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STRONG LAWS OF LARGE NUMBERS FOR B -VALUED L_q -MIXINGALE SEQUENCES AND THE q -SMOOTHNESS OF BANACH SPACE¹⁾

Доказывается сходимостъ почти на верное L_q -миксингальной последовательности со значениями в банаховом пространстве. Этот результат используется для установления нового усиленного закона больших чисел для таких последовательностей. Получена характеристизация q -сглаживаемых банаховых пространств в терминах усиленных законов больших чисел.

Ключевые слова и фразы: сходимостъ почти на верное, усиленный закон больших чисел, L_q -миксингальная последовательность со значениями в банаховом пространстве, гладкость банаховых пространств.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $(B, \|\cdot\|)$ a Banach space. A strong measurable function with values in B defined on $(\Omega, \mathcal{F}, \mathbf{P})$ is called a B -valued random variable. Throughout this paper by convergence, expectation and conditional expectation we always mean strong convergence, Bochner expectation and Bochner conditional expectation in B . C will stand for a positive constant and it is allowed to be different in various places. We omit the notation a.s. for a.s. equality or a.s. inequality.

Definition 1.1. A Banach space B is said to be q -smoothable ($1 \leq q \leq 2$), if there exists a constant $C > 0$ such that

$$\mathbf{E} \left\| \sum_{i=1}^n D_i \right\|^q \leq C \sum_{i=1}^n \mathbf{E} \|D_i\|^q, \quad \forall n \geq 1,$$

for every B -valued martingale difference sequence $\{D_n, n \geq 1\}$ with finite q -th moments.

The concept of q -smoothable Banach spaces is very important in the study of probability theory in Banach spaces. Now we know the q -smoothableness of Banach space B is equivalent to certain inequality for B -valued martingales. In 1976, Hoffmann-Jørgensen and Pisier [1] established equivalence between the strong law of large numbers for B -valued martingales and smoothness of the Banach space B .

Let $\{X_n, n \geq 1\}$ be an L_q -integrable B -valued random variable sequence and $\{\mathcal{F}_n, -\infty < n < \infty\}$ a family of sub- σ -algebras of $\mathcal{F}$ such that $\{\mathcal{F}_n\}$ is increasing in n . $\|X\|_q$ represents $(\mathbf{E} \|X\|^q)^{1/q}$, where X is a B -valued L_q -integrable random variable.

Definition 1.2. $\{X_n, \mathcal{F}_n\}$ is called a B -valued L_q -mixingale sequence ($q \geq 1$) if for sequences of nonnegative constants $\{c_n\}$ and $\{\psi_m\}$, where $\psi_m \downarrow 0$ as $m \rightarrow \infty$ we have

- (i) $\|\mathbf{E}(X_n | \mathcal{F}_{n-m})\|_q \leq c_n \psi_m$ and
- (ii) $\|X_n - \mathbf{E}(X_n | \mathcal{F}_{n+m})\|_q \leq c_n \psi_{m+1}$ for all $n \geq 1$ and $m \geq 0$.

In 1975, McLeish [2] first introduced the concept of real valued L_2 -mixingale sequences and proved some convergence theorems and the strong laws of large numbers for such sequences. Afterwards he further studied the invariance principles for these mixingales [3], [4]. Mixingales have covered wide scope and have had useful applications in showing strong consistency of estimators and kernel regression function estimation. Many authors have greater and greater interest in the study of these mixingales. But so far as we know, only a few authors have just began to explore B -valued L_q -mixingale sequences. For references we refer to [5]–[10]. In this paper we prove a.s. convergence of B -valued L_q -mixingale sequences, the result is used to establish a strong law of large numbers for B -valued L_q -mixingale sequences which has not been set up previously in the literature. Especially we characterize a q -smoothable Banach space in terms of strong laws of large numbers for B -valued L_q -mixingale sequences. This partly extends Hoffmann-Jørgensen and Pisier's result [1].

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2. Main results.

Theorem 2.1. Let B be a q -smoothable Banach space ($1 < q \leq 2$) and $\{X_n, \mathcal{F}_n\}$ a B -valued L_q -mixingale sequence such that $\psi_n = O(n^{-1}(\ln n)^{-s})$ as $n \rightarrow \infty$ for a constant $s > 1$. Then there exists a positive constant C depending only on ψ_n such that

$$\mathbf{E} \left(\max_{1 \leq i \leq n} \|S_i\|^q \right) \leq C \sum_{i=1}^n c_i^q$$

for all $n \geq 1$, where $S_i = \sum_{j=1}^i X_j$ and $\{c_i\}$ is a sequence of nonnegative constants given in Definition 1.2.

