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ZAITSEV A. YU.*

MULTIDIMENSIONAL VERSION OF A RESULT OF
SAKHANENKO IN THE INVARIANCE PRINCIPLE FOR
VECTORS WITH FINITE EXPONENTIAL MOMENTS. II¹⁾

Получен многомерный вариант результата Саханенко [18] о гауссовской аппроксимации последовательности сумм независимых случайных векторов с конечными экспоненциальными моментами.

Ключевые слова и фразы: многомерный принцип инвариантности, сильная аппроксимация, суммы независимых разнораспределенных случайных векторов.

6. Dyadic scheme. Conditions B and their consequences. In the sequel we shall always suppose that the dyadic scheme described in Part I satisfies the following additional assumptions B_1 – B_4 which we call for the sake of brevity Condition B (cf. Condition B_2 in Section 1).

B_1 . The vectors $\xi_1, \dots, \xi_{2^N}$, $X_1, \dots, X_{2^N}$ and $Y_1, \dots, Y_{2^N}$ (involved in the dyadic scheme) have the same zero means and the same covariance operators as certain vectors $\tilde{\xi}_1, \dots, \tilde{\xi}_{2^N}$:

$$\begin{aligned} E\tilde{\xi}_k &= E\xi_k = EX_k = EY_k = 0, \\ \text{cov } \tilde{\xi}_k &= \text{cov } \xi_k = \text{cov } X_k = \text{cov } Y_k \stackrel{\text{def}}{=} B_k, \end{aligned} \quad k = 1, \dots, 2^N. \quad (6.1)$$

Moreover, let $\tilde{\eta}_k$ be auxiliary random vectors which have the same moments of the first three orders as the vectors $\tilde{\xi}_k$, $k = 1, \dots, 2^N$.

B_2 . The covariance operators B_k for all $u \in \mathbb{R}^d$ and for all $k = 1, \dots, 2^N$ satisfy the condition

$$C_1^2 \|u\|^2 \leq \langle B_k u, u \rangle = \|B_k^{1/2} u\|^2 \leq C_2^2 \|u\|^2 \quad (6.2)$$

with some absolute constants C_1 and C_2 .

B_3 . The first d coordinates of the vector $U_{n,k} = A_{n,k} \tilde{U}_{n,k}$, $1 \leq k \leq 2^{N-n}$, $1 \leq n \leq N$, are defined by the relation $\bar{P}_d U_{n,k} = J_{n,k} U_{n,k}$, where

$$J_{n,k} = 2^{n/2} D_{n,k}^{-1/2}, \quad D_{n,k} = B_{(k-1)2^{n+1}} + \dots + B_{k2^n} = \text{cov } U_{n,k}. \quad (6.3)$$

*С.-Петербургское отделение Математического института им. В. А. Стеклова РАН, ул. Фонтанка, 27, 191011 С.-Петербург, Россия; e-mail: zaitsev@pdmi.ras.ru

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B₄. The last d coordinates of the vector $\mathbf{U}_{n,k} = \mathbf{A}_{n,k} \tilde{\mathbf{U}}_{n,k}$, $1 \leq k \leq 2^{N-n}$, $1 \leq n \leq N$, are defined by the relations

$$\mathbb{P}_j \mathbf{U}_{n,k} = \mathbb{P}_{j-d} \tilde{\mathcal{U}}_{n,k}, \quad j = d+1, \dots, 2d, \quad (6.4)$$

where

$$\begin{aligned} \tilde{\mathcal{U}}_{n,k} &= \tilde{\mathbf{J}}_{n,k} \tilde{\mathbf{U}}_{n,k}, \quad \tilde{\mathbf{J}}_{n,k} = 2^{n/2} \tilde{\mathbf{D}}_{n,k}^{-1/2}, \quad \tilde{\mathbf{D}}_{n,k} = \text{cov } \tilde{\mathbf{U}}_{n,k}, \\ \tilde{\mathbf{U}}_{n,k} &= 2\mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} U_{n-1,2k-1} - 2\mathbb{D}_{n-1,2k-1} \mathbb{D}_{n,k}^{-1} U_{n-1,2k}. \end{aligned} \quad (6.5)$$

Remark 6.1. It is convenient for us to suppose that C_1, C_2 are absolute constants. In this case all the quantities depending only on C_1, C_2 can be also considered as absolute constants.

Remark 6.2. The vectors $\tilde{\xi}_1, \dots, \tilde{\xi}_{2^N}$ play below the role of vectors the independent copies of which are to be constructed provided that we already have the independent Gaussian vectors $Y_1, \dots, Y_{2^N}$. However, in general case, one should apply the dyadic scheme several times and with distributions of ξ_k not necessarily coinciding with those of $\tilde{\xi}_k$, see Conditions **C_Ψ**(p) below. Moreover, the vectors $Y_1, \dots, Y_{2^N}$ involved in the dyadic scheme may be non-Gaussian, see Condition **D_Ψ**. The vectors $\tilde{\eta}_1, \dots, \tilde{\eta}_{2^N}$ will appear not always in the construction. They will be sometimes used as intermediate bounded vectors using which one can construct some smooth and bounded distributions denoted by $\Psi(\mathcal{L}(\tilde{\eta}_1)), \dots, \Psi(\mathcal{L}(\tilde{\eta}_{2^N}))$ and having the required moments of the first three orders (see (7.14) and Lemmas 7.2 and 11.1 (Part III)). In Sections 9 and 10, the vectors with distributions $\Psi(\mathcal{L}(\tilde{\eta}_1)), \dots, \Psi(\mathcal{L}(\tilde{\eta}_{2^N}))$ will be used as $Y_1, \dots, Y_{2^N}$.

Remark 6.3. The condition (6.1) automatically implies the coincidence of the corresponding first and second moments of the vectors $\mathbf{U} = \{U_{n,k}, \tilde{\mathbf{U}}_{n,k}, \mathbf{U}_{n,k}, \mathcal{U}_{n,k}, \mathcal{U}\}$ and $\mathbf{V} = \{V_{n,k}, \tilde{\mathbf{V}}_{n,k}, \mathbf{V}_{n,k}, \mathcal{V}_{n,k}, \mathcal{V}\}$, respectively, since the vectors $\mathbf{U}$ can be restored from independent vectors $X_1, \dots, X_{2^N}$ by the same linear procedure which is used for reconstruction of the vectors $\mathbf{V}$ from independent vectors $Y_1, \dots, Y_{2^N}$. From (6.1) it follows, in particular, that the mathematical expectations of all vectors $\mathbf{U}$ and $\mathbf{V}$ are equal to zero.

Remark 6.4. To define the operators $\mathbf{A}_{n,k}$, one would indicate a concrete way of constructing the coordinates of the vectors $\mathbf{U}_{n,k}^* = \mathbf{A}_{n,k} \tilde{\mathbf{U}}_{n,k}^*$. Instead of this, according to Lemma 5.1 and in order to avoid the repetition of almost identical formulas (with asterisks and without them), we defined the operators $\mathbf{A}_{n,k}$ by the coordinates $\mathbb{P}_j \mathbf{U}_{n,k}$, $j = 1, \dots, 2d$, of the vectors $\mathbf{U}_{n,k}$ (cf. (5.3)–(5.5), (5.10) and (5.24)). By (5.10) and (6.4), we have

$$\mathbf{U}_{n,k} = \mathbf{A}_{n,k} \tilde{\mathbf{U}}_{n,k} = (\mathcal{U}_{n,k}, \tilde{\mathcal{U}}_{n,k}) \in \mathbf{R}^{2d} \quad (6.6)$$

and the linear operators $\mathbf{A}_{n,k}$ is completely described by (5.10), (6.3) and (6.5). Note also that, according to (5.8)–(5.10), (5.24), and (6.3)–(6.5),

$$\mathbf{V}_{n,k} = \mathbf{A}_{n,k} \tilde{\mathbf{V}}_{n,k} = (\mathcal{V}_{n,k}, \tilde{\mathcal{V}}_{n,k}) \in \mathbf{R}^{2d}, \quad (6.7)$$

where

$$\begin{aligned}
 \tilde{\mathbf{V}}_{n,k} &= (V_{n-1,2k-1}, V_{n-1,2k}), \\
 \mathcal{V}_{n,k} &= \mathbb{J}_{n,k} V_{n,k} = \bar{\mathbb{P}}_d \mathbf{V}_{n,k}, \quad \tilde{\mathcal{V}}_{n,k} = \tilde{\mathbb{J}}_{n,k} \tilde{\mathbf{V}}_{n,k}, \\
 \mathbb{P}_j \mathbf{V}_{n,k} &= \mathbb{P}_{j-d} \tilde{\mathcal{V}}_{n,k}, \quad j = d+1, \dots, 2d, \\
 \tilde{\mathbf{V}}_{n,k} &= 2\mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} V_{n-1,2k-1} - 2\mathbb{D}_{n-1,2k-1} \mathbb{D}_{n,k}^{-1} V_{n-1,2k}.
 \end{aligned} \tag{6.8}$$

R e m a r k 6.5. We could take $\mathbb{A}_{n,k} = \mathbb{A}^\pm$, where $\mathbb{A}^\pm: \mathbf{R}^{2d} \mapsto \mathbf{R}^{2d}$ are defined as follows: for $x = (x_1, x_2, \dots, x_{2d}) \in \mathbf{R}^{2d}$,

$$\begin{aligned}
 \mathbb{P}_j \mathbb{A}^\pm x &= x_j + x_{d+j}, \quad j = 1, \dots, d, \\
 \mathbb{P}_j \mathbb{A}^\pm x &= x_j - x_{d+j}, \quad j = d+1, \dots, 2d.
 \end{aligned} \tag{6.9}$$

However we have to consider here, generally speaking, other operators, since Lemmas 3.2 and 3.3 can be applied to conditional distributions described in Section 5 only when the operators $\mathbb{A}_{n,k}$ possess some additional properties. Namely, it is necessary that $\mathcal{U}^{(j)}$ and $\mathbf{U}_{n,k}^{(j)}$ from (5.15) and (5.20) do not correlate with $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(j-1)}$ and $\mathbf{U}_{n,k}^{(1)}, \dots, \mathbf{U}_{n,k}^{(j-1)}$ respectively (see (6.14) and (6.29) below). Note that if the vectors $\xi_1, \dots, \xi_{2N}; Y_1, \dots, Y_{2N}$ have covariance operators $\mathbb{I}_d$, the conditions just mentioned are satisfied for the operators $\mathbb{A}_{n,k} = \mathbb{A}^\pm$. It is not surprising, that in this case these operators was used earlier (see [12], [9], [29]). It is clear that then $\tilde{U}_{n,k} = U_{n,k}^-$ (cf. (5.30), (6.5)).

R e m a r k 6.6. According to (5.17), (5.18), (5.21) and (6.6), under Condition **B** we have

$$\bar{\mathbb{P}}_d \mathbf{U}_{N,1} = \mathcal{U} \quad \text{and} \quad \mathbf{U}_{N,1}^{(j)} = \mathcal{U}^{(j)}, \quad \mathbf{U}_{N,1}^j = \mathcal{U}^j, \quad j = 1, \dots, d. \tag{6.10}$$

Consider some additional properties of the operators $\mathbb{J}_{n,k}, \tilde{\mathbb{J}}_{n,k}$ and $\mathbb{A}_{n,k}$ which follow from Condition **B**.

It is well known that if $\mathbb{D} = \text{cov } \xi$ for some random vector ξ and $\mathbb{A}: \mathbf{R}^d \rightarrow \mathbf{R}^m$ is some linear operator, then

$$\text{cov } \mathbb{A}\xi = \mathbb{A}\mathbb{D}\mathbb{A}^*, \tag{6.11}$$

where $\mathbb{A}^*: \mathbf{R}^m \rightarrow \mathbf{R}^d$ is the adjoint operator for $\mathbb{A}$.

From (5.19), (6.3), (6.5) and (6.11) it is easy to deduce that

$$\text{cov } \mathcal{U}_{n,k} = \text{cov}(\mathbb{J}_{n,k} U_{n,k}) = \text{cov } \tilde{\mathcal{U}}_{n,k} = \text{cov}(\tilde{\mathbb{J}}_{n,k} \tilde{U}_{n,k}) = 2^n \mathbb{I}_d, \tag{6.12}$$

for $1 \leq k \leq 2^{N-n}$, $1 \leq n \leq N$. The equality (6.12) implies, in particular, that the coordinates $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \dots, \mathcal{U}^{(d)}$ of the vector $\mathcal{U} = \mathbb{J}_{N,1} \mathbf{U}_{N,1}$ are noncorrelated and

$$\mathbb{D} \mathcal{U}^{(1)} = \mathbb{D} \mathcal{U}^{(2)} = \dots = \mathbb{D} \mathcal{U}^{(d)} = 2^N. \tag{6.13}$$

Moreover,

$$\text{cov } \mathcal{U}^{(j)} = 2^N \mathbb{I}_j, \quad j = 1, \dots, d, \quad (6.14)$$

see (5.17), (5.18) and (6.12).

