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TIME CHANGE REPRESENTATION OF STOCHASTIC INTEGRALS¹⁾

По теореме Дамбиса и Дубинса–Шварца любой стохастический интеграл $M = (\int_0^t H_s dW_s)_{t \in \mathbf{R}_+}$ по броуновскому движению может быть записан как броуновское движение со случайной заменой времени, т.е. $M = (\widehat{W}_{\widehat{T}_t})_{t \in \mathbf{R}_+}$ для некоторого броуновского движения $(\widehat{W}_\theta)_{\theta \in \mathbf{R}_+}$ и некоторой замены времени $(\widehat{T}_t)_{t \in \mathbf{R}_+}$. В [7] и [5] показано, что в этом утверждении броуновское движение можно заменить на (симметричное) α -устойчивое движение Леви. Используя процесс кумулянт семи-мартингала, мы даем короткие новые доказательства. Кроме того, мы показываем, что это утверждение не может быть распространено на другие процессы Леви.

Ключевые слова и фразы: устойчивые движения Леви, кумулянтный процесс, стохастический интеграл, замена времени.

1. Time change representations. We generally use the notation of [2]–[4]. The transposed of a vector or matrix x is denoted as x^T and its components are denoted by superscripts. Stochastic and Stieltjes integrals are written as $(H \cdot X_t) = \int_0^t H_s dX_s$. Increasing processes are identified with their corresponding Lebesgue–Stieltjes measure.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}_+}, \mathbf{P})$ be a filtered probability space as in [4, Definition I.1.2]. By *Lévy process* we refer to a process with stationary independent increments in the sense of [4, II.4.1]. Similar to [8, Definition 7.5.1], we make the following

Definition 1. Let $\alpha \in (0, 2]$. An α -stable Lévy motion is a Lévy process X such that X_1 (or equivalently any X_t) has a strictly α -stable distribution (i.e., $X_1 \sim S_\alpha(\sigma, \beta, \mu)$ for some $\alpha \in (0, 2] \setminus \{1\}$, $\sigma \in \mathbf{R}_+$, $\beta \in [-1, 1]$, $\mu = 0$ or $\alpha = 1$, $\sigma \in \mathbf{R}_+$, $\beta = 0$, $\mu \in \mathbf{R}$). We call X a *symmetric α -stable Lévy motion* if the distribution of X_1 (or equivalently any X_t) is even symmetric α -stable (i.e., $X_1 \sim S_\alpha(\sigma, 0, 0)$ for some $\alpha \in (0, 2]$, $\sigma \in \mathbf{R}_+$).

Definition 2. 1. A *time change* is a right-continuous increasing $[0, \infty]$ -valued process $(T_\theta)_{\theta \in \mathbf{R}_+}$ such that T_θ is a stopping time for any $\theta \in \mathbf{R}_+$. It is called *finite* if $T_\theta < \infty$ $\mathbf{P}$ -almost surely for any $\theta \in \mathbf{R}_+$.

2. By $\widehat{\mathcal{F}}_\theta := \mathcal{F}_{T_\theta}$ we define the *time-changed filtration* $(\widehat{\mathcal{F}}_\theta)_{\theta \in \mathbf{R}_+}$.

3. The *inverse time change* $(\widehat{T}_t)_{t \in \mathbf{R}_+}$ is defined as $\widehat{T}_t := \inf\{\theta \in \mathbf{R}_+ : T_\theta > t\}$.

4. A process X is called $(T_\theta)_{\theta \in \mathbf{R}_+}$ -*adapted* if X is constant on $[T_{\theta-}, T_\theta]$ for any $\theta \in \mathbf{R}_+$.

Remark. If X is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}_+}, \mathbf{P})$ and $(T_\theta)_{\theta \in \mathbf{R}_+}$ is a finite time change, then the process $(\widehat{X}_\theta)_{\theta \in \mathbf{R}_+}$ defined by $\widehat{X}_\theta := X_{T_\theta}$ is a semimartingale

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on $(\Omega, \mathcal{F}, (\widehat{\mathcal{F}}_\theta)_{\theta \in \mathbf{R}_+}, \mathbf{P})$ (cf., e.g., [2, (10.12)]). Many other properties are generally only preserved if X is $(T_\theta)_{\theta \in \mathbf{R}_+}$ -adapted. For details cf. [2, Chapter X].

We are concerned with the following representation property which follows from the Dambis and Dubins-Schwarz theorem (cf. [6, Theorem V.1.6]).