P r o o f. For each fixed $i \geq 1$ we have

$$\sum_{k=-m}^n \left[\mathbf{E}(X_i | \mathcal{F}_{i+k}) - \mathbf{E}(X_i | \mathcal{F}_{i+k-1}) \right] = \mathbf{E}(X_i | \mathcal{F}_{i+n}) - \mathbf{E}(X_i | \mathcal{F}_{i-m-1})$$

for any $m, n \geq 1$.

Note that $\{\mathbf{E}(X_i | \mathcal{F}_{i+n}), \mathcal{F}_{i+n}, n \geq 1\}$ is a B -valued martingale and $\{\mathbf{E}(X_i | \mathcal{F}_{i-m-1}), \mathcal{F}_{i-m-1}, m \geq 1\}$ is a B -valued reverse martingale, so $\lim_{m \rightarrow \infty} \mathbf{E}(X_i | \mathcal{F}_{i+m}) = X_i$ and $\lim_{m \rightarrow \infty} \mathbf{E}(X_i | \mathcal{F}_{i-m-1}) = 0$. Consequently we have $X_i = \sum_{k=-\infty}^{\infty} [\mathbf{E}(X_i | \mathcal{F}_{i+k}) - \mathbf{E}(X_i | \mathcal{F}_{i+k-1})]$ for each $i \geq 1$.

Let $Y_{nk} = \sum_{i=1}^n [\mathbf{E}(X_i | \mathcal{F}_{i+k}) - \mathbf{E}(X_i | \mathcal{F}_{i+k-1})]$. It follows from a change of order of summation that $S_n = \sum_{i=1}^n X_i = \sum_{k=-\infty}^{\infty} Y_{nk}$. Setting $a_0 = a_1 = a_{-1} = 1$ and $a_n = a_{-n} = (n(\ln n)^s)^{-q/p}$ for $n \geq 2$ ($1/p + 1/q = 1$), it follows by the Cauchy-Schwarz inequality that

$$\|S_n\|^q = \left\| \sum_{k=-\infty}^{\infty} Y_{nk} \right\|^q \leq \left(\sum_{k=-\infty}^{\infty} a_k^{p/q} \right)^{q/p} \left(\sum_{k=-\infty}^{\infty} a_k^{-1} \|Y_{nk}\|^q \right).$$

Hence

$$\mathbf{E} \left(\sup_{1 \leq i \leq n} \|S_i\|^q \right) \leq \left(\sum_{k=-\infty}^{\infty} a_k^{p/q} \right)^{q/p} \left(\sum_{k=-\infty}^{\infty} a_k^{-1} \mathbf{E} \left(\sup_{1 \leq i \leq n} \|Y_{ik}\|^q \right) \right).$$

For each k , $\{Y_{ik}, \mathcal{F}_{i+k}, 1 \leq i \leq n\}$ is a B -valued martingale. So $\{\|Y_{ik}\|, \mathcal{F}_{i+k}, 1 \leq i \leq n\}$ is a nonnegative real-valued submartingale. By the Doob inequality and in view of q -smoothness of B we have

$$\mathbf{E} \left(\sup_{1 \leq i \leq n} \|Y_{ik}\|^q \right) \leq p^q \mathbf{E} \|Y_{nk}\|^q \leq C \sum_{i=1}^n \mathbf{E} \left\| \mathbf{E}(X_i | \mathcal{F}_{i+k}) - \mathbf{E}(X_i | \mathcal{F}_{i+k-1}) \right\|^q$$

for all $n \geq 1$. Now $\sum_{k=-\infty}^{\infty} a_k^{p/q} = 3 + 2 \sum_{k=2}^{\infty} (k(\ln k)^s)^{-1} < \infty$. So

$$\begin{aligned} \mathbf{E} \left(\sup_{1 \leq i \leq n} \|S_i\|^q \right) &\leq C \left[\sum_{k=1}^{\infty} a_k^{-1} \sum_{i=1}^n \mathbf{E} \left\| \mathbf{E}(X_i | \mathcal{F}_{i+k}) - \mathbf{E}(X_i | \mathcal{F}_{i+k-1}) \right\|^q \right. \\ &\quad \left. + \sum_{k=-\infty}^0 a_k^{-1} \sum_{i=1}^n \left\| \mathbf{E}(X_i | \mathcal{F}_{i+k}) - \mathbf{E}(X_i | \mathcal{F}_{i+k-1}) \right\|^q \right] \\ &=: C[T_1 + T_2]. \end{aligned}$$

