence using Taylor's formula (recall that $f(0) = f'(0) = 0$), we obtain $|f(x)| \leq b_1|x|$, $|f(x)| \leq b_2x^2$, this yields

$$\left| \mathbf{E}(f(S_{k-i_{p-2}-r_m-1})) \right| \leq s_n(b_1 \wedge b_2 s_n). \quad (37)$$

We deduce from the last bound, Lemmas 5 and 4, that

$$\begin{aligned} & \sum_{k=1}^n \sum_{(p-2)} \left| \mathbf{E}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}}) \mathbf{E}(f(S_{k-i_{p-2}-r_m-1})) \right| \\ & \leq (M_{p-1,n} + s_n^{p-1}) s_n(b_1 \wedge b_2 s_n) \leq M_{p,n}(b_1, b_2) + 2s_n^p(b_1 \wedge b_2 s_n). \end{aligned} \quad (38)$$

In order to evaluate the first term of the RHS of (36), we use the following lemma.

Lemma 8. *Let (X_n) be a centered sequence of (AG) r.v.'s. Suppose that (X_n) is uniformly bounded by M . Then, for any integer $p \geq 2$, there exists a positive constant C_p such that for any $f \in \mathcal{F}_1(b_1, b_2)$:*

$$\begin{aligned} & \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-1)}^{(m), r_m} \sup_{u \in [0,1]} \left| \text{cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} \right. \\ & \quad \left. \times f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}})) \right| \mathbf{1}_{i_1 \leq A} \\ & \leq C_p \{M_{p,n}(b_1, b_2) + s_n^p(b_1 \wedge b_2 s_n)\}. \end{aligned}$$

Now, the decomposition (21) yields:

$$\begin{aligned} & \sum_{(p-2)}^{(m), r_m} \left| \text{cov}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1})) \right| \mathbf{1}_{i_1 \leq A} \\ & \leq \sum_{(p-2)}^{(m), r_m} \sum_{i_{p-1}=i_{p-2}+r_m+1}^{k-1} \sup_{u \in [0,1]} \left| \text{cov}(X_k \cdots X_{k-i_{p-2}}, X_{k-i_{p-1}} \right. \\ & \quad \left. \times f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}})) \right| \mathbf{1}_{i_1 \leq A}. \end{aligned}$$

We take the sums over m : $0 \leq m \leq p-3$, r_m : $0 \leq r_m \leq k-1$ and over k : $1 \leq k \leq n$ in the two last inequalities, and we use Lemma 8 to obtain:

$$\begin{aligned} & \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-2)}^{(m), r_m} \left| \text{cov}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1})) \right| \mathbf{1}_{i_1 \leq A} \\ & \leq C_p \{M_{p,n}(b_1, b_2) + s_n^p(b_1 \wedge b_2 s_n)\}. \end{aligned} \quad (39)$$

We deduce from (36), (38) and (39) that

$$\begin{aligned} & \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-2)}^{(m), r_m} |J_2(i_1, \dots, i_{p-2})| \mathbf{1}_{i_1 \leq A} \\ & \leq C_p \{M_{p,n}(b_1, b_2) + s_n^p(b_1 \wedge b_2 s_n)\}. \end{aligned} \quad (40)$$

Evaluation of $\sum_{(p-2)} J_1(i_1, \dots, i_{p-2}) \mathbf{1}_{i_1 \leq A}$.

The two sums $\sum_{(p-2)}^{(m), r_m} \sum_{i_{p-1}=i_{p-2}+r_m+1}^{i_{p-2}+r_m}$ are taken over the subdivisions $0 \leq i_1 \leq \dots \leq i_{p-1} \leq k-1$ such that $r_m = \max_{1 \leq q \leq p-2} (i_q - i_{q-1})$ and that $i_{p-1} - i_{p-2} \leq r_m$, hence r_m is also the big hole of the subdivision $0 \leq i_1 \leq \dots \leq i_{p-1} \leq k-1$, that is, $r_m = \max_{1 \leq q \leq p-1} (i_q - i_{q-1})$.

