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S. Louhichi, Rates of Convergence in the CLT for Some Weakly Dependent Random Variables, *Теория вероятн. и ее примен.*, 2001, том 46, выпуск 2, 345–364

DOI: <https://doi.org/10.4213/tvp3922>

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RATES OF CONVERGENCE IN THE CLT FOR SOME WEAKLY DEPENDENT RANDOM VARIABLES

Устанавливается скорость сходимости в центральной предельной теореме для некоторых слабо зависимых последовательностей при условии степенного затухания ковариаций. Эти последовательности предполагаются ассоциированными или удовлетворяющими общему свойству гауссовских процессов и положительно (или отрицательно) зависимых случайных величин.

Мы распространяем на наш случай метод Линдберга, следуя методу работы [21] для последовательностей с перемешиванием.

В ходе доказательства получены верхние границы для расстояния Дадли между распределением нормированной суммы слабо зависимых случайных величин и стандартным нормальным распределением, а также неравенства типа Розенталя для моментов частичных сумм.

Ключевые слова и фразы: ассоциированность, положительная зависимость, отрицательная зависимость, теорема Берри-Эссеена, центральная предельная теорема Линдберга, моментные неравенства, неравенства Розенталя.

1. Introduction and notation. Let $(X_n)_{n \in \mathbb{Z}}$ be a sequence of associated random variables (r.v.'s), i.e., for every finite subcollection $X_{i_1}, \dots, X_{i_n}$ and every pair of coordinatewise nondecreasing functions $h, k: \mathbb{R}^n \rightarrow \mathbb{R}$

$$\text{cov}(h(X_{i_1}, \dots, X_{i_n}), k(X_{i_1}, \dots, X_{i_n})) \geq 0,$$

whenever the covariance is defined. This definition was introduced by Esary, Proschan, and Walkup [13], mainly for the sake of applications in reliability and statistics.

In the following we assume that $X_i = 0$ for $i \leq 0$, $\mathbf{E}X_i = 0$ and $\mathbf{E}X_i^2 < \infty$. We put $S_n = X_1 + \dots + X_n$, $S_0 = 0$, $V_n = \mathbf{D}S_n$. We suppose without loss of generality that $|\text{cov}(X_i, X_j)| \leq 1$ for all $i, j \in \mathbb{N}$. Finally, we denote by π_n the deviation of the distribution of $S_n/\sqrt{V_n}$ from the normal law, that is

$$\pi_n := \sup_{x \in \mathbb{R}} \left| \mathbf{P} \left\{ \frac{S_n}{\sqrt{V_n}} \leq x \right\} - \psi(x) \right|, \quad (1)$$

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where ψ denotes the distribution of a standard normal law. Bounds of π_n are known as Berry–Esseen type estimates. The first Berry–Esseen type estimation under association was given by Wood [26]. He obtained a rate of order $O(n^{-1/5})$. Wood's rate is far from the optimal Berry–Esseen's rate $O(n^{-1/2})$ (see [18]).

The best possible rate for association was obtained by Birkel [3]. He obtained a rate of order $O(n^{-1/2} \ln n)$ if the associated sequence has a finite moment of order $(3+\rho)$, if $\inf_n V_n/n > 0$ and if the sequence $U(n)$ defined by $U(n) = \sup_{k \in \mathbb{N}} \sum_{j: |j-k| \geq n} \text{cov}(X_j, X_k)$ decreases exponentially fast to 0. Recently, Bulinski [4], [6], [7] gave Berry–Esseen type estimates for associated random fields.

The main task of this paper is to study the behavior of π_n under a power decay of the covariance of the associated process. We cannot expect the Berry–Esseen rate $O(n^{-1/2})$, since an example in [3] shows that a rate of order $O(n^{-1/2} \ln n)$ cannot be obtained under an hyperbolic decay of $U(n)$.

We prove that $\pi_n = O(n^{-1/3})$, if $\inf_n V_n/n > 0$, $\sup_{i \geq 0} \|X_i\|_{3+\rho} < \infty$ and if

$$\sup_{k \geq 0} \sum_{i=1}^{\infty} i \text{cov}^{1-1/q}(X_k, X_{k-i}) < \infty \quad (2)$$

hold for some $\rho > 0$, $1 < q < 1 + \rho$ (here and in the sequel $\|X\|_p$ denotes $(\mathbf{E}|X|^p)^{1/p}$). We do not know whether this rate is optimal, but we note that also for bounded mixing sequences, Rio [21] obtained the convergence rate $O(n^{-1/3})$ under the condition $\sum_{i=1}^{\infty} i\alpha_i < \infty$, where $(\alpha_i)_{i \in \mathbb{N}}$ denotes the strong mixing coefficients introduced by Rosenblatt [23].

If moreover the sequences are bounded, then the condition of association on the processes may be weakened. In fact, we only assume that the sequence $(X_n)_{n \in \mathbb{N}}$ satisfies the following property.

Definition 1. A sequence $(X_n)_{n \in \mathbb{N}}$ is said to be (AG) if there exists a sequence $(\tilde{X}_n)_{n \in \mathbb{N}}$ of r.v.'s such that $\mathbf{D}X_i \leq \mathbf{D}\tilde{X}_i$, for all $i \in \mathbb{N}$, and

$$|\text{cov}(h(X_i, i \in A), k(X_j, j \in B))| \leq \sum_{i \in A} \sum_{j \in B} \left\| \frac{\partial h}{\partial x_i} \right\|_{\infty} \left\| \frac{\partial k}{\partial x_j} \right\|_{\infty} |\text{cov}(\tilde{X}_i, \tilde{X}_j)|$$

holds for all disjoint finite subsets A and B of $\mathbb{N}$ and for all functions h and k having bounded first differentials.

This property is known to hold for Gaussian and associated sequences (cf., for example, [3] and [11]). We call the previous inequality the (AG) property. Random variables satisfying the (AG) property will be called (AG) r.v.'s.

Since associated sequences satisfy the (AG) property with $(X_n) = (\tilde{X}_n)$ (cf. Lemma 3-1 in [3]), we deduce that for an associated sequence $(X_n)_{n \in \mathbb{N}}$, the sequence $(g(X_n))_n$ satisfies also the (AG) property with $(\tilde{X}_n) = (\|g'\|_{\infty} X_n)$ (g' denotes the first derivative of the real-valued function g).

The (AG) property with $(X_n) = (\tilde{X}_n)$ is also satisfied by negatively dependent r.v.'s (cf. [7]); we recall that the sequence $(X_n)_{n \in \mathbf{N}}$ is negatively dependent if for all coordinatewise nondecreasing real-valued functions h and k

$$\text{cov}(h(X_i, i \in A), k(X_i, i \in B)) \leq 0$$

holds for all finite and disjoint subsets A and B of $\mathbf{N}$. We refer to [14] for more about this notion of negative dependence.

Contrary to mixing, this class of weakly dependent processes does not directly involve σ -algebras generated, but their dependence structure is contained in second order properties. This means that orthogonality is equivalent to independence. Notice that such sequences are special weakly dependent sequences as defined in [11]. This implies that nontrivial adaptations are necessary here to adapt Rio's [21] method (for strongly mixing sequences) in Lindeberg theorem.

For (AG) sequences bounded by M , we obtain a convergence rate $O(n^{-\delta/3})$, if $0 < \delta \leq 1$ and if

$$\sup_{k \geq 0} \sum_{i=1}^{\infty} i^{\delta} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| < \infty. \quad (3)$$

Also a convergence rate $O((\ln n)^{-p/3})$ holds if

$$\sup_{k \geq 0} \sum_{i=1}^{\infty} \frac{\phi(M(i+1))}{(M(i+1))^2} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| < \infty, \quad (4)$$

for some function ϕ in Φ verifying for some $p > 0$, $\phi(x) \sim x^2 \ln^p(x)$ as $x \rightarrow \infty$, where Φ is the class of twice differentiable Orlicz functions defined by

$$\Phi = \{\phi: \mathbf{R}^+ \rightarrow \mathbf{R}^+; \quad \phi \text{ is convex, } \phi(0) = \phi'(0) = 0, \\ \phi'' \text{ is nondecreasing, concave}\}. \quad (5)$$

(See Theorem 1 below.)