Theorem 1. *Let X be a standard Brownian motion. Moreover, let $M := H \cdot X$ for some $H \in L(X)$ ([2], [3]) such that $\int_0^t H_s^2 ds \rightarrow \infty$ for $t \rightarrow \infty$. Then there exist a filtration $(\widehat{\mathcal{F}}_\theta)_{\theta \in \mathbf{R}_+}$ on $(\Omega, \mathcal{F})$, a process $(\widehat{W}_\theta)_{\theta \in \mathbf{R}_+}$, and a finite time change $(\widehat{T}_t)_{t \in \mathbf{R}_+}$ such that*

- 1) $\widehat{W}$ is a $(\widehat{T}_t)_{t \in \mathbf{R}_+}$ -adapted Brownian motion on $(\Omega, \mathcal{F}, (\widehat{\mathcal{F}}_\theta)_{\theta \in \mathbf{R}_+}, \mathbf{P})$,
- 2) $M = (\widehat{W}_{\widehat{T}_t})_{t \in \mathbf{R}_+}$.

If we define the $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}_+}, \mathbf{P})$ -time change $(T_\theta)_{\theta \in \mathbf{R}_+}$ by $T_\theta := \inf\{t \in \mathbf{R}_+ : \int_0^t H_s^2 ds > \theta\}$, then we may choose $\widehat{\mathcal{F}}_\theta = \mathcal{F}_{T_\theta}$, $\widehat{W}_\theta = M_{T_\theta}$ and $\widehat{T}$ as the inverse time change of T . In particular, $\widehat{T}_t = \int_0^t H_s^2 ds$ for $t \in \mathbf{R}_+$.

The previous theorem can be generalized to symmetric α -stable Lévy motions (cf. [7] and [5]).

Theorem 2. *Let X be a symmetric α -stable Lévy motion. Moreover, let $M := H \cdot X$ for some $H \in L(X)$ such that $\int_0^t |H_s|^\alpha ds \rightarrow \infty$ for $t \rightarrow \infty$. Then there exist a filtration $(\widehat{\mathcal{F}}_\theta)_{\theta \in \mathbf{R}_+}$ on $(\Omega, \mathcal{F})$, a process $(\widehat{L}_\theta)_{\theta \in \mathbf{R}_+}$, and a finite time change $(\widehat{T}_t)_{t \in \mathbf{R}_+}$ such that*

- 1) $\widehat{L}$ is a $(\widehat{T}_t)_{t \in \mathbf{R}_+}$ -adapted Lévy process on $(\Omega, \mathcal{F}, (\widehat{\mathcal{F}}_\theta)_{\theta \in \mathbf{R}_+}, \mathbf{P})$ with $\text{Law}(\widehat{L}) = \text{Law}(X)$ and
- 2) $M = (\widehat{L}_{\widehat{T}_t})_{t \in \mathbf{R}_+}$.

If we define the $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}_+}, \mathbf{P})$ -time change $(T_\theta)_{\theta \in \mathbf{R}_+}$ by $T_\theta := \inf\{t \in \mathbf{R}_+ : \int_0^t |H_s|^\alpha ds > \theta\}$, then we may choose $\widehat{\mathcal{F}}_\theta = \mathcal{F}_{T_\theta}$, $\widehat{L}_\theta = M_{T_\theta}$ and $\widehat{T}$ as the inverse time change of T . In particular, $\widehat{T}_t = \int_0^t |H_s|^\alpha ds$ for $t \in \mathbf{R}_+$.

P r o o f. *First step.* Obviously, $(T_\theta)_{\theta \in \mathbf{R}_+}$ is a finite time change (cf. [4, I.1.28]). Choose $A_t = t$ for any $t \in \mathbf{R}_+$. Let $q \in \mathbf{Q}_+$ and $\tau_q := \inf\{t \geq q : \int_q^t |H_s|^\alpha ds > 0\}$. Note that $H = 0$ A -almost everywhere on $[q, \tau_q]$, which implies that $(\mathbf{P} \otimes A)([q, \tau_q]) = 0$. By Lemma 3, $1_{[q, \tau_q]} \cdot M = (1_{[q, \tau_q]} H) \cdot X$ is a semimartingale with characteristics $(0, 0, 0)$ and hence it is 0 up to indistinguishability. Therefore, M is constant on $[q, \tau_q]$ outside some $\mathbf{P}$ -null set. Since $(T_{\theta-}, T_\theta] = \cup\{[q, \tau_q] : q \in \mathbf{Q}_+ \text{ with } T_{\theta-} < q \leq T_\theta\}$, we have that M is constant on any interval $(T_{\theta-}, T_\theta]$, $\theta \in \mathbf{R}_+$. By right-continuity, it follows that M is $(T_\theta)_{\theta \in \mathbf{R}_+}$ -adapted.

Second step. Let $U \in L(X)$. The cumulant process of X is of the form $\mathcal{X}^X(U) = \kappa(U) \cdot A$, where $\kappa(\cdot)$ is given by $\kappa(u) = \gamma \int (e^{iux} - 1 - iux1_{[-1,1]}(x)) |x|^{-(1+\alpha)} dx$ for $\alpha \neq 2$ and $\kappa(u) = -\gamma u^2$ for $\alpha = 2$ (with some constant $\gamma \in \mathbf{R}_+$, cf., e.g., [8, p. 6] and Lemma 4). Straightforward calculations show that $\kappa(uv) = |v|^\alpha \kappa(u)$ for any $u, v \in \mathbf{R}$.