Let Ξ be an arbitrary subset of the set $\{1, 2, \dots, 2^N\}$. Write

$$U(\Xi) = \sum_{k \in \Xi} X_k, \quad \mathbb{D}(\Xi) = \sum_{k \in \Xi} \mathbb{B}_k = \text{cov } U(\Xi). \quad (6.15)$$

The relations (6.2) and (6.15) imply that

$$|\Xi| C_1^2 \|u\|^2 \leq \langle \mathbb{D}(\Xi) u, u \rangle \leq |\Xi| C_2^2 \|u\|^2, \quad \text{for all } u \in \mathbf{R}^d, \quad (6.16)$$

where $|\Xi|$ is the number of elements of the set Ξ . In particular, applying (6.16) to $u = e_j$, we get

$$|\Xi| C_1^2 \leq \langle \mathbb{D}(\Xi) e_j, e_j \rangle = \mathbf{E} \langle U(\Xi), e_j \rangle^2 \leq |\Xi| C_2^2, \quad j = 1, \dots, d. \quad (6.17)$$

According to (5.24), (6.3) and (6.15), we have

$$U_{n,k} = U(\Xi), \quad \mathbb{D}_{n,k} = \mathbb{D}(\Xi) \quad \text{for } \Xi = \{(k-1)2^n + 1, \dots, k2^n\}. \quad (6.18)$$

The relations (6.15), (6.16) and (6.18) imply that

$$2^n C_1^2 \|u\|^2 \leq \langle \mathbb{D}_{n,k} u, u \rangle = \|\mathbb{D}_{n,k}^{1/2} u\|^2 \leq 2^n C_2^2 \|u\|^2, \quad (6.19)$$

for all $u \in \mathbf{R}^d$. Putting $u = \mathbb{D}_{n,k}^{-1/2} v$, we see that

$$2^{-n} C_2^{-2} \|v\|^2 \leq \langle \mathbb{D}_{n,k}^{-1} v, v \rangle = \|\mathbb{D}_{n,k}^{-1/2} v\|^2 \leq 2^{-n} C_1^{-2} \|v\|^2, \quad (6.20)$$

for all $v \in \mathbf{R}^d$. Therefore,

$$\|\mathbb{D}_{n,k}^{1/2}\|^2 = \|\mathbb{D}_{n,k}\| \leq 2^n C_2^2 \quad \text{and} \quad \|\mathbb{D}_{n,k}^{-1/2}\|^2 = \|\mathbb{D}_{n,k}^{-1}\| \leq 2^{-n} C_1^{-2}. \quad (6.21)$$

From (6.3), (6.19) and (6.20) it follows that

$$\begin{aligned} C_2^{-2} \|v\|^2 &\leq \|\mathbb{J}_{n,k} v\|^2 \leq C_1^{-2} \|v\|^2, \\ C_1^2 \|v\|^2 &\leq \|\mathbb{J}_{n,k}^{-1} v\|^2 \leq C_2^2 \|v\|^2, \end{aligned} \quad v \in \mathbf{R}^d, \quad (6.22)$$

and, hence,

$$\|\mathbb{J}_{n,k}\| \leq C_1^{-1} \quad \text{and} \quad \|\mathbb{J}_{n,k}^{-1}\| \leq C_2. \quad (6.23)$$

Furthermore,

$$\text{cov } \mathbf{U}_{n,k} = 2^n \mathbb{I}_{2d}. \quad (6.24)$$

By virtue of (6.6) and (6.12), in order to verify (6.24) it suffices to show that the coordinates of $\mathcal{U}_{n,k}$ do not correlate with those of $\widetilde{\mathcal{U}}_{n,k}$. Indeed, according to Remark 6.3, these coordinates have zero means. Moreover, using (5.10), (5.24), (6.3), (6.5), the symmetricity of the opera-

tors $\mathbb{J}_{n,k}$, $\tilde{\mathbb{J}}_{n,k}$, $\mathbb{D}_{n,k}$, $\mathbb{D}_{n-1,2k-1}$, $\mathbb{D}_{n-1,2k}$ and the fact that $U_{n-1,2k}$ is independent of $U_{n-1,2k-1}$, we have, for all $u, v \in \mathbf{R}^d$:

$$\begin{aligned}
 \frac{1}{2} \mathbf{E} \langle \tilde{\mathcal{U}}_{n,k}, u \rangle \langle \mathcal{U}_{n,k}, v \rangle &= \frac{1}{2} \mathbf{E} \langle \tilde{U}_{n,k}, \tilde{\mathbb{J}}_{n,k} u \rangle \langle U_{n,k}, \mathbb{J}_{n,k} v \rangle \\
 &= \mathbf{E} \langle \mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} U_{n-1,2k-1}, \tilde{\mathbb{J}}_{n,k} u \rangle \langle U_{n-1,2k-1}, \mathbb{J}_{n,k} v \rangle \\
 &\quad - \mathbf{E} \langle \mathbb{D}_{n-1,2k-1} \mathbb{D}_{n,k}^{-1} U_{n-1,2k}, \tilde{\mathbb{J}}_{n,k} u \rangle \langle U_{n-1,2k}, \mathbb{J}_{n,k} v \rangle \\
 &= \mathbf{E} \langle U_{n-1,2k-1}, \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k} \tilde{\mathbb{J}}_{n,k} u \rangle \langle U_{n-1,2k-1}, \mathbb{J}_{n,k} v \rangle \\
 &\quad - \mathbf{E} \langle U_{n-1,2k}, \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k-1} \tilde{\mathbb{J}}_{n,k} u \rangle \langle U_{n-1,2k}, \mathbb{J}_{n,k} v \rangle \\
 &= \mathbf{E} \langle \mathbb{D}_{n-1,2k-1} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k} \tilde{\mathbb{J}}_{n,k} u, \mathbb{J}_{n,k} v \rangle \\
 &\quad - \mathbf{E} \langle \mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k-1} \tilde{\mathbb{J}}_{n,k} u, \mathbb{J}_{n,k} v \rangle = 0,
 \end{aligned}$$

since

$$\mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k-1} = \mathbb{D}_{n-1,2k-1} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k}. \quad (6.25)$$

The relation (6.25) immediately follows from the equality

$$\mathbb{D}_{n,k} = \mathbb{D}_{n-1,2k-1} + \mathbb{D}_{n-1,2k} \quad (6.26)$$

implying that

$$\begin{aligned}
 (\mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k-1})^{-1} &= \mathbb{D}_{n-1,2k-1}^{-1} \mathbb{D}_{n,k} \mathbb{D}_{n-1,2k}^{-1} = \mathbb{D}_{n-1,2k-1}^{-1} + \mathbb{D}_{n-1,2k}^{-1} \\
 &= \mathbb{D}_{n-1,2k}^{-1} \mathbb{D}_{n,k} \mathbb{D}_{n-1,2k-1}^{-1} = (\mathbb{D}_{n-1,2k-1} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k})^{-1}.
 \end{aligned} \quad (6.27)$$

The relations (6.6) and (6.24) automatically guarantee that the operator $\mathbb{A}_{n,k}$ is nondegenerate.

Thus, the coordinates $\mathbf{U}_{n,k}^{(1)}, \mathbf{U}_{n,k}^{(2)}, \dots, \mathbf{U}_{n,k}^{(2d)}$ of the vector $\mathbf{U}_{n,k}$ are noncorrelated and

$$\mathbf{D} \mathbf{U}_{n,k}^{(1)} = \mathbf{D} \mathbf{U}_{n,k}^{(2)} = \dots = \mathbf{D} \mathbf{U}_{n,k}^{(2d)} = 2^n. \quad (6.28)$$

Moreover (see (5.21) and (6.24)),

$$\text{cov } \mathbf{U}_{n,k}^j = 2^n \mathbb{I}_j, \quad j = 1, \dots, 2d. \quad (6.29)$$

By (6.5), (6.11), (6.25), (6.26) and by the independence of $U_{n-1,2k-1}$ and $U_{n-1,2k}$,

$$\begin{aligned}
 \tilde{\mathbb{D}}_{n,k} &= 4 \mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k-1} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k} \\
 &\quad + 4 \mathbb{D}_{n-1,2k-1} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k-1} \\
 &= 4 \mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} \mathbb{D}_{n-1,2k-1},
 \end{aligned} \quad (6.30)$$

and, hence,

$$\tilde{\mathbb{D}}_{n,k}^{-1} = \frac{1}{4} \mathbb{D}_{n-1,2k-1}^{-1} \mathbb{D}_{n,k} \mathbb{D}_{n-1,2k}^{-1}. \quad (6.31)$$

The relations (6.5), (6.21), (6.30) and (6.31) imply that

$$\begin{aligned} 2^n \|\tilde{\mathbb{J}}_{n,k}^{-1}\|^2 &= \|\tilde{\mathbb{D}}_{n,k}^{1/2}\|^2 = \|\tilde{\mathbb{D}}_{n,k}\| \leq 4 \|\mathbb{D}_{n-1,2k}\| \cdot \|\mathbb{D}_{n,k}^{-1}\| \cdot \|\mathbb{D}_{n-1,2k-1}\| \\ &\leq 2^n C_2^4 C_1^{-2} \end{aligned} \quad (6.32)$$

and, similarly,

$$2^{-n} \|\tilde{\mathbb{J}}_{n,k}\|^2 = \|\tilde{\mathbb{D}}_{n,k}^{-1/2}\|^2 = \|\tilde{\mathbb{D}}_{n,k}^{-1}\| \leq 2^{-n} C_2^2 C_1^{-4}. \quad (6.33)$$

Note that the bounds (6.32) and (6.33) can be refined with the help of (6.20), (6.27), (6.30) and (6.31).

Define $\mathbb{M}_{n,k} = \text{cov } \tilde{\mathbb{U}}_{n,k}$ and write an arbitrary vector $u = (u_1, \dots, u_{2d}) \in \mathbf{R}^{2d}$ in the form $u = (u^{(1)}, u^{(2)})$, where $u^{(1)} = (u_1, \dots, u_d)$, $u^{(2)} = (u_{d+1}, \dots, u_{2d}) \in \mathbf{R}^d$. Using (5.24), (6.3), we see that

$$\langle \mathbb{M}_{n,k} u, u \rangle = \langle \mathbb{D}_{n-1,2k-1} u^{(1)}, u^{(1)} \rangle + \langle \mathbb{D}_{n-1,2k} u^{(2)}, u^{(2)} \rangle. \quad (6.34)$$

Obviously, $\|u\|^2 = \|u^{(1)}\|^2 + \|u^{(2)}\|^2$. Therefore, from (6.19) and (6.34) it follows that

$$2^{n-1} C_1^2 \|u\|^2 \leq \langle \mathbb{M}_{n,k} u, u \rangle = \|\mathbb{M}_{n,k}^{1/2} u\|^2 \leq 2^{n-1} C_2^2 \|u\|^2. \quad (6.35)$$

Furthermore, according to (6.6) and (6.24), for all $v \in \mathbf{R}^{2d}$ we have

$$2^n \|v\|^2 = \mathbf{E} \langle \mathbb{A}_{n,k} \tilde{\mathbb{U}}_{n,k}, v \rangle^2 = \mathbf{E} \langle \tilde{\mathbb{U}}_{n,k}, \mathbb{A}_{n,k}^* v \rangle^2 = \langle \mathbb{M}_{n,k} \mathbb{A}_{n,k}^* v, \mathbb{A}_{n,k}^* v \rangle. \quad (6.36)$$

From (6.35) and (6.36) it follows that

$$\begin{aligned} c_{15}^{-2} \|v\|^2 &\leq \|\mathbb{A}_{n,k}^* v\|^2 \leq c_{16}^{-2} \|v\|^2, \\ c_{16}^2 \|v\|^2 &\leq \|(\mathbb{A}_{n,k}^*)^{-1} v\|^2 \leq c_{15}^2 \|v\|^2, \end{aligned} \quad v \in \mathbf{R}^{2d}, \quad (6.37)$$

with $c_{16} = C_1/\sqrt{2}$, $c_{15} = C_2/\sqrt{2}$. By the well-known properties of adjoint operators,

$$\|\mathbb{A}_{n,k}^*\| = \sup_{\|u\|=1, \|v\|=1} \langle \mathbb{A}_{n,k} u, v \rangle = \|\mathbb{A}_{n,k}\| \quad \text{and} \quad (\mathbb{A}_{n,k}^*)^{-1} = (\mathbb{A}_{n,k}^{-1})^*. \quad (6.38)$$

Using (6.38), we see that (6.37) implies

$$\|\mathbb{A}_{n,k}\| \leq c_{16}^{-1} \quad \text{and} \quad \|\mathbb{A}_{n,k}^{-1}\| \leq c_{15} \quad (6.39)$$

or, equivalently,

$$\begin{aligned} c_{15}^{-1} \|v\| &\leq \|\mathbb{A}_{n,k} v\| \leq c_{16}^{-1} \|v\|, \\ c_{16} \|v\| &\leq \|\mathbb{A}_{n,k}^{-1} v\| \leq c_{15} \|v\|, \end{aligned} \quad v \in \mathbf{R}^{2d}. \quad (6.40)$$

Lemma 6.1. *Let $1 \leq m = (2s+1)2^r \leq 2^N$, where s, r are nonnegative integers. Assume that the dyadic scheme satisfies Condition B. Then there exist an absolute constant $c_{17} \geq 1$ such that the following representations are valid: $S_m = \mathbb{K}U_{N,1} + \sum_{n=r+1}^N \mathbb{K}_n \tilde{\mathbb{U}}_{n,l_{n,m}}$,*

$$U_{n,l_{n,m}} = \mathbb{L}U_{N,1} + \sum_{s=n+1}^N \mathbb{L}_s \tilde{\mathbb{U}}_{s,l_{s,m}}, \quad n = r, \dots, N, \quad (6.41)$$

where linear operators $\mathbb{K}, \mathbb{K}_n: \mathbf{R}^d \rightarrow \mathbf{R}^d$ depend on m and satisfy

$$\max \{ \|\mathbb{K}\|, \|\mathbb{K}_{r+1}\|, \dots, \|\mathbb{K}_N\| \} \leq c_{17}, \quad (6.42)$$

linear operators $\mathbb{L}, \mathbb{L}_s: \mathbf{R}^d \rightarrow \mathbf{R}^d$ depend on m and n and satisfy

$$\|\mathbb{L}\| \leq c_{17} 2^{n-N}, \quad \|\mathbb{L}_s\| \leq c_{17} 2^{n-s}, \quad s = n+1, \dots, N, \quad (6.43)$$

and the integers $l_{n,m}$ are defined by (5.32).

