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SAMPLE PATH PROPERTIES OF OPERATOR-SELF-SIMILAR GAUSSIAN RANDOM FIELDS¹⁾

В статье изучаются хаусдорфова размерность образа и графика, вероятности достижения, невозвратность и другие свойства выборочных траекторий некоторых изотропных операторно-самоподобных гауссовских случайных полей $X = \{X(t), t \in \mathbf{R}^N\}$ со стационарными приращениями, включая многопараметрическое операторное дробное броуновское движение. Наши результаты показывают, что если $X(1)$, где $1 = (1, 0, \dots, 0) \in \mathbf{R}^N$, — невырождающаяся (full) случайная величина, то многие из этих свойств выборочных траекторий полностью определяются вещественными частями собственных значений показателя самоподобия D .

Ключевые слова и фразы: операторно-самоподобные гауссовские случайные поля, образ, график, хаусдорфова размерность, полярное множество, невозвратность.

1. Introduction. Self-similar processes, first studied rigorously by Lamperti [15] under the name «semi-stable», are processes that are invariant under suitable translations of time and scale. There has been an extensive literature on self-similar processes. We refer to Vervaat [29] for general properties, to Samorodnitsky and Taqqu [22, Chaps. 7, 8] for studies on Gaussian and stable self-similar processes and random fields, to Taqqu [27] for a bibliographical guide to many applications. See also Maejima [16], Kôno [13], Xiao [34] for more information related to limit theorems and sample path properties of self-similar processes.

The definition of self-similarity has been extended to allow scaling by linear operators on $\mathbf{R}^d$ and the corresponding processes are called operator-self-similar processes. See Laha and Rohatgi [14], Hudson and Mason [8] for discussions in the general framework and basic properties. Sato [23] introduced two notions of operator-self-similarity and investigated the relation between operator-self-similar processes with independent increments and

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operator-self-decomposable distributions. Maejima and Mason [19], Maejima [17], [18] constructed several classes of operator-self-similar processes such as operator fractional Brownian motion, operator-self-similar non-Gaussian stable processes and operator-self-similar operator-stable processes, and proved some related limit theorems. However, the fine properties of the sample paths of operator-self-similar processes have not been studied in the papers mentioned above.

In this paper, we will consider operator-self-similar Gaussian random fields with stationary increments including the so-called multiparameter operator fractional Brownian motion, and study their sample path properties such as the Hausdorff dimension of the image and graph, hitting probabilities, transience and the characterization of polar sets.

Let $End(\mathbf{R}^d)$ be the set of linear operators on $\mathbf{R}^d$ (endomorphisms) and let $Aut(\mathbf{R}^d)$ be the set of invertible linear operators (automorphisms) in $End(\mathbf{R}^d)$. For convenience, we will not distinguish an operator $D \in End(\mathbf{R}^d)$ from its associated matrix relative to the standard basis of $\mathbf{R}^d$.

We will use the following definition for operator-self-similar random fields, which corresponds to that of operator-self-similar processes of Sato [23], but is slightly stronger than that of Hudson and Mason [8].

An $\mathbf{R}^d$ -valued random field $X = \{X(t), t \in \mathbf{R}^N\}$ is said to be operator-self-similar (o.s.s. for short), if

- (i) it is stochastically continuous, and
- (ii) there exists a $D \in End(\mathbf{R}^d)$ such that for every $c > 0$,

$$X(c \cdot) \stackrel{d}{=} c^D X(\cdot), \quad (1.1)$$

where $X \stackrel{d}{=} Y$ means the processes X and Y have the same finite dimensional distributions and

$$c^D = \exp((\ln c)D) = \sum_{k=0}^{\infty} \frac{1}{k!} (\ln c)^k D^k.$$

The linear operator D in (1.1) will be referred to as a self-similarity exponent for the random field X and it is not necessarily unique (Hudson and Mason [8]). If an exponent $D = \alpha I_d$ for some $\alpha > 0$, where $I_d \in End(\mathbf{R}^d)$ is the identity operator, then the o.s.s. random field is simply a self-similar random field with index α .