Third step. Let $u \in \mathbf{R}$. If $\mathcal{X}^M, \mathcal{X}^{\widehat{L}}$ denote the cumulant processes of M and $\widehat{L}$, respectively, Lemmas 2 and 5 yield that $\mathcal{X}^M(u) = \mathcal{X}^X(uH)$ and $\mathcal{X}^{\widehat{L}}(u)_\theta = \mathcal{X}^M(u)_{T_\theta}$. Together, it follows that

$$\begin{aligned} \mathcal{X}^{\widehat{L}}(u)_\theta &= \mathcal{X}^M(u)_{T_\theta} = \mathcal{X}^X(uH)_{T_\theta} = \kappa(uH) \cdot A_{T_\theta} = \kappa(u) |H|^\alpha \cdot A_{T_\theta} \\ &= \kappa(u) \int_0^{T_\theta} |H_s|^\alpha ds = \kappa(u) \theta = \mathcal{X}^X(u)_\theta \end{aligned}$$

for any $\theta \in \mathbf{R}_+$. From Lemma 4 and [4, II.2.42, II.4.19], it follows that $\widehat{L}$ is a Lévy process with the same distribution as X .

Fourth step. By [2, (10.2)] and Lemma 6, $\widehat{T}$ is a finite time change and $\widehat{T}_t = \int_0^t |H_s|^\alpha ds$ for $t \in \mathbf{R}_+$. The continuity of $\widehat{T}$ implies that $\widehat{L}$ is $(\widehat{T}_t)_{t \in \mathbf{R}_+}$ -adapted. Since $t \in [T_{\widehat{T}_t-}, T_{\widehat{T}_t}]$ and M is $(T_\theta)_{\theta \in \mathbf{R}_+}$ -adapted, it follows that $\widehat{L}_{\widehat{T}_t} = M_{(T \circ \widehat{T})_t} = M_t$ for $t \in \mathbf{R}_+$.

If we consider only nonnegative integrands H , we can extend the statement to asymmetric α -stable Lévy motions (including deterministic linear functions).

Theorem 3. Let X be an α -stable Lévy motion. Moreover, let $M := H \cdot X$ for some nonnegative $H \in L(X)$ such that $\int_0^t H_s^\alpha ds \rightarrow \infty$ for $t \rightarrow \infty$. Then there exist a filtration $(\mathcal{F}_\theta)_{\theta \in \mathbf{R}_+}$ on $(\Omega, \mathcal{F})$, a process $(\tilde{L}_\theta)_{\theta \in \mathbf{R}_+}$, and a finite time change $(\hat{T}_t)_{t \in \mathbf{R}_+}$ such that

1) $\tilde{L}$ is a $(\hat{T}_t)_{t \in \mathbf{R}_+}$ -adapted Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_\theta)_{\theta \in \mathbf{R}_+}, \mathbf{P})$ with $\text{Law}(\tilde{L}) = \text{Law}(X)$ and

2) $M = (\tilde{L}_{\hat{T}_t})_{t \in \mathbf{R}_+}$.

If we define the $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}_+}, \mathbf{P})$ -time change $(T_\theta)_{\theta \in \mathbf{R}_+}$ by $T_\theta := \inf\{t \in \mathbf{R}_+ : \int_0^t H_s^\alpha ds > \theta\}$, then we may choose $\mathcal{F}_\theta = \mathcal{F}_{T_\theta}$, $\tilde{L}_\theta = M_{T_\theta}$ and $\hat{T}$ as the inverse time change of T . In particular, $\hat{T}_t = \int_0^t H_s^\alpha ds$ for $t \in \mathbf{R}_+$.

P r o o f. This is shown along the same lines as the previous theorem. Only the second step has to be modified slightly. Here,

$$\begin{aligned} \kappa(u) &= \gamma_1 \int 1_{(0, \infty)}(x) (e^{iux} - 1) x^{-(1+\alpha)} dx \\ &\quad + \gamma_2 \int 1_{(-\infty, 0)}(x) (e^{iux} - 1) |x|^{-(1+\alpha)} dx, \quad \alpha < 1, \\ \kappa(u) &= iu\mu + \gamma_1 \int (e^{iux} - 1 - iux 1_{[-1, 1]}(x)) |x|^{-(1+\alpha)} dx, \quad \alpha = 1, \\ \kappa(u) &= \gamma_1 \int 1_{(0, \infty)}(x) (e^{iux} - 1 - iux) x^{-(1+\alpha)} dx \\ &\quad + \gamma_2 \int 1_{(-\infty, 0)}(x) (e^{iux} - 1 - iux) |x|^{-(1+\alpha)} dx, \quad 1 < \alpha < 2, \\ \kappa(u) &= -\gamma_1 u^2, \quad \alpha = 2 \end{aligned}$$

(with some constants $\gamma_1, \gamma_2 \in \mathbf{R}_+$, $\mu \in \mathbf{R}$, cf. [9, Theorem 14.7] or [10, III.1c, Theorem 3]). Again, straightforward calculations show that $\kappa(uv) = v^\alpha \kappa(u)$ for any $u, v \in \mathbf{R}_+$.

