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Proof. 1) $\Rightarrow$ 2) From Theorem 2.2 and Kronecker's lemma it follows that 1) implies 2). 2) $\Rightarrow$ 1) Let $\{X_n, \mathcal{F}_n\}$ be any B -valued L_q -integrable martingale difference sequence satisfying $\sum_{i=1}^{\infty} i^{-q} \mathbf{E} \|X_i\|^q < \infty$. Set $c_n = (\mathbf{E} \|X_n\|^q)^{1/q}$, $\psi_0 = 1$, $\psi_m = 0$ for $m \geq 1$. Then

$$\left\| \mathbf{E}(X_n | \mathcal{F}_{n-m}) \right\|_q \leq c_n \psi_m, \quad \left\| X_n - \mathbf{E}(X_n | \mathcal{F}_{n-m}) \right\|_q \leq c_n \psi_{m+1}$$

for all $n \geq 1$ and $m \geq 0$ (i.e., $\{X_n, \mathcal{F}_n\}$ is a B -valued L_q -mixingale sequence); $\sum_{n=1}^{\infty} c_n^q / n^q = \sum_{n=1}^{\infty} \mathbf{E} \|X_n\|^q / n^q < \infty$. Obviously we have $\psi_n = O(n^{-1}(\ln n)^{-s})$ as $n \rightarrow \infty$ for some $s > 1$. By 2) it follows that $n^{-1} \sum_{i=1}^n X_i \rightarrow 0$. From Theorem 2.2 in [1] it follows that B is q -smoothable.

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ON MONOTONE EXTENSION OF LINEAR CONTINUOUS FUNCTIONALS

* Приводится несколько замечаний в связи с так называемой фундаментальной теоремой финансовой математики, в частности, с помощью непрерывного монотонного линейного функционала, не имеющего монотонного продолжения, построен простой контрпример к существованию мартингальной меры.

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Ключевые слова и фразы: доминирующее продолжение, лемма Фар-каша, мартингальная вероятностная мера.

Let us consider a standard L_p -space ($1 \leq p \leq \infty$) of real functions $X(\omega)$, $\omega \in \Omega$, on a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ with a *pre-order* $X \leq Y$: $X(\omega) \leq Y(\omega)$ for almost all $\omega \in \Omega$, and a linear continuous functional $x(X)$, $X \in L$, on a subspace $L \subseteq L_p$, which is *monotone*:

$$x(X_1) \leq x(X_2), \quad X_1, X_2 \in L: X_1 \leq X_2; \quad (1)$$

a question is, whether such functional has a monotone continuous extension on the whole space L_p . This question is of a particular interest in mathematical finance, in relation to existence of so-called martingale probability measure for a noncomplete market (see, for example, [1]).

Of course, a monotone version of the general Hahn-Banach extension theorem, with a dominant $p(X) = p \cdot \|X\|$: $x(X) \leq p(X)$, $X \in L$, say, would be of fundamental importance, but it does not hold true. The following counter-example related to a standard

one period market, concerns the *infinite* state space $\Omega = \{1, 2, \dots\}$ with elements $\omega = j$: $\mathbf{P}(\omega) > 0$, $j = 1, 2, \dots$, and the subspace $L \subseteq L_p$ generated by vectors

$$f_k = \lambda_k (\sin \alpha_k \cdot 1_{2k-1} - \cos \alpha_k \cdot 1_{2k}), \quad k = 1, 2, \dots,$$

where

$$1_j(\omega) = \begin{cases} 1 & \text{if } \omega = j, \\ 0 & \text{otherwise,} \end{cases}$$

and $\lambda_k > 0$, $0 < \alpha_k < \pi/2$ are some parameters (see the figure).

For the appropriate parameters $\lambda_k \rightarrow \infty$, let us consider a linear continuous functional $x(X)$, $X \in L$, defined as $x(f_k) \equiv 1$, $k = 1, 2, \dots$. To be more precise, let us consider just L_2 -space, with the *orthogonal* basis $e_k = \sin \alpha_k \cdot 1_{2k-1} - \cos \alpha_k \cdot 1_{2k}$, $k = 1, 2, \dots$, in the subspace $L \subseteq L_2$, taking the parameters λ_k such that

$$\sum_{k=1}^{\infty} \left| x \left(\frac{e_k}{\|e_k\|} \right) \right|^2 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2 \|e_k\|} < \infty$$

with

$$\|e_k\|^2 = (\sin \alpha_k)^2 (2k-1) + (\cos \alpha_k)^2 (2k) \geq \min \{ \mathbf{P}(2k-1), \mathbf{P}(2k) \}.$$