We say that X has stationary increments (s.i.), if for every $b \in \mathbf{R}^N$

$$X(\cdot + b) - X(b) \stackrel{d}{=} X(\cdot) - X(0). \quad (1.2)$$

The rest of the paper is organized as follows. In Section 2, we collect some basic facts about o.s.s. random fields and Hausdorff measure. In Section 3, we construct multiparameter operator fractional Brownian motion,

which extends the usual multiparameter fractional Brownian motion. In Section 4, we consider the Hausdorff dimension of the image and graph of certain operator-self-similar Gaussian random fields. In Section 5, we first prove some estimates on the hitting probabilities for o.s.s. Gaussian random fields, and then apply these estimates to study the polar sets and transience for such Gaussian random fields. We will find that the real parts of the eigenvalues of the self-similarity exponent D play important roles in the geometry of the sample paths. Our results not only generalize the previous results for fractional Brownian motion in $\mathbf{R}^d$ with index α ($0 < \alpha < 1$), which is an isotropic o.s.s. Gaussian random field with exponent $D = \alpha I_d$, but also give us better understanding of the role of the self-similarity index α . This paper raises many open questions about the sample path properties of o.s.s. Gaussian random fields, some of them are stated at the end of Sections 4 and 5.

We can also define o.s.s. stable and operator-stable random fields in ways that are similar to those of Maejima and Mason [19] and Maejima [17]. We refer to Hudson and Mason [9], Jurek and Mason [10] for more information on operator-stable laws. Most of the questions considered in this paper can also be asked for o.s.s. stable or operator-stable random fields. But, to the authors knowledge, few tools are available in the general cases.

We will use $K, K_1, \dots, K_{24}$ to denote unspecified positive finite constants which may not necessarily be the same in each occurrence.

2. Preliminaries. Throughout this paper, we will use $|x|$ to denote the usual Euclidean norm of $x \in \mathbf{R}^d$. For $A \in \text{End}(\mathbf{R}^d)$, let $\|A\| = \max_{|x|=1} |Ax|$ denote the operator norm of A . It is easy to see that for $A, B \in \text{End}(\mathbf{R}^d)$

$$\|AB\| \leq \|A\| \cdot \|B\| \quad (2.1)$$

and for every $A = (a_{ij}) \in \text{End}(\mathbf{R}^d)$

$$\max_{1 \leq i, j \leq d} |a_{ij}| \leq \|A\| \leq d^{3/2} \max_{1 \leq i, j \leq d} |a_{ij}|. \quad (2.2)$$

Let $\sigma(A)$ be the collection of all eigenvalues of A . We denote

$$\lambda_A = \min \{\text{Re} \lambda : \lambda \in \sigma(A)\} \quad \Lambda_A = \max \{\text{Re} \lambda : \lambda \in \sigma(A)\}.$$

The following two lemmas were proved by Maejima and Mason [19] for o.s.s. processes (i.e., $N = 1$), they can be extended to o.s.s. random fields without any difficulty.

Lemma 2.1. *Suppose $X = \{X(t), t \in \mathbf{R}^N\}$ is an operator-self-similar random field with exponent D . If $\lambda_D > 0$, then $X(0) = 0$ a.s.*

Recall that a random variable in $\mathbf{R}^d$ is called full if the support of its distribution is not contained in a proper hyperplane of $\mathbf{R}^d$. For a normal random variable Ξ in $\mathbf{R}^d$ with mean 0 it is full if and only if $\det \text{Cov}(\Xi) > 0$.

Lemma 2.2. Suppose $X = \{X(t), t \in \mathbf{R}^N\}$ is an operator-self-similar random field with exponent D and $X(0) = 0$ a.s. If $X(\mathbf{1})$ is full and $\mathbf{E}(|X(\mathbf{1})|) < \infty$, where $\mathbf{1} = (1, 0, \dots, 0)$, then $\Lambda_D \leq 1$.

We end this section by recalling briefly the definition of Hausdorff measure and Hausdorff dimension. Let Φ be the class of functions $\varphi: (0, \delta) \rightarrow (0, 1)$ which are right continuous, monotone increasing with $\varphi(0+) = 0$ and such that there exists a finite constant $K > 0$ for which $\varphi(2s)/\varphi(s) \leq K$ for $0 < s < \frac{1}{2}\delta$. For $\varphi \in \Phi$, the φ -Hausdorff measure of $E \subseteq \mathbf{R}^N$ is defined by

$$\varphi\text{-}m(E) = \liminf_{\varepsilon \rightarrow 0} \left\{ \sum_i \varphi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < \varepsilon \right\},$$

where $B(x, r)$ denotes the open (or closed) ball of radius r centered at x . It is known that $\varphi\text{-}m$ is a metric outer measure and hence every Borel set in $\mathbf{R}^N$ is $\varphi\text{-}m$ measurable. The Hausdorff dimension of E is defined by

$$\dim E = \inf \left\{ \alpha > 0 : s^\alpha\text{-}m(E) = 0 \right\}.$$

We refer to Falconer [5] for more properties of Hausdorff measure and Hausdorff dimension.