Proof. Below we shall change the subscript « $n, l_{n,m}$ » by « n », $0 \leq n \leq N$. Then the equality (5.33) may be rewritten in the form

$$S_m = \sum_{n=r}^N \beta_n U_n = \sum_{n=r+1}^{N+1} \beta_{n-1} U_{n-1}, \quad \text{where } |\beta_{n-1}| \leq \frac{3}{2}. \quad (6.44)$$

The relations (5.24), (6.5) and (6.26) imply that, for $1 \leq n \leq N$, $1 \leq k \leq 2^{N-n}$,

$$\begin{aligned} U_{n-1,2k-1} &= \frac{1}{2} \tilde{U}_{n,k} + \mathbb{D}_{n-1,2k-1} \mathbb{D}_{n,k}^{-1} U_{n,k}, \\ U_{n-1,2k} &= -\frac{1}{2} \tilde{U}_{n,k} + \mathbb{D}_{n-1,2k} \mathbb{D}_{n,k}^{-1} U_{n,k}. \end{aligned} \quad (6.45)$$

Thus, taking into account Remark 5.2, we see that, by (6.45),

$$U_{n-1} = \frac{\kappa_n}{2} \tilde{U}_n + \mathbb{D}_{n-1} \mathbb{D}_n^{-1} U_n, \quad n = r+1, \dots, N, \quad (6.46)$$

where the factors $\kappa_n = \kappa_n(m)$ can be equal either to 1 or to -1 . Substituting sequentially the equality (6.46) in itself, we get

$$\begin{aligned} U_{n-1} &= \frac{\kappa_n}{2} \tilde{U}_n + \frac{\kappa_{n+1}}{2} \mathbb{D}_{n-1} \mathbb{D}_n^{-1} \tilde{U}_{n+1} + \mathbb{D}_{n-1} \mathbb{D}_{n+1}^{-1} U_{n+1} \\ &= \dots = \sum_{s=n}^N \frac{\kappa_s}{2} \mathbb{D}_{n-1} \mathbb{D}_{s-1}^{-1} \tilde{U}_s + \mathbb{D}_{n-1} \mathbb{D}_N^{-1} U_N. \end{aligned} \quad (6.47)$$

Thus, we see that

$$U_n = \mathbb{L} U_N + \sum_{s=n+1}^N \mathbb{G}_s \tilde{U}_s, \quad n = r, \dots, N, \quad (6.48)$$

where the linear operators $\mathbb{L}$ and $\mathbb{G}_s$ (depending on m and n) are defined by

$$\mathbb{L} = \mathbb{D}_n \mathbb{D}_N^{-1}, \quad \mathbb{G}_s = \frac{\kappa_s}{2} \mathbb{D}_n \mathbb{D}_{s-1}^{-1}, \quad s = n+1, \dots, N. \quad (6.49)$$

The equalities (6.44) and (6.47) imply now that

$$\begin{aligned} S_m &= \sum_{n=r+1}^{N+1} \beta_{n-1} \mathbb{D}_{n-1} \mathbb{D}_N^{-1} U_N + \sum_{n=r+1}^N \sum_{s=n}^N \frac{\beta_{n-1} \kappa_s}{2} \mathbb{D}_{n-1} \mathbb{D}_{s-1}^{-1} \tilde{U}_s \\ &= \mathbb{K} U_N + \sum_{s=r+1}^N \mathbb{C}_s \tilde{U}_s, \end{aligned} \quad (6.50)$$

where the linear operators $\mathbb{K}$ and $\mathbb{C}_s$ (depending on m) are defined by

$$\mathbb{K} = \sum_{n=r+1}^{N+1} \beta_{n-1} \mathbb{D}_{n-1} \mathbb{D}_N^{-1}, \quad \mathbb{C}_s = \sum_{n=r+1}^s \frac{\beta_{n-1} \kappa_s}{2} \mathbb{D}_{n-1} \mathbb{D}_{s-1}^{-1} \quad (6.51)$$

and the coefficients β_{n-1} and κ_s satisfy

$$|\beta_{n-1}| \leq \frac{3}{2}, \quad |\kappa_s| = 1. \quad (6.52)$$

By (6.21), (6.49), (6.51) and (6.52), we have the bounds

$$\|\mathbb{L}\| \leq c 2^{n-N}, \quad \|\mathbb{G}_s\| \leq c 2^{n-s}, \quad (6.53)$$

$$\|\mathbb{K}\| \leq c \sum_{n=r+1}^{N+1} 2^{n-N} \leq c \quad \text{and} \quad \|\mathbb{C}_s\| \leq c \sum_{n=r+1}^s 2^{n-s} \leq c. \quad (6.54)$$

It remains to note that $U_N = U_{N,1}$, $\tilde{U}_s = \tilde{\mathbb{J}}_s^{-1} \tilde{\mathcal{U}}_s$ (see (6.5)) and define $\mathbb{K}_s = \mathbb{C}_s \tilde{\mathbb{J}}_s^{-1}$ and $\mathbb{L}_s = \mathbb{G}_s \tilde{\mathbb{J}}_s^{-1}$. Then the assertion of the lemma follows from (6.32), (6.48), (6.50), (6.53) and (6.54) provided that c_{17} is chosen to be sufficiently large.

Remark 6.7. Since the vectors T_m , $V_{n,k}$ and $\mathcal{V}_{n,k}$ can be reconstructed from the vectors $Y_1, \dots, Y_{2N}$ by means of the same linear procedure which connects the vectors S_m , $U_{n,k}$ and $\mathcal{U}_{n,k}$ with the vectors $X_1, \dots, X_{2N}$, we see that in the conditions of Lemma 6.1

$$T_m = \mathbb{K} V_{N,1} + \sum_{n=r+1}^N \mathbb{K}_n \tilde{\mathcal{V}}_{n,l_{n,m}}, \quad (6.55)$$

$$V_{n,l_{n,m}} = \mathbb{L} V_{N,1} + \sum_{s=n+1}^N \mathbb{L}_s \tilde{\mathcal{V}}_{s,l_{s,m}}, \quad n = r, \dots, N. \quad (6.56)$$

7. Dyadic scheme. Conditions $\mathbf{A}(\tau)$, $\mathbf{A}^*(\tau)$, $\mathbf{C}_\Psi(p)$, $\mathbf{D}_\Phi$, $\mathbf{D}_\Psi$ and their consequences. In order to introduce further conditions we need some notation.

Notation 7.1. Consider a one-dimensional symmetric distribution $H = \mathcal{L}(\eta)$ concentrated on the interval $[-1, 1]$ and having a three times differentiable density $w(x)$. It is evident that $\mathbf{E}\eta = 0$ and such a distribution can be easily constructed. Below we shall assume that the distribution H with properties indicated above is fixed. For the sake of definiteness, we take

$$w(x) = \begin{cases} \frac{15 \cdot 015}{256} x^4 (1 - x^2)^4, & \text{if } -1 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbf{E}\eta^2 = \frac{1}{3}$. Introduce the corresponding conjugate distributions $H_h = \overline{H}(h)$, $h \in \mathbf{R}^1$ (see (2.1)).

Lemma 7.1. *There exist absolute positive constants $c_{18}, \dots, c_{21}$ such that the following relations are true:*

$$\sup_{|h| \leq c_{18}} \sup_{|t| \geq c_{19}} |\hat{H}_h(t)| \leq e^{-c_{20}}, \quad (7.1)$$

$$\sup_{|h| \leq c_{18}} |\hat{H}_h(t)| \leq \exp\left(-\frac{1}{8}t^2\right) \quad \text{for } |t| \leq c_{19}, \quad (7.2)$$

$$\sup_{|h| \leq c_{18}} |\hat{H}_h(t)| \leq \frac{1}{1 + c_{21}t^2}, \quad \text{for } t \in \mathbf{R}^1. \quad (7.3)$$

P r o o f. Putting $w(h, x) = e^{hx} w(x)$, $h, x \in \mathbf{R}^1$, and integrating by parts, we see that (for $t \neq 0$)

$$\begin{aligned} \hat{H}_h(t) &= (\mathbf{E}e^{h\eta})^{-1} \int_{|x| \leq 1} e^{itx} w(h, x) dx \\ &= -(\mathbf{E}e^{h\eta})^{-1} \int_{|x| \leq 1} t^{-2} e^{itx} \frac{\partial^2 w(h, x)}{\partial x^2} dx. \end{aligned} \quad (7.4)$$

Furthermore,

$$\sup_{x \in \mathbf{R}^1} \sup_{|h| \leq 1} \left| \frac{\partial^2 w(h, x)}{\partial x^2} \right| \leq c_{22} \quad (7.5)$$

(constants here and below are absolute, since the distribution H is fixed). From (7.4), (7.5) it follows that

$$\sup_{|h| \leq 1} |\hat{H}_h(t)| \leq 2c_{22}t^{-2} \quad (7.6)$$

(we take into account that, by the Jensen inequality, $\mathbf{E}e^{h\eta} \geq e^{\mathbf{E}h\eta} = 1$). The inequality (7.6) implies that, for $|t| \geq c_{23} = 2c_{22}^{1/2}$,

$$\sup_{|h| \leq 1} |\hat{H}_h(t)| \leq \left(1 + \frac{t^2}{4c_{22}}\right)^{-1}, \quad (7.7)$$

$$\sup_{|h| \leq 1} \sup_{|t| \geq c_{23}} |\hat{H}_h(t)| \leq \frac{1}{2}. \quad (7.8)$$

Obviously, $H \in \mathcal{A}_1(c)$. Using Lemma 2.1b) and the relations (2.3) and (2.4), we see that the relation (7.2) is valid, if $c_{18} \leq 1$ and $c_{19} \leq c_{23}$ are sufficiently small.

Since the distributions H_h are absolutely continuous, the relation $|\hat{H}_h(t)| = 1$ can be valid for $t = 0$ only. Furthermore, the function $|\hat{H}_h(t)|$ considered as a function of two variables h and t is continuous for all $h, t \in \mathbf{R}^1$. Therefore,

$$\sup_{|h| \leq c_{18}} \sup_{c_{19} \leq |t| \leq c_{23}} |\hat{H}_h(t)| \leq c < 1. \quad (7.9)$$

The inequalities (7.8), (7.9) imply (7.1).

It is easy to see that

$$\max \left\{ \exp \left(-\frac{1}{8} t^2 \right), e^{-c_{20}} \right\} \leq \frac{1}{1 + c_{24} t^2}, \quad \text{for } |t| \leq c_{23}, \quad (7.10)$$

if c_{24} is sufficiently small. The inequality (7.3) follows now from (7.1), (7.2), (7.7) and (7.10) with $c_{21} = \min\{c_{24}, (4c_{22})^{-1}\}$. Thus, Lemma 7.1 is proved.

Below we shall need the following useful fact.

Lemma 7.2. *Assume that $0 < \alpha < 1$ and a random variable μ has values 0 with probability $1 - \alpha^3$ and α^{-1} with probability α^3 . Let $X, K \in \mathbf{R}^d$ be random vectors with*

$$\begin{aligned} \mathbf{E}X = \mathbf{E}K = 0, \quad \text{cov } X = \mathbb{B}, \quad \text{cov } K = (1 - \alpha)\mathbb{B}, \\ \mathbf{E}\|X\|^3 < \infty, \quad \mathbf{E}\|K\|^3 < \infty. \end{aligned}$$

Suppose also that X, K, μ are mutually independent and the distribution of K is symmetric. Then the random vector $W = \mu X + K$ has mean zero, covariance operator $\mathbb{B}$ and, for any $u, v, w \in \mathbf{R}^d$, we have

$$\mathbf{E}\langle W, u \rangle \langle W, v \rangle \langle W, w \rangle = \mathbf{E}\langle X, u \rangle \langle X, v \rangle \langle X, w \rangle$$

(in other words, W and X have the same moments of the first three orders).