By the C_r -inequality and the Jensen inequality we have

$$\begin{aligned} T_1 &= \sum_{k=1}^{\infty} a_k^{-1} \sum_{i=1}^n \mathbf{E} \left\| \mathbf{E}(X_i | \mathcal{F}_{i+k}) - \mathbf{E}(X_i | \mathcal{F}_{i+k-1}) \right\|^q \\ &\leq 2^{q-1} \sum_{k=1}^{\infty} a_k^{-1} \sum_{i=1}^n \left(\mathbf{E} \left\| X_i - \mathbf{E}(X_i | \mathcal{F}_{i+k}) \right\|^q + \mathbf{E} \left\| X_i - \mathbf{E}(X_i | \mathcal{F}_{i+k-1}) \right\|^q \right) \\ &\leq 2^{q-1} \sum_{k=1}^{\infty} a_k^{-1} \sum_{i=1}^n (c_i^q \psi_{k+1}^q + c_i^q \psi_k^q) \leq 2^q \sum_{k=1}^{\infty} a_k^{-1} \psi_k^q \sum_{i=1}^n c_i^q. \end{aligned}$$

Similarly we have

$$\begin{aligned} T_2 &= \sum_{k=-\infty}^0 a_k^{-1} \sum_{i=1}^n \left\| \mathbf{E}(X_i \mid \mathcal{F}_{i+k}) - \mathbf{E}(X_i \mid \mathcal{F}_{i+k-1}) \right\|^q \\ &= \sum_{k=0}^{\infty} a_{-k}^{-1} \sum_{i=1}^n \left\| \mathbf{E}(X_i \mid \mathcal{F}_{i-k}) - \mathbf{E}(X_i \mid \mathcal{F}_{i-k-1}) \right\|^q \\ &= 2^{q-1} \sum_{k=0}^{\infty} a_{-k}^{-1} \sum_{i=1}^n (c_i^q \psi_k^q + c_i^q \psi_{k+1}^q) \leq 2^q \sum_{k=0}^{\infty} a_{-k}^{-1} \psi_k^q \sum_{i=1}^n c_i^q. \end{aligned}$$

Therefore

$$T_1 + T_2 \leq 2^q \left(\psi_0^q + 2\psi_1^q + 2 \sum_{k=2}^{\infty} a_k^{-1} \psi_k^q \right) \sum_{i=1}^n c_i^q.$$

But

$$\sum_{k=2}^{\infty} a_k^{-1} \psi_k^q \leq C \sum_{k=2}^{\infty} \left(k(\ln k)^s \right)^{q/p} \left(k(\ln k)^s \right)^{-q} = C \sum_{k=2}^{\infty} \left(k(\ln k)^s \right)^{-1} < \infty.$$

Consequently $\mathbf{E}(\max_{1 \leq i \leq n} \|S_i\|^q) \leq C \sum_{i=1}^n c_i^q$ for all $n \geq 1$.

Let $S^* = \sup_{i \geq 1} \|S_i\|$. The following is an immediate consequence.

Corollary 2.1. *Let the conditions of Theorem 2.1 be fulfilled and $\sum_{i=1}^{\infty} c_i^q < \infty$. Then*

$$\mathbf{E}(S^*)^q < \infty.$$

Theorem 2.2. *Let the conditions of Theorem 2.1 be fulfilled and $\sum_{i=1}^{\infty} c_i^q < \infty$. Then there exists a B-valued L_q -integrable random variable S such that*

$$\lim_{n \rightarrow \infty} S_n = S, \quad \text{where} \quad S_n = \sum_{i=1}^n X_i.$$