3. Operator-fractional Brownian motion. Let m be a scattered Gaussian random measure on $\mathbf{R}^N$ with Lebesgue measure λ_N as its control measure, that is, $\{m(A), A \in \mathcal{E}\}$ is a centered Gaussian process on $\mathcal{E} = \{E \subset \mathbf{R}^N : \lambda_N(E) < \infty\}$ with covariance function $\mathbf{E}(m(E)m(F)) = \lambda_N(E \cap F)$.

Let $m_1, \dots, m_d$ be d independent copies of m , and define $\mathbf{m}(A) = (m_1(A), \dots, m_d(A))$. Then $\mathbf{m}$ is an $\mathbf{R}^d$ -valued process on $\mathcal{E}$. It is easy to see that $\mathbf{m}$ is self-similar with exponent $N/2$: For every $c > 0$,

$$\mathbf{m}(c \cdot) \stackrel{d}{=} c^{N/2} \mathbf{m}(\cdot) \quad (3.1)$$

and for every orthogonal operator φ on $\mathbf{R}^N$

$$\mathbf{m}(\varphi(\cdot)) \stackrel{d}{=} \mathbf{m}(\cdot). \quad (3.2)$$

Now we define stochastic integral of an $End(\mathbf{R}^d)$ -valued function $A(u)$ ($u \in \mathbf{R}^N$) over $\mathbf{m}$ as follows. If $A(u)$ is a step function of the form

$$A(u) = \sum_{i=1}^k A_i 1_{E_i}(u),$$

where $A_i \in End(\mathbf{R}^d)$, $E_1, \dots, E_k$ are disjoint sets in $\mathcal{E}$ and 1_E is the indicator function of E , then we define

$$\int_{\mathbf{R}^N} A(u) d\mathbf{m}(u) = \sum_{i=1}^k A_i \mathbf{m}(E_i).$$

If $A(u): \mathbf{R}^N \rightarrow \text{End}(\mathbf{R}^d)$ is measurable and $\int_{\mathbf{R}^N} \|A(u)\|^2 du < \infty$, then we can define the stochastic integral

$$I(A) = \int_{\mathbf{R}^N} A(u) d\mathbf{m}(u)$$

in a standard way. $I(A)$ is an $\mathbf{R}^d$ -valued Gaussian random variable with mean 0 and

$$\mathbf{E}(I(A) I'(B)) = \int_{\mathbf{R}^N} A(u) B^*(u) du, \quad (3.3)$$

where x' is the transpose vector of x and B^* is the adjoint operator of B .

The following lemma is due to Maejima and Mason [19]. In Sections 4 and 5 we will see that more precise estimates can be obtained.

Lemma 3.1. *Let $D \in \text{End}(\mathbf{R}^d)$. If $\lambda_D > 0$ and $r > 0$, then for any $\delta > 0$, there exist positive constants K_1 and K_2 such that*

$$\|r^D\| \leq \begin{cases} K_1 r^{\lambda_D - \delta} & \text{for all } r \leq 1, \\ K_2 r^{\Lambda_D + \delta} & \text{for all } r \geq 1. \end{cases}$$

Now we start to construct multiparameter operator fractional Brownian motion.

Theorem 3.1. *Let D be a linear operator on $\mathbf{R}^d$ with $0 < \lambda_D, \Lambda_D < 1$. For $t \in \mathbf{R}^N$, define*

$$\begin{aligned} X(t) = & \int_{\mathbf{R}^N} (1 - \cos\langle t, x \rangle) \left(\frac{1}{|x|} \right)^{D+NI/2} d\mathbf{m}(x) \\ & + \int_{\mathbf{R}^N} \sin\langle t, x \rangle \left(\frac{1}{|x|} \right)^{D+NI/2} d\mathbf{m}'(x), \end{aligned}$$

where $\mathbf{m}'$ is an independent copy of $\mathbf{m}$. Then the random field $X = \{X(t), t \in \mathbf{R}^N\}$ is an operator-self-similar Gaussian random field with exponent D and stationary increments. Furthermore, X is isotropic in the sense that for every $t \in \mathbf{R}^N$,

$$X(t) \stackrel{d}{\sim} |t|^D X(1),$$

where $\stackrel{d}{\sim}$ means two random variables have the same distribution, and X has a version with continuous sample paths almost surely.