We then deduce

$$\begin{aligned} & \sum_{(p-2)}^{(m), r_m} \mathbf{1}_{i_1 \leq A} \left| \mathbf{E} \left(X_k X_{k-i_1} \cdots X_{k-i_{p-2}} [f(S_{k-i_{p-2}-1}) - f(S_{k-i_{p-2}-r_m-1})] \right) \right| \\ & \leq \sum_{(p-1)}^{(m), r_m} \mathbf{1}_{i_1 \leq A} \sup_{u \in [0,1]} \left| \text{cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} \right. \\ & \quad \left. \times f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}})) \right| \\ & \quad + \sum_{(p-1)}^{(m), r_m} \left| \mathbf{E}(X_k \cdots X_{k-i_m}) \mathbf{E}(X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} \right. \\ & \quad \left. \times [f(S_{k-i_{p-1}}) - f(S_{k-i_{p-1}-1})]) \right| \mathbf{1}_{i_1 \leq A}. \end{aligned} \quad (41)$$

In order to give a bound for the sums over k , m and r_m of the RHS of (41), we argue as for the proof of (32): we use the inductive assumption to bound the second term of the RHS of (41) (as in (31)) and we use Lemma 8 instead

of Lemma 6 in order to evaluate the first term in the RHS of (41). We thus obtain:

$$\begin{aligned} & \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-2)}^{(m), r_m} |J_1(i_1, \dots, i_{p-2})| \mathbf{1}_{i_1 \leq A} \\ & \leq C_p \left\{ \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2) + M_{p,n}(b_1, b_2) + s_n^p(b_1 \wedge b_2 s_n) \right\}. \end{aligned} \quad (42)$$

Inequalities (35), (40), (42) prove that

$$\begin{aligned} & \sum_{k=1}^n \sum_{(p-2)} \mathbf{1}_{i_1 \leq A} |\mathbf{E} X_k X_{k-i_1} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})| \\ & \leq C_p \left\{ s_n^p(b_1 \wedge b_2 s_n) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2) + M_{p,n}(b_1, b_2) \right\}. \end{aligned}$$

The last inequality together with (32) proves that the quantity in (25) is bounded by the RHS of (11).

Let us now give a bound for (26).

Using again the fact $|\mathbf{E}(XY)| \leq |\text{cov}(X, Y)| + |\mathbf{E}X||\mathbf{E}Y|$, the second equality of (21), the inductive assumption and Lemma 8, we obtain:

$$\begin{aligned} & \sum_{k=1}^n \sum_{(p-2)} |\mathbf{E} X_k X_{k-i_1} \cdots X_{k-i_{p-2}} \Delta_{p-2,k}(f)| \mathbf{1}_{i_1 \leq A} \\ & \leq C_p \left\{ s_n^p(b_1 \wedge b_2 s_n) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2) + M_{p,n}(b_1, b_2) \right\}. \end{aligned}$$

This last bound together with (33) proves that the quantity in (26) is bounded by the RHS of (11), for all integer $p \geq 2$. The proof of Lemma 3 is thus finished.

We now prove Lemmas 6, 7, and 8.

P r o o f o f L e m m a 6. The (AG) property yields

$$\begin{aligned} & \left| \text{cov} \left(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} [f(S_{k-i_{p-2}-1}) - f(S_{k-i_{p-2}-r_m-1})] \right) \right| \\ & \leq C_p M^{p-2} b_1 r_m \sum_{l=r_m}^{k-1} |\text{cov}(X_1, X_l)|. \end{aligned}$$

Now, the definition of r_m , i.e., $r_m = i_{m+1} - i_m = \max_{1 \leq q \leq p-2} (i_q - i_{q-1})$, implies that $r_m \geq i_1 - i_0 = i_1$ and $j r_m \geq i_j$ for all integer j smaller than $p - 2$. Thus

$$\begin{aligned} I_{r_m} &:= \sum_{(p-2)}^{(m), r_m} M^{p-2} b_1 r_m \sum_{l=r_m}^{k-1} |\text{cov}(X_1, X_l)| \mathbf{1}_{i_1 > A} \\ &\leq M^{p-2} b_1 r_m^{p-2} \sum_{l=r_m}^{k-1} |\text{cov}(X_1, X_l)| \mathbf{1}_{r_m > A}. \end{aligned}$$

We obtain, taking the sum over r_m in the last inequality, using (28) and the trivial fact $x \mathbf{1}_{y \geq x} \leq x \wedge y$, for positive real numbers x, y ,

$$\sum_{r_m=0}^{k-1} I_{r_m} \leq \sum_{l=0}^{k-1} l^{p-2} M^{p-2} (b_1 \wedge b_2 M(l+1)) |\operatorname{cov}(X_{1+l}, X_1)|.$$

Finally, we take the sums over $m \in \{0, \dots, p-3\}$, $k \in \{1, \dots, n\}$ in the last inequality to get

$$\begin{aligned} & \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-2)}^{(m), r_m} \left| \operatorname{cov} \left(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} \right. \right. \\ & \quad \left. \left. \times [f(S_{k-i_{p-2}-1}) - f(S_{k-i_{p-2}-r_m-1})] \right) \right| \\ & \leq \sum_{k=1}^n \sum_{l=0}^{k-1} l^{p-2} M^{p-2} (b_1 \wedge b_2 M(l+1)) |\operatorname{cov}(X_{1+l}, X_1)|. \end{aligned} \quad (43)$$