The proof of Lemma 7.2 is elementary and, therefore, is omitted. One should note that Lemma 7.2 is a simplified (and modified) version of (due to Yurinskii [24]) Lemma 4 from [20, p. 85])²⁾.

Notation 7.2. Let a random vector Z have independent coordinates Z_j with $\mathcal{L}(Z_j) = H$, $j = 1, \dots, d$. Assume that the conditions of Lemma 7.2 are satisfied with

$$\alpha = \frac{2}{3}, \quad \mathcal{L}(K) = \mathcal{L}(\mathbb{B}^{1/2} Z).$$

Thus, we have

$$\text{cov } Z = \frac{1}{3} \mathbb{I}_d, \quad \text{cov } K = \frac{1}{3} \mathbb{B}, \quad (7.11)$$

$$\mathbf{P}\{\|K\| \leq \|\mathbb{B}^{1/2}\| \sqrt{d}\} = 1, \quad (7.12)$$

$$\mathbf{P}\left\{\mu = \frac{3}{2}\right\} = \frac{8}{27}, \quad \mathbf{P}\{\mu = 0\} = \frac{19}{27}. \quad (7.13)$$

In this case we shall use below the notation

$$\Psi(\mathcal{L}(X)) \stackrel{\text{def}}{=} \mathcal{L}(W) = \mathcal{L}(\mu X + K). \quad (7.14)$$

According to Lemma 7.2, the distributions $\mathcal{L}(X)$ and $\Psi(\mathcal{L}(X))$ have coinciding moments of the first three orders. However, the distribution $\Psi(\mathcal{L}(X))$ is «better» than $\mathcal{L}(X)$ because it is sufficiently smooth.

²⁾ The author is grateful to B. A. Zaleskii who attracted the author's attention to the book by Sazonov [20].

Lemma 7.3. Let a random vector $X \in \mathbf{R}^d$ with $\mathbf{E}X = 0$ satisfy the condition $\mathcal{L}(X) \in \mathcal{A}_d(\tau)$, $\tau \geq 1$. Fix absolute positive constants C_1, C_2 and suppose that the operator $\mathbb{B} = \text{cov } X$ for all $u \in \mathbf{R}^d$ satisfies the condition

$$C_1^2 \|u\|^2 \leq \langle \mathbb{B}u, u \rangle = \|\mathbb{B}^{1/2}u\|^2 \leq C_2^2 \|u\|^2. \quad (7.15)$$

Assume that the distribution $\Psi(\mathcal{L}(X))$ is defined by (7.14) and $R_h = \overline{\Psi(\mathcal{L}(X))}(h)$, $h \in \mathbf{R}^d$.

Then there exist absolute positive constants $c_{25}, c_{26}, c_{27}, c_{28}$ such that

$$\begin{aligned} \mathcal{L}(K) &= \mathcal{L}(\mathbb{B}^{1/2}Z) \in \mathcal{A}_d(c_{25}), \\ \mathcal{L}(W) &= \Psi(\mathcal{L}(X)) \in \mathcal{A}_d(c_{25}\tau), \end{aligned} \quad c_{25} \geq 1, \quad (7.16)$$

and, for all $t \in \mathbf{R}^d$,

$$\sup_{\|h\| \leq c_{26}} |\hat{R}_h(t)| \leq \min \left\{ \max \left\{ \exp(-c_{27}\|t\|^2), e^{-c_{20}} \right\}, \frac{1}{1 + c_{28}\|t\|^2} \right\}. \quad (7.17)$$

P r o o f. Using Lemma 2.2 b), we see that from the conditions of the lemma it follows that $\mathcal{L}(X) \in \mathcal{B}_d(c\tau)$ (recall that $\mathbf{E}X = 0$). Using (2.7), (7.13), we see that, for all $u, v \in \mathbf{R}^d$ and for all integer $m \geq 3$,

$$\begin{aligned} |\mathbf{E}\langle \mu X, v \rangle^2 \langle \mu X, u \rangle^{m-2}| &= \mathbf{E}\mu^m |\mathbf{E}\langle X, v \rangle^2 \langle X, u \rangle^{m-2}| \\ &\leq \frac{1}{2} m! (c\tau \|u\|)^{m-2} \mathbf{E}\langle X, v \rangle^2 \leq \frac{1}{2} m! (c\tau \|u\|)^{m-2} \mathbf{E}\langle \mu X, v \rangle^2. \end{aligned} \quad (7.18)$$

According to (7.18), (2.7), $\mathcal{L}(\mu X) \in \mathcal{B}_d(c\tau)$ (evidently, $\mathbf{E}\mu X = \mathbf{E}\mu \mathbf{E}X = 0$). By Lemma 2.2a), this implies that

$$\mathcal{L}(\mu X) \in \mathcal{A}_d(c\tau). \quad (7.19)$$

Since $\mathbf{P}\{|Z_j| \leq 1\} = 1$, $j = 1, \dots, d$, the relations (7.14), (7.15), (7.19) together with Lemmas 2.2 and 2.4 imply (7.16) (we used the completeness of classes $\mathcal{A}_d(\tau)$ with respect to convolution and their monotonicity with respect to τ). It is clear that (7.15) implies that the operators $\mathbb{B}$ and $\mathbb{B}^{1/2}$ are nondegenerate and the distributions $\mathcal{L}(K)$ and, consequently, $\Psi(\mathcal{L}(X))$ are absolutely continuous. Write $P_h = \overline{\mathcal{L}(Z)}(h)$, $h \in \mathbf{R}^d$. According to (2.2), we have

$$\hat{P}_h(t) = \prod_{j=1}^d \hat{H}_{h_j}(t_j), \quad \text{for } t \in \mathbf{R}^d.$$

Therefore, by Lemma 7.1, we obtain, for $t \in \mathbf{R}^d$,

$$\sup_{\|h\| \leq c_{18}} |\hat{P}_h(t)| \leq \max \left\{ \exp \left(-\frac{\|t\|^2}{8} \right), e^{-c_{20}} \right\}, \quad (7.20)$$

$$\sup_{\|h\| \leq c_{18}} |\hat{P}_h(t)| dt \leq \frac{1}{1 + c_{21}\|t\|^2}. \quad (7.21)$$

According to (2.2), (3.6), (7.14) and (7.16), we have, for $\|h\|\tau < 1$,

$$|\widehat{R}_h(t)| \leq (\mathbf{E}e^{(h,K)})^{-1} |\mathbf{E}e^{(h+it,K)}| = |\widehat{P}_\gamma(\mathbb{B}^{1/2}t)|, \quad \text{where } \gamma = \mathbb{B}^{1/2}h. \quad (7.22)$$

Moreover, by (7.15), (7.20)–(7.22) and since $\tau \geq 1$, we conclude that the relation (7.17) hold. Lemma 7.3 is proved.

Suppose that the dyadic scheme satisfies Condition **B** and p is integer, $0 \leq p \leq N$. Introduce the following conditions on the distributions $R^{(k)} = \mathcal{L}(\xi_k)$:

C $_\Psi(p)$. $\mathcal{L}(\xi_l) = \Psi(\mathcal{L}(\tilde{\eta}_l))$ for $l = (m-1) \cdot 2^p + 1$, $m = 1, \dots, 2^{N-p}$, $\mathcal{L}(\xi_l) = \mathcal{L}(\tilde{\xi}_l)$ otherwise.

By Lemma 7.2 and by Condition **B**₁, ξ_k and $\tilde{\xi}_k$ have in the case **C $_\Psi(p)$** the same moments of the first three orders. Condition **C $_\Psi(0)$** means that $\mathcal{L}(\xi_k) = \Psi(\mathcal{L}(\tilde{\eta}_k))$, for all $k = 1, \dots, 2^N$. Introduce also the conditions on the distributions of the vectors $Y_1, \dots, Y_{2^N}$:

D $_\Phi$. $\mathcal{L}(Y_k) = \Phi(\mathcal{L}(\tilde{\xi}_k))$, $k = 1, \dots, 2^N$;

D $_\Psi$. $\mathcal{L}(Y_k) = \Psi(\mathcal{L}(\tilde{\eta}_k))$, $k = 1, \dots, 2^N$.

According to Lemma 7.2 and Condition **B**₁, the random vectors Y_k and $\tilde{\xi}_k$ have in the case **D $_\Psi$** the same moments of the first three orders.

Introduce, finally, the conditions on the distributions involved in our scheme to belong to the classes of type $\mathcal{A}_d(\tau)$, $\tau \geq 0$:

A (τ) . $\mathcal{L}(\tilde{\xi}_k) \in \mathcal{A}_d(\tau)$ and $\mathcal{L}(\tilde{\eta}_k) \in \mathcal{A}_d(\tau)$, for $k = 1, \dots, 2^N$;

A $^*(\tau)$.

$$\mathcal{L}(\mathbf{U}_{n,k}^j) \in \overline{\mathcal{A}}_j(\tau, 4) \quad \text{for } 1 \leq k \leq 2^{N-n}, \quad 1 \leq n \leq N, \quad d \leq j \leq 2d, \quad (7.23)$$

$$\mathcal{L}(\mathbf{U}_{N,1}^j) \in \overline{\mathcal{A}}_j(\tau, 4) \quad \text{for } 1 \leq j \leq 2d. \quad (7.24)$$

Lemma 7.4. Assume that Conditions **A (τ)** , **B** and **C $_\Psi(p)$** are satisfied with $\tau \geq 1$ and $0 \leq p \leq N$. Then there exist an absolute positive constant c_{29} such that if $\tau^2 d^3(p+1)/2^{N-p} \leq c_{29}$, $1 \leq k \leq 2^{N-n}$ and $1 \leq n \leq N$, then

$$F = \mathcal{L}(\mathbf{U}_{n,k}^j) \in \mathcal{A}_j^*(c_{30}\tau, 4, 4), \quad j = 1, \dots, 2d, \quad (7.25)$$

with $c_{30} = c_{16}^{-1} \max\{c_{25}, c_{26}^{-1}, 1\}$.

Corollary 7.1. Let Conditions **A (τ)** , **B** and **C $_\Psi(p)$** be satisfied with $0 \leq p \leq N$. Then $\mathcal{L}(\mathcal{U}^{(j)}) \in \mathcal{A}(c_{30}\tau, 4, 4)$, $j = 1, \dots, d$, if $\tau \geq 1$ and $\tau^2 d^3(p+1)/2^{N-p} \leq c_{29}$.

Proof. It suffices to note that $\mathcal{U}^{(j)} = \mathbf{U}_{N,1}^j$, see Remark 6.6. It remains to apply Lemma 7.4 with $n = N$ and $k = 1$.

Proof of Lemma 7.4. Recall that $\mathbf{U}_{n,k} = \mathbf{A}_{n,k} \tilde{\mathbf{U}}_{n,k}$ (see (5.24)), where $\mathbf{A}_{n,k}$ is a linear operator defined by the relations (5.10), (6.3), (6.5) and (6.6) and satisfying (6.37), (6.39) and (6.40). Moreover, $\tilde{\mathbf{U}}_{n,k} = (U_{n-1,2k-1}, U_{n-1,2k}) \in \mathbf{R}^{2d}$, where $U_{n-1,2k-1}$ and $U_{n-1,2k}$ are independent elements of $\mathbf{R}^d$. Therefore, the relation $\mathcal{L}(\mathbf{U}_{n,k}) \in \mathcal{A}_{2d}(c_{30}\tau)$ can be easily derived from Conditions **A (τ)** and **C $_\Psi(p)$** with the help of Lemmas 2.4,

2.5, 5.1 and 7.3 (see (6.39) and (7.16)) if we take into account the completeness of classes $\mathcal{A}_d(\tau)$ with respect to convolution and their monotonicity with respect to τ . It is easy to see that $\mathbf{U}_{n,k}^j = \bar{\mathbb{P}}_j \mathbf{U}_{n,k}$, where the projector $\bar{\mathbb{P}}_j: \mathbf{R}^{2d} \rightarrow \mathbf{R}^j$ can be considered as a linear operator with $\|\bar{\mathbb{P}}_j\| = 1$ (see (5.21)). Applying Lemma 2.4 again, we obtain the relations $\mathcal{L}(\mathbf{U}_{n,k}^j) \in \mathcal{A}_j(c_{30}\tau)$, $1 \leq j \leq 2d$.