P r o o f. (i) *Integrability.* Consider the $\text{End}(\mathbf{R}^d)$ -valued functions

$$A_1(x) = (1 - \cos\langle x, t \rangle) \left(\frac{1}{|x|} \right)^{D+NI/2}, \quad A_2(x) = \sin\langle x, t \rangle \left(\frac{1}{|x|} \right)^{D+NI/2}$$

It suffices to show for each $t \in \mathbf{R}^N$

$$\int_{\mathbf{R}^N} \|A_i(x)\|^2 dx < \infty, \quad \text{for } i = 1, 2. \quad (3.4)$$

It follows from Lemma 3.1 that for $|x| \leq 1$

$$\|A_1(x)\| \leq K(|t||x|)^2 \left(\frac{1}{|x|}\right)^{\Lambda_D + N/2 + \delta} = K|t|^2 |x|^{-N/2 + 2 - \Lambda_D - \delta} \quad (3.5)$$

and for $|x| \geq 1$

$$\|A_1(x)\| \leq K|x|^{-N/2 - \lambda_D + \delta}. \quad (3.6)$$

By (3.5) and (3.6) with $\delta > 0$ small, we see clearly (3.4) holds for $i = 1$. Similarly, (3.4) is also true for $i = 2$.

(ii) *Operator-self-similarity.* For every $c > 0$,

$$\begin{aligned} X(ct) &= \int_{\mathbf{R}^N} (1 - \cos\langle ct, x \rangle) \left(\frac{1}{|x|}\right)^{D+NI/2} d\mathbf{m}(x) \\ &\quad + \int_{\mathbf{R}^N} \sin\langle ct, x \rangle \left(\frac{1}{|x|}\right)^{D+NI/2} d\mathbf{m}'(x) \\ &= \int_{\mathbf{R}^N} (1 - \cos\langle t, x \rangle) \left(\frac{c}{|x|}\right)^{D+NI/2} d\mathbf{m}\left(\frac{x}{c}\right) \\ &\quad + \int_{\mathbf{R}^N} \sin\langle t, x \rangle \left(\frac{c}{|x|}\right)^{D+NI/2} d\mathbf{m}'\left(\frac{x}{c}\right) \stackrel{d}{=} c^D X(t), \end{aligned}$$

where the last inequality follows from (3.1) and the fact that $c^{D+NI/2} \times c^{-N/2} = c^D$.

(iii) *X is Gaussian and isotropic.* It is obvious that X is Gaussian. To prove X is isotropic, we only need to consider $t \in \mathbf{R}^N \setminus \{0\}$. Let $\varphi: \mathbf{R}^N \rightarrow \mathbf{R}^N$ be the orthogonal transformation such that

$$\varphi((|t|, 0, \dots, 0)) = t.$$

Then it follows from (3.2) and (3.1) that

$$X(t) \stackrel{d}{\sim} X((|t|, 0, \dots, 0)) \stackrel{d}{\sim} |t|^D X(\mathbf{1}).$$

(iv) *Stationarity of the increments.* Since the distribution of any Gaussian random variable in $\mathbf{R}^d$ with mean 0 is completely determined by its covariance matrix, it is sufficient to prove that for every $b \in \mathbf{R}^N$ and $s, t \in \mathbf{R}^N$

$$\mathbf{E}[(X(s+b) - X(b))(X(t+b) - X(b))'] = \mathbf{E}(X(s)X'(t)). \quad (3.7)$$

It follows from (3.3) that

$$\begin{aligned} \mathbf{E}(X(s)X'(t)) &= \int_{\mathbf{R}^N} (1 - \cos\langle s, x \rangle) (1 - \cos\langle t, x \rangle) \left(\frac{1}{|x|}\right)^{D+NI/2} \left(\frac{1}{|x|}\right)^{D^*+NI/2} dx \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbf{R}^N} \sin\langle s, x \rangle \sin\langle t, x \rangle \left(\frac{1}{|x|} \right)^{D+NI/2} \left(\frac{1}{|x|} \right)^{D^*+NI/2} dx \\
& = \int_{\mathbf{R}^N} \left[1 - \cos\langle s, x \rangle - \cos\langle t, x \rangle + \cos\langle t-s, x \rangle \right] \\
& \quad \times \left(\frac{1}{|x|} \right)^{D+NI/2} \left(\frac{1}{|x|} \right)^{D^*+NI/2} dx \\
& = \frac{1}{2} \left[\text{Cov}(X(s)) + \text{Cov}(X(t)) - \text{Cov}(X(t-s)) \right], \tag{3.8}
\end{aligned}$$

where $\text{Cov}(X(s))$ is the covariance matrix of $X(s)$. Then (3.7) follows from (3.8) immediately.