The method of the proofs which is based on Lindeberg decomposition provides upper bounds of the Dudley distance between the distribution of $S_n/\sqrt{V_n}$ and the standard normal distribution. We recall that the Dudley distance $d_3(X, Y)$ between the distributions of two real-valued r.v.'s X and Y is defined by $d_3(X, Y) = \sup |\mathbf{E}(f(X) - f(Y))|$, where the supremum is taken over all $C^{(3)}$ functions on $\mathbf{R}$ such that $\sup_{x \in \mathbf{R}} \max(|f'(x)|, |f^{(2)}(x)|, |f^{(3)}(x)|) \leq 1$ (see [19] also [20]). In particular, we obtain for nondegenerate sequences of centered and associated r.v.'s having finite moment of order $(3 + \rho)$

$$d_3\left(\frac{S_n}{\sqrt{V_n}}, Y\right) \\ \leq C \left(\frac{1}{\sqrt{n}} + n V_n^{-1-(2+\rho)/(2(3+\rho))} \sum_{i=1}^{n-1} i \sup_k \text{cov}^{1-1/q}(X_k, X_{k-i}) \right), \quad (6)$$

where $1 < q < 1 + \rho$ and C denotes a positive constant not depending on n (cf. Theorem 4). For bounded, (AG) and weakly stationary sequences, we obtain

$$d_3\left(\frac{S_n}{\sqrt{V_n}}, Y\right) \leq C \left\{ \frac{1}{\sqrt{n}} + \frac{1}{V_n} \sum_{i=1}^{n-1} i |\text{cov}(X_1, X_{1+i})| \right. \\ \left. + \frac{n}{V_n} \sum_{i=0}^{n-1} \min\left(1, (i+1) \frac{M}{\sqrt{V_n}}\right) \sup_k |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| \right\}, \quad (7)$$

in this case the constant C is universal. Such bounds are useful in the study of Berry–Esseen estimates of triangular arrays of (AG) sequences, such as the study of the asymptotic normality of kernel estimators of associated r.v.'s.

Finally, we prove Rosenthal type inequalities of order $r \in]2, 3]$ for bounded sequences of (AG) r.v.'s (cf. Proposition 1). The main interest of those inequalities is that they give the right bound for integrated moments of nonparametric estimators (see [9] for more about).

We conclude this section with some remarks concerning the Lindeberg method (cf. [15]).

This method was adapted by several authors in order to obtain central limit theorems for triangular arrays. In the mixing setting it was first used by Doukhan and Portal [10]. They studied the rates of convergence in the multidimensional central limit theorem. Adapting the Lindeberg method, Rio [22] obtained the optimal Berry–Esseen rate for a class of weakly dependent r.v.'s (see also [21]). Recently, Dedecker [8] uses this approach to prove a CLT for stationary random fields under a projective criterion.

This paper is organized as follows. In the next section, we give the main results. Theorems 1 and 2 deal with the Berry–Esseen estimation under association and the (AG) property. Theorems 3 and 4 give upper bounds of the Dudley distances under the weak dependence previously considered. The proofs of the main results are given in Section 3. Proposition 2 is the main tool for the proofs. An Appendix is dedicated to prove this proposition.

2. Main results.

Theorem 1. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of (AG) r.v.'s. Suppose that $(X_n)_{n \in \mathbb{N}}$ is bounded by M and that*

$$\exists n_0 \in \mathbb{N}^* \text{ such that } \sigma^2 = \sigma_{n_0}^2 := \inf_{\{n \geq n_0, 0 \leq k < n\}} \frac{V_n - V_k}{n - k} > 0. \quad (8)$$

(i) *If condition (3) holds for some $\delta \in]0, 1]$, then, for any $n \geq n_0$*

$$\sup_{x \in \mathbb{R}} \left| \mathbf{P} \left\{ \frac{S_n}{\sqrt{V_n}} \leq x \right\} - \psi(x) \right| = O(n^{-\delta/3}). \quad (9)$$

(ii) *Suppose now that condition (4) holds for some function ϕ in Φ (defined by (5)) verifying for some positive p , $\phi(x) \sim x^2 \ln^p(x)$ as $x \rightarrow \infty$.*

Then

$$\sup_{x \in \mathbf{R}} \left| \mathbf{P} \left\{ \frac{S_n}{\sqrt{V_n}} \leq x \right\} - \psi(x) \right| = O((\ln n)^{-p/3}). \quad (10)$$

R e m a r k 1. In the mixing case, the order obtained $O(n^{-\delta/2})$ is close to the optimal rate (cf. Theorem 2 in [21]). The exponent $\delta/2$ is here replaced by $\delta/3$: the loss is due to the (AG) property (cf. Lemma 2 below).

R e m a r k 2. There is a connection between an estimate of the logarithm rate in the central limit theorem and the law of the iterated logarithm, i.e., the relation

$$\overline{\lim}_{n \rightarrow \infty} (2V_n \ln \ln V_n)^{-1/2} S_n = 1 \quad \text{a.s.},$$

as it can be seen from Theorem 7.2 in [18] (cf. also [5] or [12]).

For unbounded and associated sequences, we obtain the following Berry-Esseen estimates.

Theorem 2. Let $(X_n)_{n \in \mathbf{N}}$ be a sequence of centered and associated r.v.'s satisfying (8). Suppose that there exist some constants $\rho > 0$ and $q \in]1, 1 + \rho[$ such that condition (2) holds and that

$$\sup_i \|X_i\|_{3+\rho} < \infty. \quad (11)$$

Then

$$\sup_{x \in \mathbf{R}} \left| \mathbf{P} \left\{ \frac{S_n}{\sqrt{V_n}} \leq x \right\} - \psi(x) \right| = O(n^{-1/3}). \quad (12)$$

R e m a r k 3. For associated sequence, condition (8) is satisfied as soon as

$$\inf_{n \geq 1, k \geq 0} \frac{\mathbf{D}(S_{n+k} - S_k)}{n} > 0,$$

this follows from the inequality: $\mathbf{D}(S_{n+k} - S_k) \leq \mathbf{D}S_{n+k} - \mathbf{D}S_k$, proved for associated sequences. Condition (8) holds also if $(X_n)_{n \in \mathbf{N}}$ is a weakly stationary sequence such that $v_k := V_k - V_{k-1}$ converges to a positive real number (see (3.7) and (3.8) of [21]).

R e m a r k 4. For bounded and associated sequences fulfilling (8), the bound (12) of Theorem 2 is obtained from (9) with $\delta = 1$. In order to check this, only let q tends to infinity in (2) (recall that associated sequences satisfy the (AG) property with $\tilde{X}_i = X_i$).

R e m a r k 5. Birkel [1] obtained the rate given by (9) for associated sequence $(X_n)_{n \in \mathbf{N}}$ fulfilling $\mathbf{E}(X_i) = 0$, $\sup_i \mathbf{E}|X_i|^3 < \infty$, $\inf_n V_n/n > 0$ and

$$\sup_{r \in \mathbf{N}} \text{cov}(X_r, X_{r+n}) = O(n^{-2(3+2\delta)/(3-2\delta)}) \quad \text{for some } \delta \in]0, 1]. \quad (13)$$

Clearly, condition (3) stated for bounded and associated sequences (see Theorem 1) improves (13).

R e m a r k 6. Bulinski [7] gave convergence rates in the CLT for positively and negatively dependent random fields under a power decrease of the correlation. For bounded and associated or negatively dependent sequences, Theorem 1 here improves on Bulinski's [7] results. Bulinski's results are however available for random fields having finite moments of order $s \in]2, 3]$ and also the summation is performed over finite sets of arbitrary configuration. Moreover, employing the Cox–Grimmett coefficient Bulinski demonstrates that his results are optimal, in a sense, for rapid power-type decay of covariance. Let us note that Bulinski's method is different from ours.

The next theorem provides upper bounds on the Dudley distances between the distribution of $S_n/\sqrt{V_n}$ and the standard normal distribution of Y .

Theorem 3. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of (AG) r.v.'s. Suppose moreover that $(X_n)_{n \in \mathbb{N}}$ is a weakly stationary sequence and uniformly bounded by M . Then there exists an universal positive constant C for which the bound (7) holds.*

It is possible to omit the stationarity assumption in the associated case, as it is proved by the forthcoming result.

Theorem 4. *Let $(X_n)_{n \in \mathbb{N}}$ be a centered sequence of associated r.v.'s satisfying (8) and (11) for some $\rho > 0$. Then for any $q \in]1, 1 + \rho[$, there exists a positive constant C not depending on n , for which inequality (6) holds.*

R e m a r k 7. Let us note that no assumption of stationarity is required in Theorem 4. This is due to the inequality: $v_k = \mathbf{D}S_k - \mathbf{D}S_{k-1} \geq 0$, proved by using positive dependence (see the proof of Theorem 4).

An immediate application of Theorems 3 and 4 is the following.