It remains to verify that if c_{29} is small enough, then for $h \in \mathbf{R}^j$, $\|h\|c_{30}\tau < 1$, and for all $v \in \mathbf{R}^j$ the following inequalities hold:

$$(2\pi)^{-j} \int_T |\hat{F}_h(t)| dt \leq \frac{c_{30}^2 \tau^2 j^2}{\sigma^2 (2\pi)^{j/2} (\det \mathbb{D})^{1/2}}, \quad (7.26)$$

$$(2\pi)^{-j} \int_T |\langle t, v \rangle \hat{F}_h(t)| dt \leq \frac{\langle \mathbb{D}^{-1} v, v \rangle^{1/2}}{4(2\pi)^{j/2} (\det \mathbb{D})^{1/2}}, \quad (7.27)$$

where F_h denotes the conjugate distribution $F_h = \bar{F}(h)$ (see (2.1), (3.2) and (3.3)),

$$T = \{t \in \mathbf{R}^j: 4\|t\|c_{30}\tau j \geq 1\} \quad (7.28)$$

and σ^2 is the minimal eigenvalue of the operator $\mathbb{D} = \text{cov } \mathbf{U}_{n,k}^j$. Note that, according to (6.28) and (6.29), we have

$$\mathbb{D} = 2^n \mathbb{I}_j, \quad \sigma^2 = 2^n, \quad \langle \mathbb{D}^{-1} v, v \rangle = 2^{-n} \|v\|^2, \quad \det \mathbb{D} = 2^{nj}. \quad (7.29)$$

Introduce 2^{n-1} random vectors

$$\mathbf{X}_r = (X_r, X_{2^{n-1}+r}) \in \mathbf{R}^{2d}, \quad r = 2^n(k-1) + 1, \dots, 2^{n-1}(2k-1). \quad (7.30)$$

Obviously, these vectors are independent. According to (5.23), (5.24) and (7.30),

$$\tilde{\mathbf{U}}_{n,k} = (U_{n-1,2k-1}, U_{n-1,2k}) = \sum_{r=2^n(k-1)+1}^{2^{n-1}(2k-1)} \mathbf{X}_r. \quad (7.31)$$

Write $R_h^{(s)} = \overline{\mathcal{L}(X_s)}(h)$, for $s = 1, \dots, 2^N$, $h \in \mathbf{R}^d$, and $M_h^{(r)} \stackrel{\text{def}}{=} \overline{\mathcal{L}(\mathbf{X}_r)}(h)$, $Q_h^{(r)} \stackrel{\text{def}}{=} \overline{\mathcal{L}(\mathbb{A}_{n,k} \mathbf{X}_r)}(h)$, for $r = 2^n(k-1) + 1, \dots, 2^{n-1}(2k-1)$, $h \in \mathbf{R}^{2d}$. As usually, we consider only such h for which these distributions exist. It is easy to see that, by virtue of (3.6), for all $t \in \mathbf{R}^{2d}$,

$$\begin{aligned} \hat{Q}_h^{(r)}(t) &= \frac{\mathbf{E} \exp(\langle h + it, \mathbb{A}_{n,k} \mathbf{X}_r \rangle)}{\mathbf{E} \exp(\langle h, \mathbb{A}_{n,k} \mathbf{X}_r \rangle)} \\ &= \frac{\mathbf{E} \exp(\langle \mathbb{A}_{n,k}^* h + i \mathbb{A}_{n,k}^* t, \mathbf{X}_r \rangle)}{\mathbf{E} \exp(\langle \mathbb{A}_{n,k}^* h, \mathbf{X}_r \rangle)} = \widehat{M}_{\mathbb{A}_{n,k}^* h}^{(r)}(\mathbb{A}_{n,k}^* t). \end{aligned} \quad (7.32)$$

By (2.2) and (7.31), we have (for $j = 2d$)

$$|\hat{F}_h(t)| = \prod_{r=2^n(k-1)+1}^{2^{n-1}(2k-1)} |\hat{Q}_h^{(r)}(t)|. \quad (7.33)$$

Write an arbitrary vector $t = (t_1, \dots, t_{2d}) \in \mathbf{R}^{2d}$ in the form $t = (t^{(1)}, t^{(2)})$, where $t^{(1)} = (t_1, \dots, t_d)$, $t^{(2)} = (t_{d+1}, \dots, t_{2d}) \in \mathbf{R}^d$. Using (7.30), (3.6) and introducing a similar notation for $h = (h_1, \dots, h_{2d}) \in \mathbf{R}^{2d}$, it is easy to check that

$$\widehat{M}_h^{(r)}(t) = \widehat{R}_{h^{(1)}}^{(r)}(t^{(1)})\widehat{R}_{h^{(2)}}^{(2^{n-1}+r)}(t^{(2)}). \quad (7.34)$$

Recall that the distributions $\mathcal{L}(\xi_s) = \mathcal{L}(X_s)$, $s = 1, \dots, 2^N$, satisfy Condition $\mathbf{C}_\Psi(p)$. Hence, $R_h^{(l)} = \overline{\Psi(\mathcal{L}(\tilde{\eta}_l))}(h)$ for $l = (m-1) \cdot 2^p + 1$, $m = 1, \dots, 2^{N-p}$. Using Lemma 7.3 and Condition $\mathbf{A}(\tau)$, we see that the following estimate is valid, for such l and for all $t \in \mathbf{R}^d$:

$$\sup_{\|h\| \leq c_{26}} |\widehat{R}_h^{(l)}(t)| \leq \min \left\{ \max \left\{ \exp(-c_{27}\|t\|^2), e^{-c_{20}} \right\}, \frac{1}{1 + c_{28}\|t\|^2} \right\}. \quad (7.35)$$

Choosing c_{29} to be sufficiently small and taking into account that $\tau \geq 1$, we ensure the validity of the inequalities

$$2^{n-p-2} \geq 2d + 2 \geq j + 2, \quad (7.36)$$

$$\exp(-c_{20} \cdot 2^{n-p-3}) = \exp(-c \cdot 2^{(n-p)/2} \cdot 2^{(n-p)/2}) \leq \frac{1}{2^{n-p} \cdot 2^{(n-p)j/2}}. \quad (7.37)$$

Using (7.34)–(7.36), we get

$$\begin{aligned} \sup_{\|h\| \leq c_{26}} |\widehat{M}_h^{(l)}(t)| &\leq \prod_{\mu=1}^2 \min \left\{ \max \left\{ \exp(-c_{27}\|t^{(\mu)}\|^2), e^{-c_{20}} \right\}, \frac{1}{1 + c_{28}\|t^{(\mu)}\|^2} \right\} \\ &\leq \min \left\{ \max \left\{ \exp(-c_{27}\|t\|^2), e^{-c_{20}} \right\}, \frac{1}{1 + c_{28}\|t\|^2} \right\}, \end{aligned} \quad (7.38)$$

for $l = (m-1) \cdot 2^p + 1$, $m = 2^{n-p}(k-1) + 1, \dots, 2^{n-p-1}(2k-1)$ and for all $t \in \mathbf{R}^{2d}$. Note that such l satisfy the inequality $2^n(k-1)+1 \leq l \leq 2^{n-1}(2k-1)$ and the amount of such l equals to 2^{n-p-1} . Using (6.37), (7.32) and (7.38), we see that, for the same l and for all $t \in \mathbf{R}^{2d}$,

$$\sup_{\|h\| \leq c_{31}} |\widehat{Q}_h^{(l)}(t)| \leq \min \left\{ \max \left\{ \exp(-c_{32}\|t\|^2), e^{-c_{20}} \right\}, \frac{1}{1 + c_{33}\|t\|^2} \right\} \quad (7.39)$$

with $c_{31} = c_{26}c_{16}$, $c_{32} = c_{27}c_{15}^{-1}$, $c_{33} = c_{28}c_{15}^{-1}$.

Let us show that the inequality (7.39) implies that, for $h \in \mathbf{R}^j$, $\|h\|c_{30}\tau < 1$, and for all $t \in \mathbf{R}^j$,

$$|\widehat{F}_h(t)| \leq \left(\min \left\{ \max \left\{ \exp(-c_{32}\|t\|^2), e^{-c_{20}} \right\}, \frac{1}{1 + c_{33}\|t\|^2} \right\} \right)^{2^{n-p-1}} \quad (7.40)$$

It suffices to prove (7.40) for $j = 2d$ (for $1 \leq j < 2d$ one should apply (7.40) for $j = 2d$ and for $h \in \mathbf{R}^{2d}$, $\|h\| c_{30} \tau < 1$, $t \in \mathbf{R}^{2d}$ with $h_m = t_m = 0$, for $m = j + 1, \dots, 2d$). The restriction on h is used here to justify the existence of conjugate distributions F_h only. It remains to note that $c_{30} \geq c_{31}^{-1} = c_{26}^{-1} c_{16}^{-1}$ and to use (7.33) and (7.39).

Represent the set T defined in (7.28) in the form

$$T = T^{(1)} \cup T^{(2)}, \quad T^{(1)} = \left\{ t \in \mathbf{R}^j: \|t\|^2 \geq \frac{c_{20}}{c_{32}} \right\}, \quad T^{(2)} = T \setminus T^{(1)}. \quad (7.41)$$

Below (in the proof of (7.26) and (7.27)) we assume that $\|h\| c_{30} \tau < 1$. According to (7.28), (7.40) and (7.41), for $t \in T^{(1)}$ we have

$$|\hat{F}_h(t)| \leq \frac{1}{(1 + c_{33}\|t\|^2)^{2^{n-p-2}}} \exp(-c_{20} \cdot 2^{n-p-2}), \quad (7.42)$$

and, for $t \in T^{(2)}$,

$$|\hat{F}_h(t)| \leq \exp\left(-\frac{c_{34} \cdot 2^{n-p}}{\tau^2 j^2}\right) \exp(-c_{32} \cdot 2^{n-p-2} \|t\|^2). \quad (7.43)$$

Using (7.29), (7.36) and (7.42), we obtain

$$\begin{aligned} & (2\pi)^{-j} \int_{T^{(1)}} (1 + \|t\|^2) |\hat{F}_h(t)| dt \\ & \leq c \exp(-c_{20} \cdot 2^{n-p-2}) \int_{\mathbf{R}^j} \frac{dt}{(1 + c_{33}\|t\|^2)^j} \\ & \leq c \exp(-c_{20} \cdot 2^{n-p-2}) \left(\int_{\mathbf{R}} \frac{dt}{1 + c_{33}t^2} \right)^j \leq \frac{2^{(n-p)j/2}}{2^p \cdot (2\pi)^{j/2} (\det \mathbb{D})^{1/2}} \\ & \quad \times \exp(c_{35}(p+1)j - c_{20} \cdot 2^{n-p-3} - c_{20} \cdot 2^{n-p-3}). \end{aligned} \quad (7.44)$$

Similarly, using (7.29) and (7.43), we see that

$$\begin{aligned} & (2\pi)^{-j} \int_{T^{(2)}} |\hat{F}_h(t)| dt \leq \exp\left(-\frac{c_{34} \cdot 2^{n-p}}{\tau^2 j^2}\right) \int_{\mathbf{R}^j} \exp(-c_{32} \cdot 2^{n-p-2} \|t\|^2) dt \\ & \leq \frac{1}{2^{(n-p)j/2}} \exp\left(cj - \frac{c_{34} \cdot 2^{n-p}}{\tau^2 j^2}\right) \\ & \leq \frac{1}{2^p \cdot (2\pi)^{j/2} (\det \mathbb{D})^{1/2}} \exp\left(c_{36}(p+1)j - \frac{c_{34} \cdot 2^{n-p}}{\tau^2 j^2}\right). \end{aligned} \quad (7.45)$$

It is evident that, choosing c_{29} to be small enough, we can ensure the validity of the inequalities

$$\exp\left(c_{36}(p+1)j - \frac{c_{34} \cdot 2^{n-p}}{\tau^2 j^2}\right) \leq \frac{c_{30}^2 \tau^2 j^2}{2 \cdot 2^{n-p}} \leq \frac{1}{32}, \quad (7.46)$$

$$\exp(c_{35}(p+1)j - c_{20} \cdot 2^{n-p-3}) \leq \frac{c_{30}^2}{2 \cdot 2^{n-p}} \leq \frac{c_{30}^2 \tau^2 j^2}{2 \cdot 2^{n-p}} \leq \frac{1}{32}. \quad (7.47)$$

The relations (7.29), (7.37), (7.41) and (7.44)–(7.47) imply (7.26) and

$$(2\pi)^{-j} \int_T |\widehat{F}_h(t)| dt \leq \frac{1}{16(2\pi)^{j/2}(\det \mathbb{D})^{1/2}}. \quad (7.48)$$

For the proof of (7.27) we apply the Hölder inequality and obtain that

$$\int_T |\langle t, v \rangle \widehat{F}_h(t)| dt \leq \|v\| \left(\int_T |\widehat{F}_h(t)| dt \right)^{1/2} \left(\int_T \|t\|^2 |\widehat{F}_h(t)| dt \right)^{1/2}. \quad (7.49)$$

The first integral in the right-hand side of (7.49) is estimated in (7.48). Let us estimate the second integral. For this, using (7.29) and (7.43), we obtain

$$\begin{aligned} & (2\pi)^{-j} \int_{T(2)} \|t\|^2 |\widehat{F}_h(t)| dt \\ & \leq \int_{\mathbf{R}^j} \|t\|^2 \exp \left(-c_{32} \cdot 2^{n-p-2} \|t\|^2 - \frac{c_{34} \cdot 2^{n-p}}{\tau^2 j^2} \right) dt \\ & \leq \frac{1}{2^{n-p} \cdot 2^{(n-p)j/2}} \exp \left(cj - \frac{c_{34} \cdot 2^{n-p}}{\tau^2 j^2} \right) \\ & \leq \frac{1}{2^n (2\pi)^{j/2} (\det \mathbb{D})^{1/2}} \exp \left(c(p+1)j - \frac{c_{34} \cdot 2^{n-p}}{\tau^2 j^2} \right). \end{aligned} \quad (7.50)$$

According to (7.37), (7.41), (7.44) and (7.50), we have

$$\begin{aligned} & (2\pi)^{-j} \int_T \|t\|^2 |\widehat{F}_h(t)| dt \\ & \leq \frac{1}{2^n (2\pi)^{j/2} (\det \mathbb{D})^{1/2}} \exp \left(c_{37}(p+1)j - \frac{c_{38} \cdot 2^{n-p}}{\tau^2 j^2} \right). \end{aligned} \quad (7.51)$$

It is evident that, choosing c_{29} to be small enough, we can ensure the validity of the inequality

$$\exp \left(c_{37}(p+1)j - \frac{c_{38} \cdot 2^{n-p}}{\tau^2 j^2} \right) \leq 1. \quad (7.52)$$

The relations (7.29), (7.48), (7.49), (7.51) and (7.52) imply (7.27). This completes the proof of the lemma.