The following result shows that there are no other Lévy processes such that an analogous statement holds.

Theorem 4. Suppose that X is a real-valued Lévy process with the following property: For any $H \in \mathbf{R} \setminus \{0\}$ there exist a filtration $(\mathcal{F}_\theta)_{\theta \in \mathbf{R}_+}$ on $(\Omega, \mathcal{F})$, a process $(\tilde{L}_\theta)_{\theta \in \mathbf{R}_+}$, and a finite time change $(\hat{T}_t)_{t \in \mathbf{R}_+}$ such that

1) $\tilde{L}$ is a $(\hat{T}_t)_{t \in \mathbf{R}_+}$ -adapted Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_\theta)_{\theta \in \mathbf{R}_+}, \mathbf{P})$ with $\text{Law}(\tilde{L}) = \text{Law}(X)$ and

2) $HX = (\tilde{L}_{\hat{T}_t})_{t \in \mathbf{R}_+}$.

Then X is a symmetric α -stable Lévy motion.

If we claim the above property only for any $H \in (0, \infty)$, it follows that X is an α -stable Lévy motion.

P r o o f. *First step.* Let $H \in (0, \infty)$ and $u \in \mathbf{R}$. Note that the cumulant processes of Lévy processes are deterministic, linear in time, and characterized by the Lévy–Khinchine triplet. By Lemmas 2 and 5, the cumulant process of HX is of the form

$$\kappa(uH) A_t = \mathcal{X}^X(uH)_t = \mathcal{X}^{HX}(u)_t = \mathcal{X}^{\tilde{L}}(u)_{\hat{T}_t} = \kappa(u) A_{\hat{T}_t} \quad (1)$$

if we set $A_t = t$ and $\mathcal{X}^X(u) = \kappa(u) \cdot A$ denotes the cumulant process of X and $\tilde{L}$. The process $(A_{\hat{T}_t})_{t \in \mathbf{R}_+}$ can be written as $A_{\hat{T}_t} = a \cdot A_t + N_t$ for some $(\mathcal{F}_t)_{t \in \mathbf{R}_+}$ -predictable, nonnegative process a and some increasing process N which is orthogonal to Lebesgue measure (cf. [4, I.3.13]). Since the left-hand side of equation (1) is absolutely continuous with respect to Lebesgue measure, it follows that $\kappa(uH) \cdot A = (\kappa(u)a) \cdot A$, which implies that $\kappa(uH)$ is a multiple of $\kappa(u)$, say $\kappa(uH) = f(H)\kappa(u)$ with some function $f: (0, \infty) \rightarrow (0, \infty)$. Note that $f(HH') = f(H)f(H')$ and f is continuous because of the continuity

of κ . Therefore, $\ln(f(\exp(\cdot)))$ is a linear function, hence $f(H) = H^\alpha$ for some $\alpha \in \mathbf{R}$. Note that $\alpha = 0$ is only possible for $\kappa = 0$, in which case $X = 0$. Suppose that $\alpha \neq 0$.

Second step. Let $k_1, k_2 \in (0, \infty)$. Since $u \mapsto \exp(\kappa(u))$ is the characteristic function of X_1 , we have that

$$\begin{aligned} (P^{k_1 X_1} * P^{k_2 X_1})^\wedge(u) &= \exp(\kappa(uk_1) + \kappa(uk_2)) = \exp((k_1^\alpha + k_2^\alpha)\kappa(u)) \\ &= \exp(\kappa(u(k_1^\alpha + k_2^\alpha)^{-\alpha})) = (P^{(k_1^\alpha + k_2^\alpha)^{-\alpha} X_1})^\wedge(u), \end{aligned}$$

where $(P^Y)^\wedge$ denotes the characteristic function of Y . It follows that X_1 is strictly stable (cf. [8, Definition 1.1.1]).

Third step. If the property in the theorem holds for negative H , the same reasoning as above yields that $\kappa(u) = f(-1)\kappa(-u)$ for some $f(-1) \in \mathbf{R}$ that does not depend on u . From $\kappa(u) = f(-1)\kappa(-u) = (f(-1))^2\kappa(u)$ it follows that $f(-1) = 1$. This implies that $(P^{X_1})^\wedge(u) = (P^{X_1})^\wedge(-u)$, which in turn yields that $\text{Law}(X_1) = \text{Law}(-X_1)$. Hence, X_1 is symmetric stable.