Corollary 1. *Let $(X_n)_{n \in \mathbb{N}}$ be a centered sequence of bounded r.v.'s. Suppose moreover that the weakly stationary sequence $(X_n)_{n \in \mathbb{N}}$ satisfies the (AG) property, (8) and (3) for some $\delta \in]0, 1]$. Then $\overline{\lim}_{n \rightarrow \infty} n^{\delta/2} d_3(S_n/\sqrt{V_n}, Y) < \infty$.*

Corollary 2. *Let $(X_n)_{n \in \mathbb{N}}$ be a centered sequence of associated r.v.'s. Suppose that $(X_n)_{n \in \mathbb{N}}$ satisfies all the assumptions of Theorem 2 for some $\rho > 0$ and $q \in]1, 1 + \rho[$. Then $\overline{\lim}_{n \rightarrow \infty} n^{(2+\rho)/(2(3+\rho))} d_3(S_n/\sqrt{V_n}, Y) < \infty$.*

The next result is a moment inequality for the sum of bounded sequences satisfying the (AG) property. Let $s_n^2 := \sum_{i=1}^n \sum_{j=1}^n |\text{cov}(X_i, X_j)|$.

Proposition 1. *If the centered sequence $(X_n)_{n \in \mathbb{N}}$ is bounded by M and satisfies the (AG) property, then for any $\phi \in \Phi$ (Φ is defined by (5))*

$$\mathbf{E}(\phi(|S_n|)) \leq c \left(\phi(s_n) + \sum_{k=1}^n \sum_{i=1}^{k-1} \frac{\phi((i+1)M)}{(i+1)^2 M^2} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| \right),$$

where c denotes an universal positive constant.

Remark 8. When $\phi(x)$ equals x^r for $x \geq 0$ and for $r \in]2, 3]$, Proposition 1 yields a Rosenthal type inequality for negatively dependent r.v.'s (cf. [17] for such inequalities).

Proposition 1 applied to the centered sequence $(g(X_n))_n$ when $(X_n)_{n \in \mathbb{N}}$ is an associated sequence and g is a bounded function with a bounded first derivative, yields for $r \in]2, 3]$

$$\mathbf{E}(|S_n(g)|^r) \leq c \left(s_n^r(g) + \sum_{k=1}^n \sum_{i=1}^{k-1} (i+1)^{r-2} \|g\|_\infty^{r-2} \|g'\|_\infty^2 \text{cov}(X_k, X_{k-i}) \right),$$

where

$$S_n(g) = g(X_1) + \cdots + g(X_n), \quad s_n^2(g) = \sum_{i=1}^n \sum_{j=1}^n |\text{cov}(g(X_i), g(X_j))|.$$

This last inequality improves the corresponding one in [24] by a multiplicative factor n^ε . However the involved inequality (4.6) in [24] holds for any $r > 2$ and any unbounded function g .

We refer to [25] for versions of Rosenthal inequality for associated random fields. Birkel's moment inequalities (cf. [2]) are also deduced from Proposition 1.

3. Proofs. The method of the proofs is based on Rio [21] but the details are quite different. The main step in Rio's proofs is to evaluate

$$\Delta_k(f) = \mathbf{E} \left[f(S_k) - f(S_{k-1}) - \frac{v_k}{2} f''(S_{k-1}) \right], \quad (14)$$

where $v_n = \mathbf{D}S_n - \mathbf{D}S_{n-1}$ and f is a function that belongs to the set

$$\mathcal{F}(b_2, b_3) := \left\{ f: \mathbf{R} \rightarrow \mathbf{R}; \|f''\|_\infty \leq b_2, \|f^{(3)}\|_\infty \leq b_3 \right\}.$$

So the purpose of the following proposition is to control $\Delta_k(f)$ for associated or (AG) sequences. All the results are shown to be deduced from the forthcoming proposition.

From now on, C stands for a positive constant that does not depend on the considered sequence. It may, however, take different values in each appearance. Define, for some $p > 1$, $\|X\|_p := \sup_i \|X_i\|_p$.

Proposition 2. *If the centered sequence $(X_n)_{n \in \mathbb{N}}$ is bounded by M and satisfies the (AG) property, then for any function $f \in \mathcal{F}(b_2, b_3)$*

$$|\Delta_k(f)| \leq C \sum_{i=0}^{k-1} \min(b_2, b_3(i+1)M) |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})|. \quad (15)$$

If $(X_n)_{n \in \mathbb{N}}$ is a sequence of centered and associated r.v.'s fulfilling (11) for some $\rho > 0$, then for any $f \in \mathcal{F}(b_2, b_3)$ and for any p, q such that $1 < p <$

$(3 + \rho)(1 + \rho)/(2 + \rho)$ and $1 < q < 1 + \rho$, there holds

$$\begin{aligned} |\Delta_k(f)| \leq C & \left\{ b_3 \|X\|_{3+\rho} \sum_{i=1}^{k-1} i \operatorname{cov}(X_k, X_{k-i}) \right. \\ & + b_3 \|X\|_{3+\rho}^{1+2/q} \sum_{i=1}^{k-1} i \operatorname{cov}^{1-1/q}(X_k, X_{k-i}) \\ & + b_2^{1/(3+\rho)} b_3^{(2+\rho)/(3+\rho)} \|X\|_{3+\rho}^{2/p+(2+\rho)/(3+\rho)} \\ & \times \left. \sum_{i=1}^{k-1} i \operatorname{cov}^{1-1/p}(X_k, X_{k-i}) + b_3 \mathbf{E}|X_k|^3 \right\}. \end{aligned} \quad (16)$$

3.1. Berry–Esseen type estimates. We recall the notation (3.5) of [21]. For $m = \lfloor n/3 \rfloor$, let

$$\begin{aligned} \phi_{n,1} &= \phi_m := \mathbf{E}(e^{itS_m}), \quad \phi_{n,2} = \mathbf{E}(e^{it(S_{2m}-S_m)}), \\ \phi_{n,3} &= \mathbf{E}(e^{it(S_n-S_{2m})}), \quad \text{and} \quad V := \mathbf{D}S_m + \mathbf{D}(S_{2m} - S_m) + \mathbf{D}(S_n - S_{2m}). \end{aligned}$$

It follows from Esseen’s inequality that for any $T > 0$,

$$\begin{aligned} \sup_{x \in \mathbf{R}} \left| \mathbf{P}\{S_n \leq x\} - \psi\left(\frac{x}{\sqrt{V_n}}\right) \right| & \leq C \int_{-T}^T \left| \frac{\phi_n(t) - \exp(-Vt^2/2)}{|t|} \right| dt \\ & + \frac{C}{T\sqrt{V}} + \sup_{x \in \mathbf{R}} \left| \psi\left(\frac{x}{\sqrt{V}}\right) - \psi\left(\frac{x}{\sqrt{V_n}}\right) \right|. \end{aligned} \quad (17)$$

We first evaluate $\sup_{x \in \mathbf{R}} |\psi(x/\sqrt{V}) - \psi(x/\sqrt{V_n})|$. Inequality (3.25) of [21] yields, if (8) holds,

$$\sup_{x \in \mathbf{R}} \left| \psi\left(\frac{x}{\sqrt{V}}\right) - \psi\left(\frac{x}{\sqrt{V_n}}\right) \right| \leq \frac{C}{n} \sum_{i=1}^{\infty} \min(i, n) \sup_{k \geq i} |\operatorname{cov}(X_k, X_{k-i})|. \quad (18)$$

Now define, for some t in $\mathbf{R}^+$,

$$M(t) = \sup_k \sum_{i=1}^{\infty} \min(t(i+1), 1) |\operatorname{cov}(\tilde{X}_k, \tilde{X}_{k-i})|.$$

The forthcoming lemma gives bounds of $M(t)$ as soon as the assumptions of Theorems 1 and 2 are fulfilled.

Lemma 1. *Let $(X_n)_{n \in \mathbf{N}}$ be a sequence of (AG) r.v.’s. For t small enough, the following properties hold.*

- 1) *If (3) holds for some $\delta \in]0, 1]$, then $M(t) = O(t^\delta)$.*
- 2) *If (4) holds for some ϕ in Φ , then $M(t) = O(t^{-2}/\phi(Mt^{-1}))$. Hence $M(t) = O(|\ln t|^{-p})$.*
- 3) *Suppose now that $\sup_k \sum_{i=1}^{\infty} (i+1) |\operatorname{cov}(\tilde{X}_k, \tilde{X}_{k-i})| < \infty$. Then $M(t) = O(t)$.*

P r o o f. The proofs of 1) and 3) are immediate. The bound 2) follows from Lemma A.1 in [21].