P r o o f. For each $m' > m$ it follows by the Chebyshev inequality and Theorem 2.1 that $\mathbf{P}\{\max_{m < n \leq m'} \|S_n - S_m\| > \varepsilon\} \leq \varepsilon^{-q} C(\sum_{i=m}^{m'} c_i^q)$. So that

$$\mathbf{P}\left\{\max_{n > m} \|S_n - S_m\| > \varepsilon\right\} \leq \varepsilon^{-q} C\left(\sum_{i=m}^{\infty} c_i^q\right) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Therefore $\lim_{n \rightarrow \infty} S_n$ exists almost surely. We write $\lim_{n \rightarrow \infty} S_n = S$. By Fatou's lemma we have

$$\mathbf{E}\|S\|^q = \mathbf{E}\left(\lim_{n \rightarrow \infty} \|S_n\|^q\right) \leq \liminf_{n \rightarrow \infty} \mathbf{E}\|S_n\|^q \leq \mathbf{E}(S^*)^q < \infty.$$

Corollary 2.2. *Let $\{X_n, \mathcal{F}_n\}$ be a real L_2 -integrable φ -mixing sequence (definition cf., e.g., [11]) satisfying the following conditions:*

- 1) $\mathbf{E}X_n^2 = O(n^{-(1+\alpha)})$ as $n \rightarrow \infty$ for some $\alpha > 0$,
- 2) $\varphi(n) = O(n^{-2}(\ln n)^{-2s})$ as $n \rightarrow \infty$ for some $s > 0$.

Then there exists a real L_2 -integrable random variable S such that $\lim_{n \rightarrow \infty} S_n = S$, where $S_n = \sum_{i=1}^n X_i$.

P r o o f. Set $c_n = 2(\mathbf{E}X_n^2)^{1/2}$, $\psi_n = \varphi^{1/2}(n)$. Then $\{X_n, \mathcal{F}_n\}$ is a real L_2 -mixingale sequence, $\sum_{i=1}^{\infty} c_i^2 = 4 \sum_{i=1}^{\infty} \mathbf{E}X_i^2 \leq C \sum_{i=1}^{\infty} i^{-(1+\alpha)} < \infty$, $\psi_n = \varphi^{1/2}(n) = O(n^{-1}(\ln n)^{-s})$. By Theorem 2.2 we have Corollary 2.2.

Corollary 2.3. *Let B be a Banach space and $1 < q \leq 2$. Then the following two statements are equivalent.*

- 1) B is q -smoothable.
- 2) *For any B-valued L_q -mixingale sequence $\{X_n, \mathcal{F}_n\}$ satisfying $\sum_{i=1}^{\infty} c_i^q/i^q < \infty$ and $\psi_n = O(n^{-1}(\ln n)^{-s})$ as $n \rightarrow \infty$ for some $s > 0$, we have $n^{-1} \sum_{i=1}^n X_i \rightarrow 0$ a.s.*

Proof. 1) $\Rightarrow$ 2) From Theorem 2.2 and Kronecker's lemma it follows that 1) implies 2). 2) $\Rightarrow$ 1) Let $\{X_n, \mathcal{F}_n\}$ be any B -valued L_q -integrable martingale difference sequence satisfying $\sum_{i=1}^{\infty} i^{-q} \mathbf{E} \|X_i\|^q < \infty$. Set $c_n = (\mathbf{E} \|X_n\|^q)^{1/q}$, $\psi_0 = 1$, $\psi_m = 0$ for $m \geq 1$. Then

$$\left\| \mathbf{E}(X_n | \mathcal{F}_{n-m}) \right\|_q \leq c_n \psi_m, \quad \left\| X_n - \mathbf{E}(X_n | \mathcal{F}_{n-m}) \right\|_q \leq c_n \psi_{m+1}$$

for all $n \geq 1$ and $m \geq 0$ (i.e., $\{X_n, \mathcal{F}_n\}$ is a B -valued L_q -mixingale sequence); $\sum_{n=1}^{\infty} c_n^q / n^q = \sum_{n=1}^{\infty} \mathbf{E} \|X_n\|^q / n^q < \infty$. Obviously we have $\psi_n = O(n^{-1}(\ln n)^{-s})$ as $n \rightarrow \infty$ for some $s > 1$. By 2) it follows that $n^{-1} \sum_{i=1}^n X_i \rightarrow 0$. From Theorem 2.2 in [1] it follows that B is q -smoothable.

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ON MONOTONE EXTENSION OF LINEAR CONTINUOUS FUNCTIONALS

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