It is an open question whether the previous theorem still holds if we remove the constraint that $\tilde{L}$ is $(\tilde{T}_t)_{t \in \mathbf{R}_+}$ -adapted.

2. Tools from stochastic calculus.

Lemma 1. *Let Z be a semimartingale such that Z, Z_- are $(\mathbf{C} \setminus \{0\})$ -valued. Then there exists an up to indistinguishability unique $\mathbf{C}$ -valued semimartingale X such that $X_0 = 0$ and $Z = Z_0 \mathcal{E}(X)$. It is given by $X = (1/Z_-) \cdot Z$.*

Proof. The existence and explicit form of X follows from $1 + (Z_-/Z_0) \cdot ((1/Z_-) \cdot Z) = 1 + (Z_- (1/Z_-)) \cdot Z/Z_0 = Z/Z_0$. Now let X be any semimartingale with $X_0 = 0$ and $Z = Z_0 \mathcal{E}(X)$. Then $X = ((1/Z_-) Z_-) \cdot X = (1/Z_-) \cdot (Z_- \cdot X) = (1/Z_-) \cdot Z$, which yields the uniqueness.

Definition 3. We call the process X in the previous lemma *stochastic logarithm* of Z and write $\mathcal{L}(Z) := X$.

Let X be a $\mathbf{R}^d$ -valued semimartingale whose characteristics (B, C, ν) relative to some truncation function $h: \mathbf{R}^d \rightarrow \mathbf{R}^d$ are given in the form $B = b \cdot A$, $C = c \cdot A$, $\nu = A \otimes F$, where $A \in \mathcal{A}_{\text{loc}}^+$ is a predictable process, b is a predictable $\mathbf{R}^d$ -valued process, c is a predictable $\mathbf{R}^{d \times d}$ -valued process whose values are nonnegative, symmetric matrices, and F is a transition kernel from $(\Omega \times \mathbf{R}_+, \mathcal{P})$ into $(\mathbf{R}^d, \mathcal{B}^d)$. By [4, Proposition II.2.9] such a representation always exists.

Definition 4. Suppose $\mathbf{R}^d$ -valued process $U \in L(X)$. The *cumulant process* $\mathcal{K}^X(U)$ of X in U is defined as the predictable part of finite variation in the canonical decomposition of the special semimartingale $\mathcal{L}(\exp(iU^\top \cdot X))$ (in the sense of [4, I.4.22]).

Remark. Note that $\exp(iU^\top \cdot X)$ is bounded and hence is a complex special semimartingale. Therefore, $\mathcal{L}(\exp(iU^\top \cdot X)) = \exp(-iU^\top \cdot X_-) \cdot \exp(iU^\top \cdot X)$ is a special semimartingale as well (cf. [2, (2.51)]).

Lemma 2. *Let $H \in L(X)$ and $U \in L(H^\top \cdot X)$. Then $\mathcal{K}^{H^\top \cdot X}(U) = \mathcal{K}^X(UH)$.*

Proof. From $H \in L(X)$, $U \in L(H^\top \cdot X)$, it follows that $UH \in L(X)$ (cf. [1, Proposition 5.1]). The assertion is now readily obtained from $\exp(iU \cdot (H^\top \cdot X)) = \exp(i(UH)^\top \cdot X)$.

Lemma 3. *Let $H \in L(X)$. Then the characteristics $(\tilde{B}, \tilde{C}, \tilde{\nu})$ of $H^\top \cdot X$ relative to some truncation function $h_1: \mathbf{R} \rightarrow \mathbf{R}$ are of the form $\tilde{B} = \tilde{b} \cdot A$, $\tilde{C} = \tilde{c} \cdot A$, $\tilde{\nu} = A \otimes \tilde{F}$, where*

$$\begin{aligned} \tilde{b}_t &= H_t^\top b_t + \int \left(h_1(H_t^\top x) - H_t^\top h(x) \right) F_t(dx), \\ \tilde{c}_t &= H_t^\top c_t H_t, \\ \tilde{F}_t(G) &= \int 1_G(H_t^\top x) F_t(dx) \quad \text{for any } G \in \mathcal{B} \text{ with } 0 \notin G. \end{aligned}$$

P r o o f. For locally bounded integrands the lemma is proved in [4, IX.5.3]. Since $H \in L_{\text{loc}}^1(X^c)$, it follows that $\tilde{C} = \langle H^\top \cdot X^c, H^\top \cdot X^c \rangle = (H^\top cH) \cdot A$ (cf. [1, Proposition 5.2] and [3, 1a]). Moreover, $\Delta(H^\top \cdot X) = H^\top \Delta X$ implies that $1_G(x) * \mu^{H^\top \cdot X} = 1_G(H^\top x) * \mu^X$ and hence $1_G(x) * \tilde{\nu} = 1_G(H^\top x) * \nu$ for any $G \in \mathcal{B}$ with $0 \notin G$. By [1, Proposition 5.2], there exist a set $\Delta \in \mathcal{P} \otimes \mathcal{B}^d$ and a predictable process $\bar{B} \in \mathcal{V}^d$ such that

$$\begin{aligned} X &= X_0 + X^c + x1_{\Delta^c}(x) * (\mu^X - \nu) + x1_{\Delta}(x) * \mu^X + \bar{B}, \\ H^\top \cdot X &= H^\top \cdot X^c + H^\top x1_{\Delta^c}(x) * (\mu^X - \nu) + H^\top x1_{\Delta}(x) * \mu^X + H^\top \cdot \bar{B}. \end{aligned}$$