From (18) and from some calculations like in Lemma 1, we deduce that $\sup_{x \in \mathbf{R}} |\psi(x/\sqrt{V}) - \psi(x/\sqrt{V_n})|$ has an order $O(n^{-\delta})$ (respectively, $O(\ln^{-p} n)$ and $O(n^{-1})$) if the (AG) sequence satisfies (3) (respectively, (4) and (2)).

Hence, taking into account Esseen's inequality, Theorems 1 and 2 are proved if we control $|\phi_n(t) - \exp(-Vt^2/2)|$. For this, write

$$\left| \phi_n(t) - \exp\left(-\frac{Vt^2}{2}\right) \right| \leq \left| \phi_n(t) - \prod_{i=1}^3 \phi_{n,i}(t) \right| + \left| \prod_{i=1}^3 \phi_{n,i}(t) - \exp\left(-\frac{Vt^2}{2}\right) \right|.$$

So we have two terms to evaluate.

(i) A bound for $|\phi_n(t) - \prod_{i=1}^3 \phi_{n,i}(t)|$ is given from the following lemma.

Lemma 2. *If the centered sequence $(X_n)_{n \in \mathbf{N}}$ is (AG), then for any $l \in]1, n[$,*

$$\left| \text{cov}(\exp(itS_l), \exp(it(S_n - S_l))) \right| \leq 4t^2 \sup_i \sum_{j=1}^{\infty} \min(j, n) |\text{cov}(\tilde{X}_i, \tilde{X}_{i+j})|.$$

P r o o f. The (AG) property applied with $h(x_1, \dots, x_l) = f(t(x_1 + \dots + x_l))$, $k(x_{l+1}, \dots, x_n) = f(t(x_{l+1} + \dots + x_n))$ ($f(x)$ is either $\cos x$ or $\sin x$) yields

$$\begin{aligned} \left| \text{cov}(\exp(itS_l), \exp(it(S_n - S_l))) \right| &\leq 4t^2 \sum_{j=1}^l \sum_{i=l+1}^n |\text{cov}(\tilde{X}_i, \tilde{X}_j)| \\ &\leq 4t^2 \sup_i \sum_{j=1}^{\infty} \min(j, n) |\text{cov}(\tilde{X}_i, \tilde{X}_{i+j})|. \end{aligned}$$

We deduce from Lemmas 1 and 2 that $|\phi_n(t) - \prod_{i=1}^3 \phi_{n,i}(t)|$ has an order $O(t^2 n^{1-\delta})$ (respectively, $O(t^2 n \ln^{-p} n)$ and $O(t^2)$) if the (AG) sequence satisfies (3) (respectively, (4) and (2)).

(ii) In order to bound $|\prod_{i=1}^3 \phi_{n,i}(t) - \exp(-Vt^2/2)|$, we shall use the following propositions.

Proposition 3. *Let $(X_n)_{n \in \mathbf{N}}$ be a sequence of centered r.v.'s satisfying condition (8). Suppose that $(X_n)_{n \in \mathbf{N}}$ is bounded by M and satisfies the (AG) property. Then there exists a positive constant C such that for any $t \in]0, \sigma^{-1}[$, $\delta \in]0, 1[$ and $n \geq n_0$,*

$$\left| \phi_n(t) - \exp\left(-\frac{V_n t^2}{2}\right) \right| \leq C \sup_k \sum_{i=1}^{k-1} \min(M |t|(i+1), 1) |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})|. \quad (19)$$

We deduce from Proposition 3, that the left-hand side of (19) is of order $O(t^\delta)$ (respectively, $O(|\ln t|^{-p})$) if condition (3) (respectively, (4)) is satisfied.

For unbounded sequences of associated r.v.'s, we obtain the following.

Proposition 4. *Suppose that all the assumptions of Theorem 2 hold for $\rho > 0$ and q in $]1, 1 + \rho[$. Then there exists a positive constant C such that for any $t \in]0, \sigma^{-1}]$ and for any $n \geq n_0$,*

$$\left| \phi_n(t) - \exp\left(-\frac{V_n t^2}{2}\right) \right| \leq C t^{(2+\rho)/(3+\rho)}. \quad (20)$$

Let us come back to the proof of Theorems 1 and 2 (the proof of Propositions 3 and 4 is given in the Subsection 3.1.1).

We suppose first that the sequence is (AG) and that (3) holds; then inequality (3.11) of [21] yields, for any $t \in]0, \sigma^{-1}]$,

$$\left| \prod_{i=1}^3 \phi_{n,i}(t) - \exp\left(-\frac{V t^2}{2}\right) \right| \leq C \left\{ t^{3\delta} + t^\delta \exp\left(-\frac{t^2 n \sigma^2}{4}\right) \right\}.$$

Hence, we obtain from a previous bound $|\phi_n(t) - \exp(-V t^2/2)| \leq C \{t^{3\delta} + t^\delta \exp(-t^2 n \sigma^2/4) + t^2 n^{1-\delta}\}$.

The last inequality together with (17) yields:

$$\sup_{x \in \mathbf{R}} \left| \mathbf{P}\{S_n \leq x\} - \psi\left(\frac{x}{\sqrt{V}}\right) \right| \leq C \left(T^{3\delta} + T^2 n^{1-\delta} + n^{-\delta/2} + \frac{1}{T\sqrt{n}} \right).$$

Thus the bound (9) is obtained with $T = n^{-1/2+\delta/3}$.

We suppose now that the (AG) sequence satisfy (ii) of Theorem 1. Then (19) together with (17) and (3.11), (3.21) of [21], yields for n large enough and for $T \geq n^{-1/2}$,

$$\sup_{x \in \mathbf{R}} \left| \mathbf{P}\{S_n \leq x\} - \psi\left(\frac{x}{\sqrt{V}}\right) \right| \leq C \left(\frac{1}{T\sqrt{n}} + \frac{T^2 n}{\ln^p n} + \ln^{-p} n + \frac{\ln(nT^2)}{|\ln T|^{3p}} \right).$$

So, we get the bound (10) if we take $T = (\ln^{p/3} n)/\sqrt{n}$ in the last inequality.

Finally, suppose that the sequence is associated and assume that all the conditions of Theorem 2 hold. Then, as before, (3.11) of [21] together with (20) and (17) yield:

$$\begin{aligned} & \sup_{x \in \mathbf{R}} \left| \mathbf{P}\{S_n \leq x\} - \psi\left(\frac{x}{\sqrt{V}}\right) \right| \\ & \leq C \left(T^{3(2+\rho)/(3+\rho)} + T^2 + n^{-(2+\rho)/2(3+\rho)} + \frac{1}{T\sqrt{n}} \right). \end{aligned}$$

Hence the bound (12) follows with $T = n^{-1/6}$.

3.1.1. Proof of Propositions 3 and 4. We have, for all $n \geq n_0$:

$$\begin{aligned} & \left| \phi_n(t) - \exp\left(-\frac{V_n t^2}{2}\right) \right| \\ & \leq \sum_{k=1}^n \exp\left(-\frac{(V_n - V_k) t^2}{2}\right) \left| \phi_k(t) - \exp\left(-\frac{v_k t^2}{2}\right) \phi_{k-1}(t) \right| \\ & \leq C_\sigma (t^{-2} \wedge n) \max_{k \leq n} \left| \phi_k(t) - \exp\left(-\frac{v_k t^2}{2}\right) \phi_{k-1}(t) \right|. \end{aligned} \quad (21)$$

Clearly,

$$\begin{aligned} \max_{k \leq n} \left| \phi_k(t) - \exp\left(-\frac{v_k t^2}{2}\right) \phi_{k-1}(t) \right| &\leq \max_{k \leq n} \left| \phi_k(t) - \left(1 - \frac{v_k t^2}{2}\right) \phi_{k-1}(t) \right| \\ &+ \max_{k \leq n} \left| \exp\left(-\frac{v_k t^2}{2}\right) - \left(1 - \frac{v_k t^2}{2}\right) \right|. \end{aligned} \quad (22)$$

Hence Proposition 2 applied when $f(x)$ is either $\cos(tx)$ or $\sin(tx)$, implies that the quantity $|\phi_k(t) - (1 - v_k t^2/2) \phi_{k-1}(t)|$ is of order $O(t^{2+(2+\rho)/(3+\rho)})$ if the sequence is associated and if (8), (11) and (2) are satisfied. For (AG) sequences fulfilling condition (8), Proposition 2 yields the bound:

$$\begin{aligned} \max_{k \leq n} \left| \phi_k(t) - \left(1 - \frac{v_k t^2}{2}\right) \phi_{k-1}(t) \right| \\ \leq C t^2 \max_{k \leq n} \sum_{i=1}^{k-1} \min(1, |t|(i+1)M) |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})|. \end{aligned} \quad (23)$$

Propositions 3 and 4 follow then if we prove that the quantity $|\exp(-v_k t^2/2) - (1 - v_k t^2/2)|$ has a same order as $|\phi_k(t) - (1 - v_k t^2/2) \phi_{k-1}(t)|$. For this we use the following lemma.