8. Estimation of the rate of approximation in constructing the vectors with needed moments up to the third order.

Theorem 8.1. *Let Conditions $\mathbf{A}^*(\tau)$, $\mathbf{B}$, $\mathbf{D}_\Phi$ be satisfied for the dyadic scheme described in Sections 5–7. Then there exist absolute positive constants c_{39}, c_{40}, c_{41} such that*

$$\mathbf{E} \exp \left(\frac{c_{39} \Delta(X, Y)}{d^{5/2} \tau} \right) \leq \exp (c_{40}(N+1) \ln^* d) \quad (8.1)$$

provided that $\tau d^{3/2} \leq c_{41}$, where $\Delta(X, Y)$ is defined in (1.1) with $s = 2^N$.

Below (up to the formulation of Theorem 8.2) we shall prove Theorem 8.1. Therefore, we assume that its conditions are fulfilled.

Lemma 8.1. *There exist absolute positive constants c_{41}, c_{42} such that*

a) *If $\tau d^{3/2}/2^{N/2} \leq c_{41}$, then*

$$\|U_{N,1} - V_{N,1}\| \leq c_{42}\tau \left(d^2 + d^{3/2} \cdot 2^{-N} \|U_{N,1}\|^2 \right) \quad (8.2)$$

provided that $\|U_{N,1}\| \leq c_{41} \cdot 2^N / (d^{3/2}\tau)$;

b) *If $1 \leq n \leq N$, $1 \leq k \leq 2^{N-n}$, $\tau d^{3/2}/2^{n/2} \leq c_{41}$, then*

$$\|\widetilde{\mathcal{U}}_{n,k} - \widetilde{\mathcal{V}}_{n,k}\| \leq c_{42}\tau \left(d^2 + d^{3/2} \cdot 2^{-n} \|\mathbf{U}_{n,k}\|^2 \right) \quad (8.3)$$

provided that $\|\mathbf{U}_{n,k}\| \leq c_{41} \cdot 2^n / (d^{3/2}\tau)$.

In the proof of this lemma we need the following auxiliary assertion which is useful for the application of Lemmas 3.2 and 3.3 to conditional distributions involved in the dyadic scheme.

Lemma 8.2. *Let a continuous distribution function $F(\cdot)$ and an arbitrary distribution function $G(\cdot)$ satisfy for $z \in B \in \mathfrak{B}_1$ the inequality*

$$G(z - f(z) + w_1) < F(z + w_2) < G(z + f(z) + w_1)$$

with some $f: B \rightarrow \mathbf{R}^1$ and $w_1, w_2 \in \mathbf{R}^1$. Let $\eta \in \mathbf{R}^1$, $0 < G(\eta) < 1$ and $\xi = F^{-1}(G(\eta))$, where $F^{-1}(x) = \sup\{u: F(u) \leq x\}$, $0 < x < 1$. Then

$$|\xi - \eta| < f(\xi - w_2) + |w_2 - w_1|, \quad \text{if } \xi - w_2 \in B. \quad (8.4)$$

We omit the elementary proof of this lemma.

Proof of Lemma 8.1. Let us prove at first the statement a). By condition **B** (see (6.13) and (6.14)), the coordinates $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(d)}$ of the vectors $\mathcal{U} = \mathbb{J}_{N,1} U_{N,1}$ and $\mathcal{U}^{(j)} = \overline{\mathbb{P}}_j(\mathcal{U})$, $j = 1, \dots, d$, are noncorrelated and have mean zero and the same variance 2^N . According to Condition **D**_Φ, this implies that the distribution functions $G^{(j)}$ defined in the dyadic scheme are Gaussian with mean zero and variance 2^N . Moreover, by Remarks 6.3 and 6.6, we have $\mathcal{L}(\mathcal{V}^j) = \Phi(\mathcal{L}(\mathcal{U}^{(j)}))$ and $\mathcal{U}^{(j)} = \mathbf{U}_{N,1}^j$. Thus, using Condition **A**^{*}(τ), we see that the distributions $\mathcal{L}(\mathcal{U}^{(j)})$, $\mathcal{L}(\mathcal{V}^j)$ satisfy in the j -dimensional situation the conditions of Lemma 3.2 with $\text{cov } \xi = \text{cov } \mathcal{U}^{(j)} = 2^N \mathbb{I}_j$, $\mathbb{B} = \text{cov } \mathcal{U}^{j-1} = 2^N \mathbb{I}_{j-1}$ (the last equality for $j \geq 2$).

Taking into account (5.14), (5.15) and applying Lemmas 3.2, 8.2 in the j -dimensional case, $j = 1, \dots, d$, we obtain that

$$|\mathcal{U}^{(1)} - \mathcal{V}^{(1)}| \leq c\tau \left(1 + \frac{|\mathcal{U}^{(1)}|^2}{2^N} \right), \quad (8.5)$$

provided that $\tau/2^{N/2} \leq c_{43}$, $|\mathcal{U}^{(1)}| \leq c_{43} \cdot 2^N / \tau$. Moreover,

$$|\mathcal{U}^{(j)} - \mathcal{V}^{(j)}| \leq c\tau \left(j^{3/2} + j \frac{\|\mathcal{U}^{j-1}\|}{2^{N/2}} \left(1 + \frac{|\mathcal{U}^{(j)} - y_j|^2}{2^{N/2}} \right) + \frac{|\mathcal{U}^{(j)} - y_j|^2}{2^N} \right) + |y_j|, \quad (8.6)$$

provided that

$$\frac{\tau j^{3/2}}{2^{N/2}} \leq c_{43}, \quad |\mathcal{U}^{(j)} - y_j| \leq \frac{c_{43} \cdot 2^N}{j\tau}, \quad \frac{|\mathcal{U}^{j-1}|}{2^{N/2}} \leq \frac{c_{43} \cdot 2^{N/2}}{j^{3/2}\tau}, \quad 2 \leq j \leq d, \quad (8.7)$$

where

$$|y_j| \leq c_{44}\tau \|\mathcal{U}^{j-1}\|^2 2^{-N}, \quad 2 \leq j \leq d. \quad (8.8)$$

Obviously,

$$|\mathcal{U}^{(1)}| \leq \max \left\{ \|\mathcal{U}^{j-1}\|, |\mathcal{U}^{(j)}| \right\} \leq \|\mathcal{U}^{(j)}\| \leq \|\mathcal{U}\|, \quad 2 \leq j \leq d, \quad (8.9)$$

see (5.17) and (5.18). Using (8.5), (8.6), (8.8) and (8.9), we see that one can choose c_{44} to be so large and c_{45} to be so small that

$$|\mathcal{U}^{(j)} - \mathcal{V}^{(j)}| \leq c_{44}\tau(d^{3/2} + d\|\mathcal{U}\|^2 2^{-N}), \quad (8.10)$$

provided that

$$\frac{\tau d^{3/2}}{2^{N/2}} \leq c_{45}, \quad \|\mathcal{U}\| \leq \frac{c_{45} \cdot 2^N}{d^{3/2}\tau}, \quad 1 \leq j \leq d. \quad (8.11)$$

Taking into account (5.13), (5.17) and (6.23), we see that

$$\|\mathcal{U}\| \leq C_1^{-1} \|U_{N,1}\|, \quad \|U_{N,1} - V_{N,1}\| \leq C_2 \|\mathcal{U} - \mathcal{V}\|. \quad (8.12)$$

The assertion a) follows from (8.10)–(8.12) if we choose c_{41} to be sufficiently small and c_{42} sufficiently large.

Let us prove now statement b). By condition **B** (see (6.28) and (6.29)), the coordinates $\mathbf{U}_{n,k}^{(1)}, \dots, \mathbf{U}_{n,k}^{(2d)}$ of the vectors $\mathbf{U}_{n,k} = \mathbf{A}_{n,k} \tilde{\mathbf{U}}_{n,k}$ and $\mathbf{U}_{n,k}^j = \tilde{\mathbf{P}}_j(\mathbf{U}_{n,k})$, $j = 1, \dots, 2d$, are noncorrelated and have the same variance 2^n . By Condition **D** $_{\Phi}$, this implies that the distribution functions $\tilde{G}_{n,k}^{(j)}$, $j = d+1, \dots, 2d$, defined in the dyadic scheme are Gaussian with mean zero and variance 2^n . Moreover, by Remark 6.3, $\mathcal{L}(\mathbf{V}_{n,k}^j) = \Phi(\mathcal{L}(\mathbf{U}_{n,k}^j))$, $j = 1, \dots, 2d$. Thus, using Condition **A** $^*(\tau)$, we see that the distributions $\mathcal{L}(\mathbf{U}_{n,k}^j), \mathcal{L}(\mathbf{V}_{n,k}^j)$, $j = d+1, \dots, 2d$, satisfy in the j -dimensional situation the conditions of Lemma 3.2 with $\text{cov } \xi = \text{cov } \mathbf{U}_{n,k}^j = 2^n \mathbb{I}_j$, $\mathbb{B} = \text{cov } \mathbf{U}_{n,k}^{j-1} = 2^n \mathbb{I}_{j-1}$.

Taking into account (5.20) and applying Lemmas 3.2, 8.2 in the j -dimensional case, $j = d+1, \dots, 2d$, we obtain that

$$|\mathbf{U}_{n,k}^{(j)} - \mathbf{V}_{n,k}^{(j)}| \leq c\tau \left(j^{3/2} + j \frac{\|\mathbf{U}_{n,k}^{j-1}\|}{2^{n/2}} \left(1 + \frac{|\mathbf{U}_{n,k}^{(j)} - y_j|}{2^{n/2}} \right) + \frac{|\mathbf{U}_{n,k}^{(j)} - y_j|^2}{2^n} \right) + |y_j|, \quad (8.13)$$

provided that

$$\frac{\tau j^{3/2}}{2^{n/2}} \leq c_{46}, \quad |\mathbf{U}_{n,k}^{(j)} - y_j| \leq \frac{c_{46} \cdot 2^n}{j\tau}, \quad \frac{|\mathbf{U}_{n,k}^{j-1}|}{2^{n/2}} \leq \frac{c_{46} \cdot 2^{n/2}}{j^{3/2}\tau}, \quad (8.14)$$

where

$$|y_j| \leq c_{47} \|\mathbf{U}_{n,k}^{j-1}\|^2 2^{-n}, \quad d+1 \leq j \leq 2d. \quad (8.15)$$

Obviously,

$$\max \left\{ \|\mathbf{U}_{n,k}^{j-1}\|, |\mathbf{U}_{n,k}^{(j)}| \right\} \leq \|\mathbf{U}_{n,k}^j\| \leq \|\mathbf{U}_{n,k}\|, \quad d+1 \leq j \leq 2d, \quad (8.16)$$

see (5.17) and (5.18). Using (8.13), (8.15), and (8.16), we see that one can choose c_{47} to be so large and c_{48} to be so small that

$$|\mathbf{U}_{n,k}^{(j)} - \mathbf{V}_{n,k}^{(j)}| \leq c_{47}\tau \left(d^{3/2} + d \|\mathbf{U}_{n,k}\|^2 2^{-n} \right), \quad (8.17)$$

provided that

$$\frac{\tau d^{3/2}}{2^{n/2}} \leq c_{48}, \quad \|\mathbf{U}_{n,k}\| \leq \frac{c_{48} \cdot 2^n}{d^{3/2}\tau}, \quad d+1 \leq j \leq 2d. \quad (8.18)$$

The assertion b) follows from (6.4)–(6.8), (8.17) and (8.18) provided that we choose c_{41} to be sufficiently small and c_{42} sufficiently large.