Since X can also be written in its canonical semimartingale representation $X = X_0 + X^c + h(x) * (\mu^X - \nu) + (x - h(x)) * \mu^X + B$, it follows from [1, Proposition 5.3] that $\bar{B} = B + (x1_{\Delta^c}(x) - h(x)) * \nu = (b + \int (x1_{\Delta^c}(x) - h(x)) F(dx)) \cdot A$. Similarly, the canonical semimartingale representation of $H^\top \cdot X$ equals $H^\top \cdot X = H^\top \cdot X^c + h_1(H^\top x) * (\mu^X - \nu) + (H^\top x - h_1(H^\top x)) * \mu^X + \tilde{B}$, which yields, again using [1, Proposition 5.3], that

$$\begin{aligned} \tilde{B} &= H^\top \cdot \bar{B} + (h_1(H^\top x) - H^\top x1_{\Delta^c}(x)) * \nu \\ &= \left(H^\top \left(b + \int (x1_{\Delta^c}(x) - h(x)) F(dx) \right) \right) \cdot A \\ &\quad + \left(\int (h_1(H^\top x) - H^\top x1_{\Delta^c}(x)) F(dx) \right) \cdot A \\ &= \left(H^\top b + \int (h_1(H^\top x) - H^\top h(x)) F(dx) \right) \cdot A. \end{aligned}$$

This proves the assertion.

Lemma 4. Let $U \in L(X)$. Then $\mathcal{X}^X(U) = \kappa(U) \cdot A$, where

$$\kappa(U)_t = iU_t^\top b_t - \frac{1}{2}U_t^\top c_t U_t + \int (e^{iU_t^\top x} - 1 - iU_t^\top h(x)) F_t(dx).$$

P r o o f. *First step.* From Lemmas 2 and 3 it follows that $\mathcal{X}^X(U) = \mathcal{X}^{U^\top \cdot X}(1)$ and $\kappa(U) = \kappa^{U^\top \cdot X}(1)$, where $\kappa^{U^\top \cdot X}$ is defined as κ but for the process $U^\top \cdot X$ instead of X . Therefore, it suffices to prove the statement for $\mathbf{R}^1$ -valued X with $X_0 = 0$ and $U = 1$.

Second step. Let $Y := iX$. Applying Itô's formula (e.g. as in [1, Lemma 5.5]) yields that $\exp(iX) = \exp(Y) = \mathcal{E}(Y + \frac{1}{2}\langle Y^c, Y^c \rangle + (e^x - 1 - x) * \mu^Y)$ and hence $L := \mathcal{L}(\exp(iX)) = Y + \frac{1}{2}\langle Y^c, Y^c \rangle + (e^x - 1 - x) * \mu^Y$. Note that $Y = i(B + X^c + h(x) * (\mu^X - \nu) + (x - h(x)) * \mu^X)$. Together, we obtain that

$$L = iB + iX^c + ih(x) * (\mu^X - \nu) - \frac{1}{2}\langle X^c, X^c \rangle + (e^{ix} - 1 - ih(x)) * \mu^X.$$

On the other hand, the canonical decomposition of L is $L = V + L^c + x * (\mu^L - \nu^L) = V + L^c + (e^{ix} - 1) * (\mu^X - \nu)$, where $V \in \mathcal{V}$ is predictable. From [1, Proposition 5.3] it follows that $V = iB - \frac{1}{2}C + (e^{ix} - 1 - ih(x)) * \nu = \kappa(1) \cdot A$. Therefore $\mathcal{X}^X(1) = \kappa(1) \cdot A$ as claimed.

R e m a r k. In particular, the cumulant process $\mathcal{X}^X(U)$ for $U \in L(X)$ can be obtained from the cumulant processes $\mathcal{X}^X(u)$, $u \in \mathbf{R}^d$. Also note that by [4, II.2.42] the cumulant processes $\mathcal{X}^X(u)$, $u \in \mathbf{R}^d$, uniquely determine the characteristics (B, C, ν) of X .