Lemma 3 (see [21]). *If $|v| \leq x + y$, for some $x, y \geq 0$, then $|\exp(v) - 1 - v| \leq (x + y^2/2) \sup(1, \exp(v))$.*

Now, let t be a positive real number. Clearly

$$\begin{aligned} v_k &= \left(\mathbf{E} X_k^2 \mathbf{1}_{|X_k| \leq t-1} + 2 \sum_{i=1}^{k-1} \mathbf{1}_{i\|X\|_2|t| \leq 1} \text{cov}(X_k, X_{k-i}) \right) \\ &+ \left(\mathbf{E} X_k^2 \mathbf{1}_{|X_k| > t-1} + 2 \sum_{i=1}^{k-1} \mathbf{1}_{i\|X\|_2|t| > 1} \text{cov}(X_k, X_{k-i}) \right) =: y_k + x_k. \end{aligned}$$

We first give a bound for x_k . It is easily seen that,

$$\begin{aligned} x_k &\leq \mathbf{E} X_k^2 \min(1, |t||X_k|) + \sum_{i=1}^{k-1} \min(1, |t|(i+1)\|X\|_2) |\text{cov}(X_k, X_{k-i})| \\ &=: M_{3,k}(t, \|X\|_2), \end{aligned}$$

also,

$$y_k^2 t^4 \leq 4t^4 \left((\mathbf{E} X_k^2 \mathbf{1}_{|X_k| \leq t-1})^2 + \left(\sum_{i=1}^{k-1} \mathbf{1}_{i\|X\|_2|t| \leq 1} \text{cov}(X_k, X_{k-i}) \right)^2 \right)$$

On the one hand, the Hölder inequality yields:

$$\begin{aligned} (\mathbf{E} X_k^2 \mathbf{1}_{|X_k| \leq t-1})^2 &\leq \mathbf{E} X_k^4 \mathbf{1}_{|X_k| \leq t-1} \leq |t|^{-1} \mathbf{E} |X_k|^3 \mathbf{1}_{|X_k| \leq |t|-1} \\ &\leq |t|^{-2} \mathbf{E} |X_k|^2 \min(|X_k|, |t|, 1). \end{aligned}$$

On the other hand (denoting by $\|X\|_2 = \sup_i \|X_i\|_2$),

$$\begin{aligned} & \left(\sum_{i=1}^{k-1} \mathbf{1}_{\|X\|_2|t|\leq 1} \text{cov}(X_k, X_{k-i}) \right)^2 \\ & \leq 2\|X\|_2^2 \sum_{j=1}^{k-1} \sum_{i=1}^j \mathbf{1}_{\|X\|_2|t|\leq 1} |\text{cov}(X_k, X_{k-j})| \\ & \leq 2\|X\|_2^2 \sum_{j=1}^{k-1} j \mathbf{1}_{\|X\|_2|t|\leq 1} |\text{cov}(X_k, X_{k-j})| \\ & \leq 2|t|^{-2} \sum_{j=1}^{k-1} j|t| \|X\|_2 \mathbf{1}_{\|X\|_2|t|\leq 1} |\text{cov}(X_k, X_{k-j})| \\ & \leq 2|t|^{-2} \sum_{j=1}^{k-1} \min(j|t| \|X\|_2, 1) |\text{cov}(X_k, X_{k-j})|. \end{aligned}$$

The previous inequalities, together with Lemma 3, yield

$$\begin{aligned} & \left| \exp\left(-\frac{v_k t^2}{2}\right) - \left(1 - \frac{v_k t^2}{2}\right) \right| \leq t^2 M_{3,k}(t, \|X\|_2) \\ & \leq t^2 \sum_{i=1}^{k-1} \min(t\|X\|_2(i+1), 1) |\text{cov}(X_k, X_{k-i})|. \end{aligned}$$

The last inequality together with inequalities (21)–(23) complete the proof of Propositions 3 and 4.

3.2. Estimation of the Dudley distance. Let $(Y_i)_{i \in \mathbb{N}}$ be a centered sequence of independent and normal r.v.'s. For (AG) sequences, we suppose that $\mathbf{D} Y_i = V_n/n$ for all i . While for associated sequences, we suppose that $\mathbf{D} Y_i = v_i$ (recall that the association property yields $v_i = V_i - V_{i-1} \geq 0$).

Assume furthermore that $(Y_i)_{i \in \mathbb{N}}$ is independent of $(X_i)_{i \in \mathbb{N}}$. For any $k \in [0, n]$, let $T_{k+1,n} = \sum_{i=k+1}^n Y_i$, with the convention that $T_{n+1,n} = 0$, where $T_n = Y_1 + \dots + Y_n$. Let $f \in \mathcal{F}(b_2, b_3)$. Clearly, $\mathbf{D} T_n = V_n$ and $\mathbf{E}(f(S_n) - f(T_n)) = \sum_{k=1}^n [\Delta_{1,k}(f) - \Delta_{2,k}(f)]$, where

$$\begin{aligned} \Delta_{1,k}(f) &= \mathbf{E} \left[f(S_{k-1} + X_k + T_{k+1,n}) - f(S_{k-1} + T_{k+1,n}) \right. \\ & \quad \left. - \frac{v_k}{2} f''(S_{k-1} + T_{k+1,n}) \right], \\ \Delta_{2,k}(f) &= \mathbf{E} \left[f(S_{k-1} + Y_k + T_{k+1,n}) - f(S_{k-1} + T_{k+1,n}) \right. \\ & \quad \left. - \frac{v_k}{2} f''(S_{k-1} + T_{k+1,n}) \right]. \end{aligned}$$

C o n t r o l o f $|\Delta_{2,k}(f)|$. Since $(Y_i)_{i \in \mathbb{N}}$ is independent of $(X_i)_{i \in \mathbb{N}}$, Taylor's formula yields:

$$|\Delta_{2,k}(f)| \leq b_3 \mathbf{E}|Y_k|^3 + b_2 |v_k - \mathbf{E}Y_k^2|. \tag{24}$$

When Y_k is a normal law with $\mathbf{D} Y_k = V_n/n$ and $\mathbf{E} Y_k = 0$, there holds

$$|\Delta_{2,k}(f)| \leq C \left(b_3 \left(\frac{V_n}{n} \right)^{3/2} + \frac{b_2}{n} |nv_k - V_n| \right).$$

If moreover $(X_i)_{i \in \mathbf{N}}$ is a weakly stationary sequence, then $|nv_k - V_n| \leq 2 \sum_{r=1}^{n-1} (r + n \mathbf{1}_{r \geq k}) |\mathbf{E} X_1 X_{1+r}|$ and $\sum_{k=1}^n |nv_k - V_n| \leq 4n \sum_{r=1}^{n-1} r |\mathbf{E} X_1 X_{1+r}|$. Hence we get for (AG) sequences:

$$\sum_{k=1}^n |\Delta_{2,k}(f)| \leq C \left(b_3 n \left(\frac{V_n}{n} \right)^{3/2} + b_2 \sum_{r=1}^{n-1} r |\mathbf{E} X_1 X_{1+r}| \right).$$

In the associated case, Y_k is supposed to be a normal law with $\mathbf{D} Y_k = v_k$ and $\mathbf{E} Y_k = 0$, then (24) yields: $|\Delta_{2,k}(f)| \leq C b_3 v_k^{3/2}$. Hölder's inequality yields: $v_k^{3/2} \leq C \sum_{i=1}^{k-1} i \operatorname{cov}(X_k, X_{k-i})$. Hence we obtain for associated sequences

$$\sum_{k=1}^n |\Delta_{2,k}(f)| \leq C b_3 n \sum_{i=1}^{n-1} i \sup_{k \geq i} \operatorname{cov}(X_k, X_{k-i}).$$

C o n t r o l o f $|\Delta_{1,k}(f)|$. Since $T_{k+1,n}$ is independent of the sequence $(X_i)_{i \in \mathbf{N}}$, we obtain (using also Fubini's theorem and the invariance by translations of the class $\mathcal{F}(b_2, b_3)$)

$$|\Delta_{1,k}(f)| \leq \sup_{g \in \mathcal{F}(b_2, b_3)} |\Delta_k(g)|,$$

where $\Delta_k(f)$ is defined by (14). Proposition 2 gives then bounds of $|\Delta_k(f)|$ for associated or (AG) sequences.