P r o o f o f T h e o r e m 8.1. Write

$$\Delta = \Delta(X, Y) = \max_{1 \leq k \leq 2^N} \|S_k - T_k\|, \quad (8.19)$$

$$c_{49} = \min \{c_{41}, c_{16}c_{41}\sqrt{2}, C_2^2\}, \quad y \stackrel{\text{def}}{=} \frac{c_{49}}{d^{3/2}\tau} \leq \frac{C_2^2}{\tau}. \quad (8.20)$$

Let us prove that there exists a c_{50} such that

$$\mathbf{P}\{\Delta \geq x\} \leq (15d)^{N+1} \exp \left(-\frac{x}{c_{50}d^{5/2}\tau} \right), \quad x > 0, \quad (8.21)$$

provided that $\tau d^{3/2} \leq c_{41}$. Fix some $x > 0$, and choose the integer M satisfying the conditions

$$x < 8y \cdot 2^M \leq 2x. \quad (8.22)$$

Write

$$P = \max\{0, M\}, \quad L = \min\{N, P\} \quad (8.23)$$

and

$$\Delta_1 = \max_{0 \leq k < 2^{N-L}} \max_{1 \leq s \leq 2^L} \|S_{k \cdot 2^L + s} - S_{k \cdot 2^L}\|, \quad (8.24)$$

$$\Delta_2 = \max_{0 \leq k < 2^{N-L}} \max_{1 \leq s \leq 2^L} \|T_{k \cdot 2^L + s} - T_{k \cdot 2^L}\|, \quad (8.25)$$

$$\Delta_3 = \max_{0 \leq k \leq 2^{N-P}} \|S_{k \cdot 2^L} - T_{k \cdot 2^L}\|, \quad (8.26)$$

where the maximums are taken over integer values of k and s . It is evident that

$$\Delta \leq \Delta_1 + \Delta_2 + \Delta_3. \quad (8.27)$$

Define the event

$$A = \left\{ \omega: \max\{\Delta_1, \Delta_2\} < z \right\}, \quad \text{where} \quad z = \max \left\{ y \cdot 2^{L-1}, \frac{x}{8} \right\} \quad (8.28)$$

(we shall always assume that all considered random vectors are measurable mappings of elementary event $\omega \in \Omega$). For the complementary event we shall use the notation $\bar{A} = \Omega \setminus A$. Recall that all d -dimensional Gaussian distributions belong to classes $\mathcal{A}_d(\tau)$ for all $\tau \geq 0$. Applying Lemma 2.6 with $B^2 \leq C_2^2 \cdot 2^L$ (see (6.2) and (6.16)) and using (8.20), (8.24), (8.25) and (8.28), we obtain that

$$\begin{aligned} \mathbf{P}\{\bar{A}\} = 1 - \mathbf{P}\{A\} &\leq 2^N \cdot 4d \max \left\{ \exp \left(-\frac{z^2}{4dB^2} \right), \exp \left(-\frac{z}{4\tau\sqrt{d}} \right) \right\} \\ &\leq 2^N \cdot 4d \exp \left(-\frac{cx}{d^{5/2}\tau} \right). \end{aligned} \quad (8.29)$$

If $M > N$ then $L = N < M = P$, $\Delta_3 = 0$, $z = x/8$, and, for $\omega \in A$, it is easy to see that $\Delta \leq x/4$ (one should use (8.22), (8.23), (8.26), (8.27) and (8.28)). Hence, $\mathbf{P}\{\Delta \geq x\} \leq 1 - \mathbf{P}\{A\}$ and one can conclude via (8.29).

Let now $M \leq N$. Then, by (8.22), (8.23) and (8.28), we have

$$0 \leq L = P \leq N, \quad M \leq L, \quad z \leq y \cdot 2^L. \quad (8.30)$$

Note that if $L = P = 0$ then $\Delta = \Delta_3$ and if $L = P = M$ then $z = x/8$ (see (8.22), (8.26), (8.28) and (8.30)). Thus, by (8.27) and (8.28), in both possible cases we have

$$\Delta \leq \frac{x}{4} + \Delta_3, \quad \text{for } \omega \in A. \quad (8.31)$$

Fix an integer k satisfying $1 \leq k \leq 2^{N-L}$ and denote for simplicity

$$j = j(k) \stackrel{\text{def}}{=} k \cdot 2^L. \quad (8.32)$$

By Lemma 6.1 and Remark 6.7, we have

$$\begin{aligned} \|S_{k \cdot 2^L} - T_{k \cdot 2^L}\| &= \|S_j - T_j\| \leq c_{17} \|U_{N,1} - V_{N,1}\| \\ &\quad + c_{17} \sum_{n=L+1}^N \|\widetilde{\mathcal{U}}_{n,l_{n,j}} - \widetilde{\mathcal{V}}_{n,l_{n,j}}\|, \end{aligned} \quad (8.33)$$

where $l_{n,j}$ are integers defined by $(l_{n,j} - 1) \cdot 2^n < j \leq l_{n,j} \cdot 2^n$.

Recall that $U_{n,l} = S_{l \cdot 2^n} - S_{(l-1) \cdot 2^n}$. By virtue of (8.20), (8.24), (8.28) and (8.30), for $\omega \in A$ we have

$$|U_{L,l}| \leq \|U_{L,l}\| \leq \Delta_1 \leq z \leq y \cdot 2^L = \frac{c_{49} \cdot 2^L}{d^{3/2}\tau}, \quad 1 \leq l \leq 2^{N-L}, \quad (8.34)$$

and, by (5.2), (5.22)–(5.24), $U_{L,l}$ are sums over blocks consisting of 2^L summands. Moreover, $U_{n,l}$, $L+1 \leq n \leq N$, $1 \leq l \leq 2^{N-n}$, are sums over blocks containing each 2^n summands. These sums can be represented as a sum of 2^{n-L} sums over blocks containing each 2^L summands and satisfying (8.34). Therefore, for $\omega \in A$, $L+1 \leq n \leq N$, $1 \leq l \leq 2^{N-n}$ we have (see (5.24), (6.39), (8.20) and (8.34))

$$|U_{n,l}| \leq \|U_{n,l}\| \leq 2^{n-L} z \leq y \cdot 2^n = \frac{c_{49} \cdot 2^n}{d^{3/2\tau}} \leq \frac{c_{41} \cdot 2^n}{d^{3/2\tau}}, \quad (8.35)$$

$$\|\tilde{U}_{n,l}\| = \left(\|U_{n-1,2l-1}\|^2 + \|U_{n-1,2l}\|^2 \right)^{1/2} \leq \frac{c_{49} \cdot 2^n}{\sqrt{2}d^{3/2\tau}}, \quad (8.36)$$

$$\|\mathbf{U}_{n,l}\| = \|\mathbf{A}_{n,l}\tilde{\mathbf{U}}_{n,l}\| \leq \|\mathbf{A}_{n,l}\| \cdot \|\tilde{\mathbf{U}}_{n,l}\| \leq \frac{c_{16}^{-1}c_{49} \cdot 2^n}{\sqrt{2}d^{3/2\tau}} \leq \frac{c_{41} \cdot 2^n}{d^{3/2\tau}}. \quad (8.37)$$

Using (8.20), (8.35) and (8.37), we see that the conditions of Lemma 8.1 are satisfied for τ , $U_{N,1}$ and $\mathbf{U}_{n,l}$, if $\omega \in A$, $\tau d^{3/2} \leq c_{41}$, $L+1 \leq n \leq N$, $1 \leq l \leq 2^{N-n}$. By (6.39), (8.33), (8.36), (8.37) and by Lemma 8.1, for $\omega \in A$ we have

$$\begin{aligned} \|S_j - T_j\| &\leq c\tau \left(d^2 + d^{3/2} \cdot 2^{-N} \|U_{N,1}\|^2 \right) + \sum_{n=L+1}^N c\tau \left(d^2 + d^{3/2} 2^{-n} \|\mathbf{U}_{n,l_{n,j}}\|^2 \right) \\ &\leq c\tau \left(d^2(N+1) + d^{3/2} \left(2^{-N} \|U_{N,1}\|^2 \right. \right. \\ &\quad \left. \left. + \sum_{n=L}^{N-1} \frac{\|U^{(n)}\|^2 + \|U_{(n)}\|^2}{2^n} \right) \right), \end{aligned} \quad (8.38)$$

where

$$U^{(n)} = U_{n,l_{n,j}}, \quad U_{(n)} = U_{n,\tilde{l}_{n,j}}, \quad (8.39)$$

$$\tilde{l}_{n-1,j} = \begin{cases} 2l_{n,j}, & \text{if } l_{n-1,j} = 2l_{n,j} - 1, \\ 2l_{n,j} - 1, & \text{if } l_{n-1,j} = 2l_{n,j}, \end{cases} \quad L < n \leq N, \quad (8.40)$$

(it is not hard to verify that $l_{n-1,j}$ can be equal either to $2l_{n,j}$ or to $2l_{n,j} - 1$ if $l_{n,j}$ is fixed). According to Remark 5.2, $U^{(n)}$, $L \leq n \leq N$, are the sums over the blocks of 2^n summands which contains X_j . The sum $U_{(n)}$ does not contain X_j and

$$U^{(n+1)} = U^{(n)} + U_{(n)}, \quad L \leq n < N, \quad (8.41)$$

(see (5.24)). The equality (8.41) implies

$$U^{(n)} = U^{(L)} + \sum_{s=0}^{n-L-1} U_{(L+s)}, \quad L \leq n \leq N. \quad (8.42)$$

It is important that all summands in the right-hand side of (8.42) are the sums of disjoint blocks of independent summands. Therefore, they are themselves independent. This argument will be used below in similar situations.

Put $\beta = 1/\sqrt{2}$. Then, using (8.42) and the Hölder inequality, one can easily derive that, for $L \leq n \leq N$,

$$\|U^{(n)}\|^2 \leq c_{51} \left(\beta^{-(n-L)} \|U^{(L)}\|^2 + \sum_{s=0}^{n-L-1} \beta^{-(n-L-1)+s} \|U_{(L+s)}\|^2 \right), \quad (8.43)$$

with $c_{51} = \sum_{j=0}^{\infty} \beta^j = \sqrt{2}/(\sqrt{2} - 1)$. It is easy to see that

$$\sum_{n=L}^N 2^{-n} \beta^{-(n-L)} \|U^{(L)}\|^2 \leq c_{51} \cdot 2^{-L} \|U^{(L)}\|^2. \quad (8.44)$$

Moreover,

$$\begin{aligned} & \sum_{n=L+1}^N \sum_{s=0}^{n-L-1} 2^{-n} \beta^{-(n-L-1)+s} \|U_{(L+s)}\|^2 \\ &= \sum_{s=0}^{N-L-1} \sum_{n=L+1+s}^N 2^{-n} \beta^{-(n-L-1)+s} \|U_{(L+s)}\|^2 \\ &\leq c_{51} \sum_{s=0}^{N-L-1} 2^{-(L+1+s)} \|U_{(L+s)}\|^2. \end{aligned} \quad (8.45)$$

It is clear that the inequalities (8.43)–(8.45) imply

$$\begin{aligned} & 2^{-N} \|U_{N,1}\|^2 + \sum_{n=L}^{N-1} 2^{-n} (\|U^{(n)}\|^2 + \|U_{(n)}\|^2) \\ &\leq c_{51} \left(\frac{\|U^{(L)}\|^2}{2^L} + \sum_{s=0}^{N-L-1} \frac{\|U_{(L+s)}\|^2}{2^{L+1+s}} \right) + \sum_{n=L}^{N-1} \frac{\|U_{(n)}\|^2}{2^n} \\ &\leq c \left(\frac{\|U^{(L)}\|^2}{2^L} + \sum_{n=L}^{N-1} \frac{\|U_{(n)}\|^2}{2^n} \right). \end{aligned} \quad (8.46)$$

From (8.38) and (8.46) it follows that for $\omega \in A$ we have

$$\begin{aligned} \|S_j - T_j\| &\leq c_{52} \tau \left(d^2(N+1) + d^{3/2} \left(\frac{\|U^{(L)}\|^2}{2^L} + \sum_{n=L}^{N-1} \frac{\|U_{(n)}\|^2}{2^n} \right) \right) \\ &\leq c_{52} d^{5/2} \tau \left(N+1 + \frac{|U^{(L)}|^2}{2^L} + \sum_{n=L}^{N-1} \frac{|U_{(n)}|^2}{2^n} \right). \end{aligned} \quad (8.47)$$

Write (for $0 \leq n \leq N$, $1 \leq l \leq 2^{N-n}$)

$$W_{n,l} = \begin{cases} 2^{-n} C_2^{-2} |U_{n,l}|^2, & \text{if } |U_{n,l}| \leq y \cdot 2^n, \\ 0, & \text{otherwise.} \end{cases} \quad (8.48)$$

Let us show that

$$\mathbf{E} \exp(t W_{n,l}) \leq 2d + 1, \quad \text{for } 0 \leq t \leq \frac{1}{8}. \quad (8.49)$$

Indeed, integrating by parts, we obtain

$$\begin{aligned} \mathbf{E} \exp(tW_{n,l}) &= 1 + \int_0^\gamma te^{tu} \mathbf{P}\{W_{n,l} \geq u\} du \\ &\leq 1 + \frac{1}{8} \int_0^\gamma e^{u/8} \mathbf{P}\{|U_{n,l}| \geq C_2 \cdot 2^{n/2} \sqrt{u}\} du, \end{aligned} \quad (8.50)$$

where $\gamma = y^2 \cdot 2^n C_2^{-2}$. Taking into account (5.24), (6.2), (6.16), (8.20) and using Lemma 2.6 with $B^2 \leq C_2^2 \cdot 2^n$, we obtain that

$$\begin{aligned} \mathbf{P}\{|U_{n,l}| \geq C_2 \cdot 2^{n/2} \sqrt{u}\} &\leq 2d \exp\left(-\min\left\{\frac{2^n C_2^2 u}{4 \cdot 2^n C_2^2}, \frac{2^{n/2} C_2 \sqrt{u}}{4\tau}\right\}\right) \\ &\leq 2d \exp\left(-\min\left\{\frac{u}{4}, \frac{C_2^2 u}{4y\tau}\right\}\right) = 2de^{-u/4}, \end{aligned} \quad (8.51)$$

if $0 \leq u \leq \gamma = y^2 \cdot 2^n C_2^{-2}$. The relation (8.49) immediately follows from (8.50) and (8.51).