Lemma 5. Let $(T_\theta)_{\theta \in \mathbf{R}_+}$ be a finite time change such that X is $(T_\theta)_{\theta \in \mathbf{R}_+}$ -adapted. Then the process $(\hat{X}_\theta)_{\theta \in \mathbf{R}_+}$ defined by $\hat{X}_\theta := X_{T_\theta}$ is a semimartingale on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_\theta)_{\theta \in \mathbf{R}_+}, \mathbf{P})$ whose characteristics $(\hat{B}, \hat{C}, \hat{\nu})$ are of the form

$$\hat{B}_\theta = B_{T_\theta}, \quad \hat{C}_\theta = C_{T_\theta}, \quad 1_G * \hat{\nu}_\theta = 1_G * \nu_{T_\theta}$$

for $\theta \in \mathbf{R}_+$, $G \in \mathcal{B}^d$ with $0 \notin G$. Moreover, its cumulant process satisfies

$$\mathcal{K}^{\hat{X}}(u)_\theta = \mathcal{K}^X(u)_{T_\theta} \quad \text{for } \theta \in \mathbf{R}_+, u \in \mathbf{R}^d.$$

P r o o f. The statement on $\hat{C}$ follows from [2, (10.17)a]. The $(T_\theta)_{\theta \in \mathbf{R}_+}$ -adaptedness of X implies that $1_G * \mu_{T_\theta-}^X = \sum_{t < T_\theta} 1_G(\Delta X_t) = \sum_{t \leq T_\theta} 1_G(\Delta X_t) = 1_G * \mu_{T_\theta}^X$, hence μ^X is $(T_\theta)_{\theta \in \mathbf{R}_+}$ -adapted in the sense of [2, (10.25)]. Since $\Delta \hat{X}_\theta 1_{\{\Delta \hat{X}_\theta \neq 0\}} = \Delta X_{T_\theta-} 1_{\{\Delta X_{T_\theta-} \neq 0\}}$, we have that $1_G * \mu_{\hat{\theta}}^{\hat{X}} = \sum_{\hat{\theta} \leq \theta} 1_G(\Delta \hat{X}_{\hat{\theta}}) = \sum_{t < T_\theta} 1_G(\Delta X_t) = 1_G * \mu_{T_\theta}^X$ for $\theta \in \mathbf{R}_+$, $G \in \mathcal{B}^d$ with $0 \notin G$. It follows from [2, (10.27)] that $1_G * \hat{\nu}_\theta = 1_G * \nu_{T_\theta}$. Moreover, we have $\hat{X}_\theta - (x - h(x)) * \mu_{\hat{\theta}}^{\hat{X}} = X_{T_\theta} - (x - h(x)) * \mu_{T_\theta}^X$. Therefore, the statement on $\hat{B}$ follows from [2, (10.17)b]. Finally, observe that by Lemma 4,

$$\begin{aligned} \mathcal{K}^{\hat{X}}(u)_\theta &= iu^\top \hat{B}_\theta - \frac{1}{2} u^\top \hat{C}_\theta u + (e^{iu^\top x} - 1 - iu^\top h(x)) * \hat{\nu}_\theta \\ &= iu^\top B_{T_\theta} - \frac{1}{2} u^\top C_{T_\theta} u + (e^{iu^\top x} - 1 - iu^\top h(x)) * \nu_{T_\theta} = \mathcal{K}^X(u)_{T_\theta}. \end{aligned}$$

R e m a r k. Even though the semimartingale property is preserved under arbitrary finite time changes, Lemma 5 generally ceases to hold. For example, even a continuous process may have fixed times of discontinuity after a time change. Under $(T_\theta)_{\theta \in \mathbf{R}_+}$ -adaptedness, however, many properties are preserved (cf. [2, Chapter X]).

We end with a characterisation of $L(X)$ for α -stable Lévy motions X which can be found in [5] (cf. also [7]). We give a more direct proof here.

Lemma 6. *Let X be a nonconstant α -stable Lévy motion and H a predictable real-valued process. Then $H \in L(X)$ if and only if $\int_0^t |H_s|^\alpha ds < \infty$ $\mathbf{P}$ -almost surely for any $t \in \mathbf{R}_+$.*

P r o o f. Since $H = H' + H''$ with $H'_t := H_t 1_{\{|H_t| > 1\}} + 1_{\{|H_t| \leq 1\}}$ and $H''_t := (H_t - 1) 1_{\{|H_t| \leq 1\}}$, it suffices to consider the case $|H| \geq 1$. By (B, C, ν) we denote the characteristics of X relative to $h: \mathbf{R} \rightarrow \mathbf{R}$, i.e., $B_t = bt$, $C_t = ct$, $\nu = \lambda \otimes F$, where (b, c, F) is the Lévy–Khinchine triplet of $\text{Law}(X_1)$. By [1, Proposition 5.2], $H \in L(X)$ holds if and only if $H \in L_{\text{loc}}^1(X^c)$, $Hx 1_{\{|Hx| \leq 1\}} \in G_{\text{loc}}(\mu^X)$, $|Hx 1_{\{|Hx| > 1\}}| * \mu^X \in \mathcal{V}$, and $H \in L_s(\tilde{B})$, where $\tilde{B} = B + (x 1_{\{|Hx| \leq 1\}} - h(x)) * \nu$. By $\alpha \in (0, 2]$ we denote the index of stability of X .