The proof of Theorems 3 and 4 follows then from the previous estimations with $b_2 = V_n^{-1}$ and $b_3 = V_n^{-3/2}$.

3.3. Proof of Proposition 1. The proof of this proposition follows from inequality (15) exactly as in the proof of Theorem 3 in [21].

Appendix: Proof of Proposition 2. The proof of Proposition 2 is done in two steps.

S t e p 1: The Lindeberg-Rio decomposition.

Lemma 4. *If $(X_n)_{n \in \mathbf{N}}$ is a centered sequence of r.v.'s, then for any integer a_k in $[1, k]$ and for all twice differentiable real-valued function f*

$$\begin{aligned} |\Delta_k(f)| &\leq \left| \mathbf{E} \left(f(S_k) - f(S_{k-1}) - X_k f'(S_{k-1}) - \frac{1}{2} X_k^2 f''(S_{k-1}) \right) \right| \\ &\quad + |\mathbf{E} X_k f'(S_{k-a_k})| \\ &\quad + \sum_{i=1}^{a_k-1} \left| \operatorname{cov} \left(X_k, X_{k-i} \int_0^1 [f''(S_{k-i-1} + u X_{k-i}) - f''(S_{k-i-1})] du \right) \right| \\ &\quad + \sum_{i=1}^{a_k-1} \left| \operatorname{cov} (X_k, X_{k-i} [f''(S_{k-i-1}) - f''(S_{k-2i-1})]) \right| \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{a_k-1} \left| \text{cov}(X_k X_{k-i}, f''(S_{k-2i-1})) \right| + \frac{1}{2} \left| \text{cov}(X_k^2, f''(S_{k-1})) \right| \\
& + \left| \sum_{i=1}^{a_k-1} \mathbf{E} X_k X_{k-i} \mathbf{E} f''(S_{k-2i-1}) - \sum_{i=1}^{k-1} \mathbf{E} X_k X_{k-i} \mathbf{E} f''(S_{k-1}) \right|. \quad (25)
\end{aligned}$$

Lemma 4 follows using the relations $g(S_n) - g(0) = \sum_{k=1}^n [g(S_k) - g(S_{k-1})]$ and $g(S_k) - g(S_{k-1}) = X_k \int_0^1 g'(S_{k-1} + uX_k) du$, for convenient functions g .

Step 2: Covariance inequalities. Now we evaluate the covariance quantities of Lemma 4. We will use repeatedly the following inequalities:

$$x \mathbf{1}_{x \leq 1} \leq \min(x, 1), \quad \mathbf{1}_{1 \leq x} \leq \min(x, 1) \quad \text{for } x \geq 0. \quad (26)$$

1. Proof of (15). Suppose here that the sequence $(X_n)_{n \in \mathbb{N}}$ is (AG) and bounded by M . Let a_k (defined in Lemma 4) equal to $\min(b_2 b_3^{-1} M^{-1}, k)$.

It follows from Taylor's formula and the (AG) property that:

$$\begin{aligned}
& \left| \mathbf{E} \left(f(S_k) - f(S_{k-1}) - X_k f'(S_{k-1}) - \frac{1}{2} X_k^2 f''(S_{k-1}) \right) \right| \\
& \leq \mathbf{E} X_k^2 \min(b_2, b_3 |X_k|) \leq (b_2 \wedge b_3 M) \mathbf{D} \tilde{X}_k. \quad (27)
\end{aligned}$$

Now we apply the (AG) property with $k(x_1, \dots, x_{k-1}) = f''(x_1 + \dots + x_{k-1})$ and with a function h (with bounded first derivative) that equals to x^2 if $|x| \leq M$ and to a suitable constant otherwise. We obtain

$$\begin{aligned}
\left| \text{cov}(X_k^2, f''(S_{k-1})) \right| & \leq 2 \min \left(b_2 \mathbf{E} X_k^2, b_3 M \sum_{i=1}^{k-1} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| \right) \\
& \leq 2 \min(b_2, b_3 M) \sum_{i=0}^{k-1} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})|. \quad (28)
\end{aligned}$$

In a similar way, the (AG) property yields

$$\begin{aligned}
|\mathbf{E} X_k f'(S_{k-a_k})| & \leq b_2 \sum_{i=a_k}^{k-1} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| \\
& \leq \sum_{i=1}^{k-1} b_2 \mathbf{1}_{b_2 b_3^{-1} M^{-1} \leq i} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| \leq \sum_{i=1}^{k-1} \min(b_2, i M b_3) |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})|,
\end{aligned}$$

also,

$$\begin{aligned}
& \sum_{i=1}^{a_k-1} \int_0^1 \left| \text{cov}(X_k, X_{k-i} [f''(S_{k-i-1} + uX_{k-i}) - f''(S_{k-i-1})]) \right| du \\
& \leq 2 M b_3 \sum_{i=1}^{a_k-1} \left(|\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| + \sum_{j=i+1}^{k-1} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-j})| \right) \\
& \leq 2 M b_3 \sum_{i=1}^{k-1} \mathbf{1}_{i \leq b_2 b_3^{-1} M^{-1}} \left(|\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| + \sum_{j=i+1}^{k-1} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-j})| \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} 2Mb_3 \mathbf{1}_{i \leq b_2 b_3^{-1} M^{-1}} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-j})| \\
&\leq \sum_{j=1}^{k-1} 2 \min(Mb_3 j, b_2) |\text{cov}(\tilde{X}_k, \tilde{X}_{k-j})|. \tag{29}
\end{aligned}$$

In the same way, we bound the quantities

$$\begin{aligned}
&\sum_{i=1}^{a_{k-1}} \left| \text{cov} \left(X_k, X_{k-i} [f''(S_{k-i-1}) - f''(S_{k-2i-1})] \right) \right|, \\
&\sum_{i=1}^{a_{k-1}} \left| \text{cov} (X_k X_{k-i}, f''(S_{k-2i-1})) \right|.
\end{aligned}$$

Inequality (15) follows if we bound the last quantity of (25). For this write

$$\begin{aligned}
&\left| \sum_{i=1}^{a_{k-1}} \mathbf{E} X_k X_{k-i} \mathbf{E} f''(S_{k-2i-1}) - \sum_{i=1}^{k-1} \mathbf{E} X_k X_{k-i} \mathbf{E} f''(S_{k-1}) \right| \\
&= \left| \sum_{i=1}^{k-1} \mathbf{E} X_k X_{k-i} \left(\mathbf{1}_{i \leq b_2 b_3^{-1} M^{-1}} \mathbf{E} f''(S_{k-2i-1}) - \mathbf{E} f''(S_{k-1}) \right) \right| \\
&\leq \sum_{i=1}^{k-1} |\mathbf{E} X_k X_{k-i}| \min(b_2, b_3 M i) \\
&\leq \sum_{i=1}^{k-1} |\text{cov}(\tilde{X}_k, \tilde{X}_{k-i})| \min(b_2, b_3 M i), \tag{30}
\end{aligned}$$

the last inequalities are obtained from the (AG) property and from the fact

$$|\mathbf{1}_{i \leq b_2 b_3^{-1} M^{-1}} \mathbf{E} f''(S_{k-2i-1}) - \mathbf{E} f''(S_{k-1})| \leq \min(b_2, b_3 M i).$$

Now, inequalities (27)–(30), together with (25), yield the bound (15).

2. **P r o o f o f (16).** We suppose now that the sequence $(X_n)_{n \in \mathbf{N}}$ is associated.

For a positive real number x , we define the functions $Z_i(x) := Z_{X_i}(x) := \mathbf{1}_{X_i \geq x} - \mathbf{1}_{X_i \leq -x}$ and $Z(x) := Z_X(x)$. Let $X^+ = \max(0, X)$ and $X^- = \max(0, -X)$. Then

$$X = X^+ - X^- = \int_0^\infty [\mathbf{1}_{X^+ \geq x} - \mathbf{1}_{X^- \geq x}] dx = \int_0^\infty Z(x) dx \quad \text{and} \tag{31}$$

$$|Z(x)| \leq 1, \quad \mathbf{E} |Z(x)| = \mathbf{P}\{|X| \geq x\}, \tag{32}$$

these last inequalities will be used on several occasions. We note also that $Z(x)$ is a nondecreasing function of the r.v. X . Hence by $(\mathcal{P}_4)$ of [13] the vector $(Z(x), Y, X)$ is associated whenever (X, Y) fulfills the association property.