The relations (8.34), (8.35) and (8.48) imply that, for $L \leq n \leq N$, $1 \leq l \leq 2^{N-n}$, $\omega \in A$,

$$2^{-n}|U_{n,l}|^2 = C_2^2 W_{n,l}. \quad (8.52)$$

Thus, according to (8.39), we can rewrite (8.47) in the form

$$\|S_j - T_j\| \leq c_{52} d^{5/2} \tau \left(N + 1 + W^{(L)} + \sum_{n=L}^{N-1} W_{(n)} \right), \quad \omega \in A, \quad (8.53)$$

where

$$W^{(L)} = W_{L,l_{L,j}}, \quad W_{(n)} = W_{n,\tilde{l}_{n,j}}. \quad (8.54)$$

Using that the random variables $W^{(L)}, W_{(L)}, \dots, W_{(N-1)}$ are independent and applying (8.26), (8.32), (8.49), (8.53) and (8.54), we obtain

$$\mathbf{E} e^{h\Delta_3} \mathbf{1}_A \leq 2^N \cdot (2d+1)^{N+1} e^{(N+1)/8}, \quad \text{where } h = (8c_{52} d^{5/2} \tau)^{-1} \quad (8.55)$$

and $\mathbf{1}_A$ is the indicator function of the event A . By (8.31),

$$\mathbf{P}\{\Delta \geq x\} \leq \mathbf{P}\{\bar{A}\} + \mathbf{P}\left\{\Delta_3 \geq \frac{3x}{4}, A\right\} \leq \mathbf{P}\{\bar{A}\} + e^{-3hx/4} \mathbf{E} e^{h\Delta_3} \mathbf{1}_A. \quad (8.56)$$

Now (8.21) follows via (8.29), (8.55) and (8.56).

Let the quantities $\varepsilon, x_0 > 0$ be defined by the relations

$$\varepsilon = (2c_{50} d^{5/2} \tau)^{-1}, \quad e^{\varepsilon x_0} = (15d)^{N+1}. \quad (8.57)$$

Integrating by parts and using (8.21) and (8.57), we obtain

$$\mathbf{E} e^{\varepsilon \Delta} = \int_0^\infty \varepsilon e^{\varepsilon x} \mathbf{P}\{\Delta \geq x\} dx + 1,$$

$$\int_0^{x_0} \varepsilon e^{\varepsilon x} \mathbf{P}\{\Delta \geq x\} dx \leq \int_0^{x_0} \varepsilon e^{\varepsilon x} dx = e^{\varepsilon x_0} - 1 = (15d)^{N+1} - 1,$$

$$\int_{x_0}^{\infty} \varepsilon e^{\varepsilon x} \mathbf{P}\{\Delta \geq x\} dx \leq \int_{x_0}^{\infty} \varepsilon e^{-\varepsilon(x-x_0)} dx = 1,$$

and, hence, $\mathbf{E}e^{\varepsilon\Delta} \leq (15d)^{N+1} + 1 \leq (16d)^{N+1}$. Together with (8.19) and (8.57), this completes the proof of Theorem 8.1.

Theorem 8.2. *Let $\tau \geq 1$ and Conditions $\mathbf{A}(\tau)$, $\mathbf{B}$, $\mathbf{D}_\Phi$ and $\mathbf{C}_\Psi(p)$ be satisfied with $p = 0$ for the dyadic scheme described in Sections 5–7. Then there exist absolute positive constants c_{53}, c_{54} such that*

$$\mathbf{E} \exp\left(\frac{c_{53}\Delta(X, Y)}{d^{5/2}\tau}\right) \leq \exp\left(c_{54}(N+1)\ln^* d\right), \quad (8.58)$$

where $\Delta(X, Y)$ is defined in (1.1) with $s = 2^N$.

P r o o f. By Lemma 7.4 and by the condition $\mathbf{A}(\tau)$,

$$\mathcal{L}(\mathbf{U}_{n,k}^j) \in \mathcal{A}_j^*(c_{30}\tau, 4, 4), \quad 1 \leq k \leq 2^{N-n}, j = 1, \dots, 2d, \quad (8.59)$$

for any positive integer n satisfying $\tau^2 d^3 / 2^n \leq c_{29}$ and $1 \leq n \leq N$. Put

$$c_{55} = \min\{c_{29}, c_{41}^2 c_{30}^{-2}, c_{30}^{-2}, \frac{1}{2}\}. \quad (8.60)$$

Let the integer M be defined by the relation

$$\frac{d^3 \tau^2}{c_{55}} \leq 2^M < \frac{2d^3 \tau^2}{c_{55}}. \quad (8.61)$$

It is clear that, by (8.60) and (8.61) and since $\tau \geq 1$, we have $M \geq 1$. Let $M \leq N$. Write

$$\begin{aligned} \tilde{X}_m &= 2^{-M/2}(X_{(m-1) \cdot 2^{M+1}} + \dots + X_{m \cdot 2^M}) = 2^{-M/2}U_{M,m}, \\ \tilde{Y}_m &= 2^{-M/2}(Y_{(m-1) \cdot 2^{M+1}} + \dots + Y_{m \cdot 2^M}) = 2^{-M/2}V_{M,m}, \end{aligned} \quad (8.62)$$

$m = 1, \dots, 2^{N-M}$. It is easy to see that, according to Lemma 5.2 and Remark 5.1, the random vectors $2^{-M/2}U_{n,k}$, $M \leq n \leq N$, $1 \leq k \leq 2^{N-n}$, are the quantile transformations of the vectors $2^{-M/2}V_{M,1} = \tilde{Y}_1, \dots, 2^{-M/2}V_{M,2^{N-M}} = \tilde{Y}_{2^{N-M}}$ associated with the operators $\mathbf{A}_{n,k}$, $M \leq n \leq N$, $1 \leq k \leq 2^{N-n}$. One can verify that the corresponding reduced dyadic scheme (considered with the change of 2^N by 2^{N-M} and with multiplying of all vectors by $2^{-M/2}$) satisfies the Conditions $\mathbf{B}$ and $\mathbf{D}_\Phi$. For example, Condition $\mathbf{B}_2$ for $\text{cov } \tilde{X}_m$, $1 \leq m \leq 2^{N-M}$, obviously follows from (6.3), (6.19) and (8.62).

Let us show that the reduced dyadic scheme satisfies Condition $\mathbf{A}^*(\tau_0)$, where

$$\tau_0 = c_{30}\tau \cdot 2^{-M/2}. \quad (8.63)$$

Indeed, using (8.59)–(8.61) and Lemma 3.1, we see that

$$\mathcal{L}(2^{-M/2}\mathbf{U}_{n,k}^j) \in \mathcal{A}_j^*(\tau_0, 4, 4), \quad n = M, \dots, N, \quad (8.64)$$

$1 \leq k \leq 2^{N-n}$, $j = 1, \dots, 2d$. By (6.29), the minimal eigenvalues of covariance operators of the vectors $2^{-M/2}\mathbf{U}_{n,k}^j$ from (8.64) cannot be less than 1. Moreover, by (8.60), (8.61) and (8.63) we have $\tau_0 d^{1/2} \leq 1$. Therefore, from (3.4), (3.5) and (8.64) it follows that

$$\mathcal{L}(2^{-M/2}\mathbf{U}_{n,k}^j) \in \overline{\mathcal{A}}_j(\tau_0, 4), \quad n = M, \dots, N, \quad 1 \leq k \leq 2^{N-n}, \quad j = 1, \dots, 2d. \quad (8.65)$$

The relations (8.65) provide the validity of the required Condition $\mathbf{A}^*(\tau_0)$.

The relations (8.60), (8.61) and (8.63) imply that $\tau_0 d^{3/2} \leq c_{41}$. Thus, the reduced dyadic scheme satisfies the conditions of Theorem 8.1 with the change of 2^N by 2^{N-M} and τ by τ_0 . By this theorem,

$$\mathbf{E} \exp \left(2^{M/2} b \Delta(\tilde{X}, \tilde{Y}) \right) \leq \exp \left(c_{40}(N+1) \ln^* d \right), \quad (8.66)$$

where

$$\Delta(\tilde{X}, \tilde{Y}) = \max_{1 \leq k \leq 2^{N-M}} \left\| \sum_{i=1}^k \tilde{X}_i - \sum_{i=1}^k \tilde{Y}_i \right\|, \quad (8.67)$$

$$b = \frac{c_{56}}{d^{5/2}\tau} \quad \text{with some } c_{56} < \frac{1}{2}. \quad (8.68)$$

Note that Theorem 8.1 yields (8.66) with

$$b = \frac{c_{39}}{d^{5/2}\tau_0 \cdot 2^{M/2}} = \frac{c_{39}}{c_{30}d^{5/2}\tau}. \quad (8.69)$$

This implies (8.66) with b defined in (8.68) and $c_{56} = \min\{c_{39}c_{30}^{-1}, \frac{1}{3}\}$.

Write, by analogy with the proof of Theorem 8.1,

$$\Delta_1 = \max_{0 \leq k < 2^{N-M}} \max_{1 \leq s \leq 2^M} |S_{k \cdot 2^M + s} - S_{k \cdot 2^M}|, \quad (8.70)$$

$$\Delta_2 = \max_{0 \leq k < 2^{N-M}} \max_{1 \leq s \leq 2^M} |T_{k \cdot 2^M + s} - T_{k \cdot 2^M}|, \quad (8.71)$$

$$\Delta_3 = \max_{1 \leq k \leq 2^{N-M}} \|S_{k \cdot 2^M} - T_{k \cdot 2^M}\|, \quad (8.72)$$

where the maximums are taken over integer values of k and s . It is evident that from (1.1), (5.7), (5.23), (8.70)–(8.72) it follows that

$$\Delta(X, Y) \leq d^{1/2}\Delta_1 + d^{1/2}\Delta_2 + \Delta_3. \quad (8.73)$$

Hence, by the Hölder inequality

$$\begin{aligned} & \mathbf{E} \exp \left(b \frac{\Delta(X, Y)}{3} \right) \\ & \leq \left(\mathbf{E} \exp (b d^{1/2} \Delta_1) \mathbf{E} \exp (b d^{1/2} \Delta_2) \mathbf{E} \exp (b \Delta_3) \right)^{1/3}. \end{aligned} \quad (8.74)$$

Moreover, the relations (1.1), (8.62), (8.67), (8.72) imply

$$\Delta_3 = 2^{M/2} \Delta(\tilde{X}, \tilde{Y}). \quad (8.75)$$

From (8.68) it follows that $0 < d^{1/2} b \tau < \frac{1}{2}$. Recall that all d -dimensional Gaussian distributions belong to all classes $\mathcal{A}_d(\tau)$, $\tau \geq 0$. Using Lemma 2.6 with $t = d^{1/2} b$, $B^2 \leq 2^M C_2^2$ (see (6.3) and (6.19)), (8.61), (8.70), (8.71) and (8.68), we have

$$\mathbf{E} \exp(d^{1/2} b \Delta_l) \leq 2^{N-M} \cdot 3d \exp(4db^2 \cdot 2^M C_2^2) \leq cd \cdot 2^N, \quad l = 1, 2. \quad (8.76)$$

Now the inequality (8.58) can be easily derived from (8.66), (8.68), (8.74)–(8.76) by a suitable choice of c_{53} and c_{54} .

Let now $M > N$. From (8.61) it follows that $2^N < 2d^3 \tau^2 / c_{55}$. We shall use the rough bound

$$\Delta(X, Y) \leq d^{1/2} \Delta_4 + d^{1/2} \Delta_5, \quad (8.77)$$

where

$$\Delta_4 = \max_{1 \leq k \leq 2^N} |S_k|, \quad \Delta_5 = \max_{1 \leq k \leq 2^N} |T_k|. \quad (8.78)$$

Applying Lemma 2.6 with $t = (3\tau d^{3/2})^{-1}$, $B^2 \leq 2^N C_2^2$ (see (6.3) and (6.19)), we obtain the inequality

$$\mathbf{E} \exp(t \Delta_l) \leq 3d \exp(4t^2 2^N C_2^2) \leq cd, \quad l = 4, 5. \quad (8.79)$$

Now the statement of Theorem 8.2 in the case $M > N$ can be easily derived from (8.77)–(8.79) by an application of the Hölder inequality.

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