Case 1. $\alpha = 2$. In this case $\mu^X = 0$, $B = 0$, and X is a continuous local martingale. Therefore, $H \in L(X)$ if and only if $H \in L_{\text{loc}}^1(X)$, i.e., if and only if $H^2 \cdot \langle X, X \rangle \in \mathcal{V}$. Since $\langle X, X \rangle_t = ct$, the claim follows.

Case 2. $\alpha \neq 2$. Note that $X^c = 0$. Moreover, $|Hx 1_{\{|Hx| > 1\}}| * \mu^X \leq |Hx 1_{\{|x| > 1\}}| * \mu^X \in \mathcal{V}$ holds automatically because $|Hx 1_{\{|x| > 1\}}| * \mu_t^X$ is a finite sum for fixed t . By [4, II.1.33c], $Hx 1_{\{|Hx| \leq 1\}} \in G_{\text{loc}}(\mu^X)$ if and only if $H^2 x^2 1_{\{|Hx| \leq 1\}} * \nu \in \mathcal{V}$. But note that $\nu = \lambda \otimes F$ and $\int H_t^2 x^2 1_{\{|H_t x| \leq 1\}} F(dx) = |H_t|^\alpha \int y^2 1_{\{|y| \leq 1\}} F(dy)$ because the density of F is proportionate to $|x|^{-(1+\alpha)}$ on $(0, \infty)$ (respectively $(-\infty, 0)$). Since $\int y^2 1_{\{|y| \leq 1\}} F(dy) < \infty$, it follows that $H^2 x^2 1_{\{|Hx| \leq 1\}} * \nu \in \mathcal{V}$ if and only if $F = 0$ or $\int_0^\cdot |H_t|^\alpha dt \in \mathcal{V}$. For the drift part, we have to distinguish several cases.

Firstly, let $\alpha = 1$. If we choose a symmetric h , then $(x 1_{\{|Hx| \leq 1\}} - h(x)) * \nu = 0$ because F is symmetric. Therefore, $\tilde{B} = B$ is a deterministic, linear function. This implies that $H \in L_s(\tilde{B})$ if and only if $\tilde{B} = 0$ or $\int_0^\cdot |H_t| dt \in \mathcal{V}$.

Secondly, let $\alpha \in (1, 2)$. By [9, Theorem 14.7], we can use $h(x) = x$ and have $B = 0$ for this choice of h . Consequently, $\tilde{B} = -x 1_{\{|Hx| > 1\}} * \nu$. Therefore, $H \in L_s(\tilde{B})$ holds if $|Hx 1_{\{|Hx| > 1\}}| * \nu \in \mathcal{V}$. Since $\int |y| 1_{\{|y| > 1\}} F(dy) < \infty$, a similar reasoning as above yields that $|Hx 1_{\{|Hx| > 1\}}| * \nu \in \mathcal{V}$ if and only if $F = 0$ or $\int_0^\cdot |H_t|^\alpha dt \in \mathcal{V}$.

Finally, let $\alpha \in (0, 1)$. By [9, Theorem 14.7], we can use $h(x) = 0$ and have $B = 0$ for this choice of h . Therefore, $\tilde{B} = x 1_{\{|Hx| \leq 1\}} * \nu$ in this case. Note that $\int |y| 1_{\{|y| \leq 1\}} F(dy) < \infty$. As for $\alpha > 1$, it follows that $H \in L_s(\tilde{B})$ holds if $\int_0^\cdot |H_t|^\alpha dt \in \mathcal{V}$.

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**МНОГОЧЛЕНЫ, ОРТОГОНАЛЬНЫЕ ОТНОСИТЕЛЬНО
ПОЛИНОМИАЛЬНОГО РАСПРЕДЕЛЕНИЯ, И
ФАКТОРИАЛЬНО-СТЕПЕННОЙ ФОРМАЛИЗМ**

Ключевые слова и фразы: биномиальное распределение, пуассоновское распределение, полиномиальное распределение, многочлены Кравчука, многочлены Пуассона–Шарлье, факториально-степенные многочлены, ортогональные относительно полиномиального распределения многочлены, комбинаторные тождества, тождество Лерха.

Введение. Первоначальный вариант этого сообщения был направлен в редакцию журнала «Теория вероятностей и ее применения» еще в начале 1985 года. Однако в том варианте отсутствовали указания на иллюстрации таких применений полученных результатов и предложенной комбинаторной техники, которые могли бы представлять чисто вероятностный интерес. Поэтому автор решил в конечном счете ограничиться публикацией формулировок главных результатов в тезисах [6]. Впоследствии к ним были даны комментарии в докладе на IV Всемирном конгрессе Общества им. Бернулли в Вене (август 1996 г.), а в докладе на Международном

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