The following two lemmas are the main tools for the proof of (16).

Lemma 5. Let (X_1, X_2) be an associated random vector. Let p and δ be some fixed positive real numbers such that $1 < p < \delta^{-1}$. If $\|X\|_\alpha := \max(\|X_1\|_\alpha, \|X_2\|_\alpha) < \infty$, for $\alpha > 2(1 - p\delta)^{-1}$, then there exists a positive constant $C_{p,\delta,\alpha}$ that depends only on p , δ and α such that

$$\int_0^\infty \int_0^\infty \text{cov}^{1-\delta}(Z_1(x), Z_2(y)) dx dy \leq C_{p,\delta,\alpha} \|X\|_\alpha^{2/p} \text{cov}^{1-1/p}(X_1, X_2). \quad (33)$$

Suppose now that $\|X_2\|_\alpha < \infty$, $\|X_1\|_\gamma < \infty$ for some $\alpha > 1$ and $\gamma > \alpha[(\alpha - 1)(1 - p\delta)]^{-1}$, then there exists a positive constant $C_{p,\alpha}$ that depends only on p and α for which

$$\int_0^\infty \text{cov}^{1-\delta}(Z_1(x), X_2) dx \leq C_{p,\alpha} \|X_2\|_\alpha^{1/p-\delta} \|X_1\|_\gamma^{1/p} \text{cov}^{1-1/p}(X_1, X_2). \quad (34)$$

Lemma 6. Let (X_1, X_2) be an associated random vector. Let X_3 be an arbitrary r.v.'s such that $\|X_3\|_q < +\infty$, for some $q > 1$. Then there exists a constant $C_{q,p,\alpha}$ such that for any positive real numbers A_1, A_2

$$\begin{aligned} & \int_0^\infty \int_0^\infty \min(A_1 \mathbf{P}\{|X_3| > z\}, A_2 \text{cov}(Z_1(x), X_2)) dx dz \\ & \leq C_{q,p,\alpha} A_1^{1/q} A_2^{1-1/q} \|X_1\|_\gamma^{1/p} \|X_2\|_\alpha^{1/p-1/q} \|X_3\|_q \text{cov}^{1-1/p}(X_1, X_2), \end{aligned}$$

where $1 < p < q$, $\alpha > 1$ and $\gamma > \alpha[(\alpha - 1)(1 - p/q)]^{-1}$.

We prove Lemmas 5 and 6 at the end of the paper and we continue the proof of inequality (16). Lemma 5 applied with $\delta = (3 + \rho)^{-1}$ and $\alpha = 3 + \rho$ implies for some $p := q \in]1, 1 + \rho[$:

$$\begin{aligned} & \|X\|_{3+\rho} \int_0^\infty \int_0^\infty \text{cov}^{(2+\rho)/(3+\rho)}(Z_k(x), Z_{k-i}(y)) dx dy \\ & \leq C_\rho \|X\|_{3+\rho}^{1+2/q} \text{cov}^{1-1/q}(X_k, X_{k-i}). \end{aligned} \quad (35)$$

In the same way, Lemma 6 applied with $\alpha = q = \gamma = 3 + \rho$ implies for some p in $]1, (3 + \rho)(1 + \rho)/(2 + \rho)[$:

$$\begin{aligned} & \int_0^\infty \int_0^\infty \min(b_2 \mathbf{P}\{|X_{k-i}| \geq y\}, b_3 \text{cov}(Z_k(x), S_{k-i})) dx dy \\ & \leq C_\rho b_2^{1/(3+\rho)} b_3^{(2+\rho)/(3+\rho)} \|X\|_{3+\rho}^{2/p+(2+\rho)/(3+\rho)} \sum_{j=i}^{k-1} \text{cov}^{1-1/p}(X_k, X_{k-j}). \end{aligned} \quad (36)$$

In order to prove (16), it suffices then to bound the covariance quantities of Lemma 4 by the left-hand side of (35) and (36). For this we suppose in the sequel that $a_k = k$ (recall that a_k is defined in Lemma 4).

Control of $A_1 := |\text{cov}(X_k, X_{k-i}[f''(S_{k-i-1} + uX_{k-i}) - f''(S_{k-i-1})])|$.

It follows clearly from Fubini's theorem and from the integral representation (31) that:

$$\begin{aligned} & \text{cov}(X_k, X_{k-i}[f''(S_{k-i-1} + uX_{k-i}) - f''(S_{k-i-1})]) \\ & = \int_0^\infty \int_0^\infty \text{cov}(Z_k(x), Z_{k-i}(y)[f''(S_{k-i-1} + uX_{k-i}) - f''(S_{k-i-1})]) dx dy \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \int_0^\infty \left\{ \text{cov}(Z_k(x), Z_{k-i}(y)) \right. \\
&\quad \times \left[f''(S_{k-i-1} + uX_{k-i}) - f''(S_{k-i-1} + u\phi_M(X_{k-i})) \right] \\
&\quad + \text{cov}(Z_k(x), Z_{k-i}(y)) \\
&\quad \times \left[f''(S_{k-i-1} + u\phi_M(X_{k-i})) - f''(S_{k-i-1}) \right] \Big\} dx dy \\
&=: \int_0^\infty \int_0^\infty (I_1(x, y) + I_2(x, y)) dx dy,
\end{aligned}$$

where ϕ_M is the nondecreasing function bounded by M and defined by

$$\phi_M(x) := x\mathbf{1}_{|x| \leq M} + M\mathbf{1}_{x \geq M} - M\mathbf{1}_{x \leq -M}. \quad (37)$$

1. We bound first $I_1(x, y)$. Since $f \in \mathcal{F}(b_2, b_3)$, we obtain

$$\begin{aligned}
|I_1(x, y)| &\leq 2b_3 \mathbf{E} |X_{k-i} - \phi_M(X_{k-i})| \leq 2b_3 \mathbf{E} |X_{k-i}| \mathbf{1}_{|X_{k-i}| \geq M} \\
&\leq 2b_3 \frac{\|X_{k-i}\|_{3+\rho}^{3+\rho}}{M^{2+\rho}}.
\end{aligned} \quad (38)$$

2. Now we evaluate $|I_2(x, y)|$. Since $f \in \mathcal{F}(b_2, b_3)$, (32) yields

$$|I_2(x, y)| \leq 4b_2 \mathbf{P}\{|X_{k-i}| \geq y\}. \quad (39)$$

Another bound of $|I_2(x, y)|$ follows from the association of the vector $(Z_k(x), Z_{k-i}(y), S_{k-i-1}, X_{k-i})$, indeed the (AG) property applied with $h(x) = x$ and $k(x, y, z) = x[f''(y + u\phi_M(z)) - f''(y)]$ yields

$$|I_2(x, y)| \leq b_3 M \text{cov}(Z_k(x), Z_{k-i}(y)) + 2b_3 \text{cov}(Z_k(x), S_{k-i}). \quad (40)$$

Both inequalities (39), (40) and the elementary bound $\min(a, c + d) \leq c + \min(a, d)$, for $a, c, d \geq 0$, yield

$$\begin{aligned}
|I_2(x, y)| &\leq \min(b_2 \mathbf{P}\{|X_{k-i}| \geq y\}, b_3 \text{cov}(Z_k(x), S_{k-i})) \\
&\quad + b_3 M \text{cov}(Z_k(x), Z_{k-i}(y)).
\end{aligned} \quad (41)$$

Inequalities (38) and (41) yield then, if we choose

$$M = \frac{\|X_{k-i}\|_{3+\rho}^{3+\rho}}{\text{cov}^{1/(3+\rho)}(Z_k(x), Z_{k-i}(y))},$$

that

$$\begin{aligned}
A_1 &\leq \int_0^\infty \int_0^\infty \min(b_2 \mathbf{P}\{|X_{k-i}| \geq y\}, b_3 \text{cov}(Z_k(x), S_{k-i})) dx dy \\
&\quad + b_3 \|X_{k-i}\|_{3+\rho} \int_0^\infty \int_0^\infty \text{cov}^{(2+\rho)/(3+\rho)}(Z_k(x), Z_{k-i}(y)) dx dy.
\end{aligned} \quad (42)$$