Obviously, monotone relation (1) on any subspace is equivalent to $x(X) \geq 0$, $X \in L$: $X \geq 0$. In our case, we have $X \in L$:

$$X(\omega) = \begin{cases} x_k \sin \alpha_k & \text{if } \omega = 2k-1 \\ -x_k \cos \alpha_k & \text{if } \omega = 2k \end{cases}, \quad k = 1, 2, \dots,$$

and there are no variables $X \geq 0$ in L , but $X = 0$; so our linear functional is *monotone* on the subspace L . And, if it would be an extension $x(X) \geq 0$, $X \in L_2$: $X \geq 0$, then, for $x(1_j) = c_j \geq 0$, $j = 1, 2, \dots$, it should be

$$\sum_{j=1}^{\infty} \left| x \left(\frac{1_j}{\|1_j\|} \right) \right|^2 = \sum_{j=1}^{\infty} \frac{c_j^2}{\mathbf{P}(j)} < \infty,$$

which would not hold true for sufficiently small parameters $\alpha_k \rightarrow 0$:

$$\sin \alpha_k \leq \frac{1}{\lambda_k}, \quad k = 1, 2, \dots,$$

say, since $c_{2k-1} \geq 1$, according to $x(f_k) = \lambda_k (\sin \alpha_k \cdot c_{2k-1} - \cos \alpha_k \cdot c_{2k}) = 1$, $k = 1, 2, \dots$.

Recall that for the *finite* state space Ω , the monotone extension always does exist. A very simple proof can be given with application of the known Farkas lemma¹⁾, say. Namely taking the state space as $\Omega = \{1, \dots, n\}$, with its elements $\omega = j$: $\mathbf{P}(\omega) > 0$, $j = 1, \dots, n$, and linear functionals

$$a_j(X) = X(j), \quad X \in L_p \quad (j = 1, \dots, n),$$

we see that conditions $a_j(X) \geq 0$, $j = 1, \dots, n$, for $X \in L$ imply $x(X) \geq 0$; hence, according to the Farkas lemma $x(X) = \sum_{j=1}^n c_j a_j(X)$, $X \in L$, with some coefficients $c_j \geq 0$, $j = 1, \dots, n$, and

$$x(X) = \sum_{j=1}^n c_j a_j(X) = \sum_{j=1}^n c_j X(j), \quad X \in L_p,$$

represents the required monotone extension.

Here, it seems worthy to demonstrate, how the martingale measure appears on the probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ considered, with respect to a filtration $\mathfrak{A}_t$, $t = 0, 1, \dots, T$, and certain monotone linear operators $x(X)$, $X \in L$, on the corresponding subspaces L :

$$L_p(\mathfrak{A}_{t-1}) \subseteq L \subseteq L_p(\mathfrak{A}_t), \quad t = 1, \dots, T,$$

where $L_p(\mathfrak{B})$ stays for the subspace of all $\mathfrak{B}$ -measurable variables $X \in L_p$; to be precise, the operators

$$x: X \rightarrow x(X) \in L_p(\mathfrak{A}_{t-1})$$

are such that $x(X) \geq 0$ for $X \in L$: $X \geq 0$ and $x(1_A X) = 1_A x(X)$, $A \in \mathfrak{A}_{t-1}$. The algebra $\mathfrak{A}_{t-1} = \mathfrak{B}$ of ω -sets is generated by some partition $\sum_{k=1}^m B_k = \Omega$, and, with respect to conditional probability measures $\mathbf{P}(A|B)$, $A \in \mathfrak{A}$, (with respect to $B = B_k$, $k = 1, \dots, m$), the operators $x(X)$, $X \in L$, can be treated as monotone *linear functionals*, with the *constant* values

$$x(X) = 1_B x(X) = x(1_B X), \quad \omega \in B.$$

Hence, we can apply their *monotone* extensions on the corresponding *subspaces* $L_p(\mathfrak{A}_t)$. These extensions lead to conditional probability measures $\mathbf{P}^0(A|B)$, $A \in \mathfrak{A}_t$, with their conditional expectations defined as

$$\mathbf{E}^0(1_B X | B) = R x(1_B X), \quad X \in L_p(\mathfrak{A}_t), \tag{2}$$

with the «discount factor» $R = [x(1)]^{-1}$. In this way, for $t = 1$ (with the trivial σ -algebra $\mathfrak{A}_0$), we obtain the certain probability measure $\mathbf{P}^0(A)$, $A \in \mathfrak{A}_1$. And consequently, for all t (up to $t = T$), with $\mathbf{P}^0(A)$, $A \in \mathfrak{A}_{t-1}$ (on the σ -algebra $\mathfrak{A}_{t-1} = \mathfrak{B}$) in hands, we can extend this probability measure to

$$\mathbf{P}^0(A) \stackrel{\text{def}}{=} \sum_{k=1}^m \mathbf{P}^0(A | B_k) \mathbf{P}^0(B_k), \quad A \in \mathfrak{A}_t,$$

for $t = T$ (with the algebra $\mathfrak{A} = \mathfrak{A}_T$), it represents the *martingale measure*, which is characterized by equations (2) — cf. [3].

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¹⁾ This lemma of Farkas (1902) is of a common use in Convex Analysis — see, for example, [4].