We now use the trivial fact

$$\begin{aligned} \operatorname{cov}(X, YZ) &= \operatorname{cov}(XY, Z) + \mathbf{E}(XY) \mathbf{E}(Z) - \mathbf{E}(X) \operatorname{cov}(Y, Z) \\ &\quad - \mathbf{E}(X) \mathbf{E}(Y) \mathbf{E}(Z), \end{aligned}$$

to get

$$\begin{aligned} & |\operatorname{cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-r_m-1}))| \\ & \leq |\operatorname{cov}(X_k \cdots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1}))| \\ & \quad + |\mathbf{E}(X_k \cdots X_{k-i_{p-2}}) \mathbf{E}(f(S_{k-i_{p-2}-r_m-1}))| \\ & \quad + |\mathbf{E}(X_k \cdots X_{k-i_m}) \operatorname{cov}(X_{k-i_{m+1}} \cdots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1}))| \\ & \quad + |\mathbf{E}(X_k \cdots X_{k-i_m}) \mathbf{E}(X_{k-i_{m+1}} \cdots X_{k-i_{p-2}}) \mathbf{E}(f(S_{k-i_{p-2}-r_m-1}))| \\ & =: I_1(i_1, \dots, i_{p-2}) + I_2(i_1, \dots, i_{p-2}) + I_3(i_1, \dots, i_{p-2}) + I_4(i_1, \dots, i_{p-2}). \end{aligned}$$

Lemmas 5, 4 and inequality (37) yield

$$\begin{aligned} & \sum_{k=1}^n \sum_{(p-2)} I_2(i_1, \dots, i_{p-2}) + \sum_{k=1}^n \sum_{(p-2)} I_4(i_1, \dots, i_{p-2}) \\ & \leq C_p \sum_{l=0}^{k-1} l^{p-2} M^{p-2} (b_1 \wedge b_2 M(l+1)) |\operatorname{cov}(X_{1+l}, X_1)| \\ & \quad + C_p s_n^p (b_1 \wedge b_2 s_n). \end{aligned} \quad (44)$$

The (AG) property yields

$$|\operatorname{cov}(X_k \cdots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1}))| \leq C_p M^{p-2} b_1 \sum_{l=r_m}^{k-1} |\operatorname{cov}(X_1, X_l)|. \quad (45)$$

Hence

$$\sum_{k=1}^n \sum_{m=2}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-2)}^{(m), r_m} \mathbf{1}_{i_1 > A} I_1(i_1, \dots, i_{p-2}) \leq C_p M_{p,n} (b_1, b_2). \quad (46)$$

Using the (AG) property (as for (45)), Lemmas 5 and 4, we obtain

$$\sum_{k=1}^n \sum_{(p-2)} \mathbf{1}_{i_1 > A} I_3(i_1, \dots, i_{p-2}) \leq C_p \{M_{p,n}(b_1, b_2) + s_n^p(b_1 \wedge b_2 s_n)\}. \quad (47)$$

Finally the proof of Lemma 6 is complete by adding inequalities (43), (44), (46) and (47).

P r o o f o f L e m m a 7. The (AG) property and the fact that $r_m \geq i_1$ yield

$$\begin{aligned} & \left| \text{cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} \Delta_{p-2,k}(f)) \right| \mathbf{1}_{i_1 > A} \\ & \leq C_p M^{p-2} b_1 \sum_{l=r_m}^{k-1} |\text{cov}(X_1, X_l)| \mathbf{1}_{l > A}. \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{k=1}^n \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{(p-2)}^{(m), r_m} \left| \text{cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-2}} \Delta_{p-2,k}(f)) \right| \mathbf{1}_{i_1 > A} \\ & \leq C_p \sum_{k=1}^n \sum_{r_m=0}^{k-1} M^{p-2} b_1 \sum_{l=r_m}^{k-1} |\text{cov}(X_1, X_l)| \mathbf{1}_{l > A} \leq C_p M_{p,n}(b_1, b_2). \end{aligned}$$

P r o o f o f L e m m a 8. We write

$$\begin{aligned} f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}}) &= [f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}}) - f'(S_{k-i_{p-1}-1})] \\ &\quad + f'(S_{k-i_{p-1}-1}), \end{aligned}$$

so the covariance quantity of Lemma 8 is decomposed into two terms:

$$\left| \text{cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-1})) \right| \mathbf{1}_{i_1 \leq A}$$

and

$$\begin{aligned} & \left| \text{cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} \right. \\ & \quad \left. \times [f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}}) - f'(S_{k-i_{p-1}-1})] \right) \mathbf{1}_{i_1 \leq A} \end{aligned}$$

which are evaluated as the covariance quantities of Lemmas 6 and 7.

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