C o n t r o l o f A_2 : $\text{cov}(X_k, X_{k-i}[f''(S_{k-i-1}) - f''(S_{k-2i-1})])$. We write

$$\begin{aligned}
A_2 &= \int_0^\infty \int_0^\infty \left\{ \text{cov}(Z_k(x), Z_{k-i}(y) [f''(S_{k-i-1}) - f''(S_{k-2i-1} + \tilde{S}_M)]) \right. \\
&\quad \left. + \text{cov}(Z_k(x), Z_{k-i}(y) [f''(S_{k-2i-1} + \tilde{S}_M) - f''(S_{k-2i-1})]) \right\} dx dy \\
&=: \int_0^\infty \int_0^\infty [I_1(x, y) + I_2(x, y)] dx dy,
\end{aligned}$$

where $\tilde{S}_M := \sum_{j=k-2i}^{k-i-1} \phi_M(X_j)$, we note that $\tilde{S}_M$ is bounded by iM . As in (38), we obtain

$$|I_1(x, y)| \leq 2b_3 i \sup_k \mathbf{E} |X_k - \phi_M(X_k)| \leq 2b_3 i \frac{\|X\|_{3+\rho}^{3+\rho}}{M^{2+\rho}}. \quad (43)$$

By the same arguments as in (41), we obtain

$$|I_2(x, y)| \leq \min(4b_2 \mathbf{P}\{|X_{k-i}| \geq y\}, b_3 \operatorname{cov}(Z_k(x), S_{k-i})) \\ + 2b_3 i M \operatorname{cov}(Z_k(x), Z_{k-i}(y)).$$

Hence we obtain for a suitable choice of M

$$|A_2| \leq 2b_3 i \|X\|_{3+\rho} \int_0^\infty \int_0^\infty \operatorname{cov}^{(2+\rho)/(3+\rho)}(Z_k(x), Z_{k-i}(y)) dx dy \\ + 4 \int_0^\infty \int_0^\infty \min(b_2 \mathbf{P}\{|X_{k-i}| \geq y\}, b_3 \operatorname{cov}(Z_k(x), S_{k-i})) dx dy. \quad (44)$$

C o n t r o l o f $A_3 := \operatorname{cov}(X_k X_{k-i}, f''(S_{k-2i-1}))$. As before we write, $A_3 = \int_0^\infty \int_0^\infty \operatorname{cov}(Z_k(x) Z_{k-i}(y), f''(S_{k-2i-1})) dx dy =: \int_0^\infty \int_0^\infty I_3(x, y) dx dy$. We provide two different bounds of the quantity $|I_3(x, y)|$. The first bound is $|I_3(x, y)| \leq 2b_2 \min(\mathbf{P}\{|X_k| \geq x\}, \mathbf{P}\{|X_{k-i}| \geq y\})$. The second bound is obtained by using the association of the vector $(Z_k(x), Z_{k-i}(y), S_{k-2i-1})$: the (AG) property applied with $h(x, y) = xy$ and with $k(x) = f''(x)$ yields $|I_3(x, y)| \leq b_3 \operatorname{cov}(Z_k(x) + Z_{k-i}(y), S_{k-2i-1}) \leq b_3 \operatorname{cov}(Z_k(x) + Z_{k-i}(y), S_{k-i})$. The previous two bounds of $|I_3(x, y)|$ and the elementary inequality $\min(a, b, e + f) \leq \min(a, e) + \min(b, f)$, for $a, b, e, f \geq 0$, yield:

$$|A_3| \leq 2 \int_0^\infty \int_0^\infty \min(b_2 \mathbf{P}\{|X_{k-i}| \geq y\}, b_3 \operatorname{cov}(Z_k(x), S_{k-i})) dx dy \\ + 2 \int_0^\infty \int_0^\infty \min(b_2 \mathbf{P}\{|X_k| \geq x\}, b_3 \operatorname{cov}(Z_{k-i}(y), S_{k-i})) dx dy. \quad (45)$$

In the same way, we evaluate $\operatorname{cov}(X_k^2, f''(S_{k-1}))$.

C o n t r o l o f $A_4 := \mathbf{E} X_k X_{k-i} \mathbf{E} f''(S_{k-2i-1}) - \mathbf{E} X_k X_{k-i} \mathbf{E} f''(S_{k-i})$. Clearly

$$|A_4| \leq 2 \mathbf{E} X_k X_{k-i} \min(b_2, i b_3 \mathbf{E}|X|) \quad (46)$$

(here $\mathbf{E}|X| = \sup_i \mathbf{E}|X_i|$). Hence Lemma 4, together with inequalities (42), (44)–(46) and (35), (36) yields the proof of (16).

Let us now prove Lemmas 5 and 6.

P r o o f o f L e m m a 5. Hölder's inequality and some elementary estimations yield for any $p \in]1, \delta^{-1}[$,

$$\int_0^\infty \int_0^\infty \operatorname{cov}^{1-\delta}(Z_1(x), Z_2(y)) dx dy \\ \leq \operatorname{cov}^{1-1/p}(X_1, X_2) \left(\int_0^\infty \int_0^\infty \operatorname{cov}^{1-p\delta}(Z_1(x), Z_2(y)) dx dy \right)^{1/p} \\ \leq \operatorname{cov}^{1-1/p}(X_1, X_2)$$

$$\begin{aligned} & \times \left(\int_0^\infty \int_0^\infty \left[\min \left(\mathbf{P}\{|X_1| \geq x\}, \mathbf{P}\{|X_2| \geq y\} \right) \right]^{1-p\delta} dx dy \right)^{1/p} \\ & \leq 2^{1/p} \text{cov}^{1-1/p}(X_1, X_2) \left(\int_0^\infty x \mathbf{P}^{1-p\delta}\{|X| \geq x\} dx \right)^{1/p}, \end{aligned} \quad (47)$$

where $\mathbf{P}\{|X| \geq x\} = \max(\mathbf{P}\{|X_1| \geq x\}, \mathbf{P}\{|X_2| \geq x\})$.

Suppose now that $\|X\|_\alpha < \infty$ for $\alpha > 2(1-p\delta)^{-1}$, then for all $M > 0$

$$\begin{aligned} \int_0^\infty x \mathbf{P}^{1-p\delta}\{|X| \geq x\} dx & \leq \frac{M^2}{2} + \int_M^\infty x \mathbf{P}^{1-p\delta}\{|X| \geq x\} dx \\ & \leq \frac{M^2}{2} + C \frac{(\mathbf{E}|X|^\alpha)^{1-p\delta}}{M^{\alpha(1-p\delta)-2}}. \end{aligned}$$

Hence if we choose $M = \|X\|_\alpha$ we obtain $\int_0^\infty x \mathbf{P}^{1-p\delta}\{|X| \geq x\} dx \leq C\|X\|_\alpha^2$, where C is some positive constant depending only on δ , α and p .

Inequality (33) is then obtained from the last inequality and from (47).

Now to prove (34), we use again Hölder's inequality

$$\begin{aligned} & \int_0^\infty \text{cov}^{1-\delta}(Z_1(x), X_2) dx \\ & \leq \text{cov}^{1-1/p}(X_1, X_2) \left(\int_0^\infty \text{cov}^{1-p\delta}(Z_1(x), X_2) dx \right)^{1/p} \\ & \leq \|X_2\|_\alpha^{1/p-\delta} \text{cov}^{1-1/p}(X_1, X_2) \left(\int_0^\infty \mathbf{P}^{(1-1/\alpha)(1-p\delta)}\{|X_1| \geq x\} dx \right)^{1/p} \\ & \leq C_{\alpha,\delta,p} \|X_2\|_\alpha^{1/p-\delta} \|X_1\|_\gamma^{1/p} \text{cov}^{1-1/p}(X_1, X_2), \end{aligned}$$

which yields (34) as soon as $\gamma > \alpha[(\alpha-1)(1-p\delta)]^{-1}$.

P r o o f o f L e m m a 6. Clearly

$$\begin{aligned} & \int_0^\infty \int_0^\infty \min \left(A_1 \mathbf{P}\{|X_3| > z\}, A_2 \text{cov}(Z_1(x), X_2) \right) dx dz \\ & \leq \int_0^\infty \int_0^\infty \min \left(A_1 \|X_3\|_q^q z^{-q}, A_2 \text{cov}(Z_1(x), X_2) \right) dx dz. \end{aligned} \quad (48)$$

Writing the above integral as sums of integrals allows us to suppress the minimum and thus the quantity (48) is bounded by

$$C A_1^{1/q} A_2^{1-1/q} \|X_3\|_q \int_0^\infty \text{cov}^{1-1/q}(Z_1(x), X_2) dx.$$

The proof of Lemma 6 follows then from the last bound and from (34).

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