(v) *Sample path continuity.* Let $0 < \delta < \lambda_D$. It follows from (ii)–(iv) and Lemma 3.1 that for $|t-s| \leq 1$ and any integer $k \geq 1$ we have

$$\begin{aligned}
\mathbf{E}(|X(t) - X(s)|^k) &= \mathbf{E}(|t-s|^D |X(1)|^k) \leq K_1 |t-s|^{(\lambda_D - \delta)k} \mathbf{E}(|X(1)|^k) \\
&\leq K_3 |t-s|^{(\lambda_D - \delta)k}, \tag{3.9}
\end{aligned}$$

where the last inequality follows from the easily verified fact that $\mathbf{E}(X_j(1)) = 0$, $\mathbf{D}(X_j(1)) < \infty$ ($j = 1, \dots, d$) and K_3 is a constant depending on k and D only. Now the sample path continuity follows immediately from (3.9) and a continuity lemma of Garsia [6]. This finishes the proof of Theorem 3.1.

The Gaussian random field X defined in Theorem 3.1 will be called the multiparameter operator fractional Brownian motion in $\mathbf{R}^d$ with self-similarity exponent D .

R e m a r k. If $D = \alpha I_d$ ($0 < \alpha < 1$), then the multiparameter operator fractional Brownian motion X is the ordinary d -dimensional fractional Brownian motion of index α . More generally, if

$$D = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha_d \end{pmatrix},$$

where $0 < \alpha_j < 1$ ($1 \leq j \leq d$), then the component processes

$$X_j = \{X_j(t), t \in \mathbf{R}^N\} \quad (j = 1, \dots, d)$$

of the Gaussian random field X defined in Theorem 3.1 are independent fractional Brownian motions of indices α_j respectively. In this case, X is called a Gaussian random field with fractional Brownian motion components. The sample path properties of such Gaussian random fields have been studied by Cuzick [3], Adler [1], Talagrand [25], [26], Xiao [30], [32], [33]. Their results and arguments are heavily applied in the rest of the present paper.

4. Hausdorff dimension of the image and graph. Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an isotropic, o.s.s. Gaussian random field in $\mathbf{R}^d$ with exponent D and stationary increments. A typical example of such Gaussian random fields is the multiparameter operator fractional Brownian motion in $\mathbf{R}^d$ with self-similarity exponent D . Throughout this paper, we will assume that all the eigenvalues of D have positive real parts (hence by Lemma 2.1 $X(0) = 0$) and the eigenvalues $\alpha_j = \lambda_j + i\beta_j$ ($j = 1, \dots, d$) of D are arranged such that

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_d < 1. \quad (4.1)$$

An $(l \times l)$ -matrix $J = (a_{ij})$ is called a Jordan cell matrix with eigenvalue λ if for $1 \leq i \leq l$

$$a_{ij} = \begin{cases} \lambda & \text{if } j = i, \\ 1 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is known (see, e.g., [7, Chap. 6]) that every $D \in \text{End}(\mathbf{R}^d)$ has a real canonical form, i.e., there exists a real invertible $(d \times d)$ matrix B such that $\tilde{D} = B^{-1}DB$ is composed of diagonal blocks which are either Jordan cell matrices or blocks of the form

$$\begin{pmatrix} D_2 & I_2 & 0 & \cdots & 0 & 0 \\ 0 & D_2 & I_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & D_2 & I_2 \\ 0 & 0 & 0 & \cdots & 0 & D_2 \end{pmatrix}, \quad (4.2)$$

where

$$D_2 = \begin{pmatrix} \lambda & -\beta \\ \beta & \lambda \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For any Borel set $F \subseteq \mathbf{R}^N$, we consider the Hausdorff dimension of the image $X(F)$ and the graph set $GrX(F) = \{(t, X(t)): t \in F\}$. If X is an ordinary fractional Brownian motion or a Gaussian random field with fractional Brownian motion components, these problems were solved by Kahane [11], Cuzick [3], Xiao [30]. The main theorem in this section is Theorem 4.1.

We first prove the following result about the modulus of continuity for the components X_j ($j = 1, \dots, d$) of X . With a more careful analysis based on the form of the exponent D , one can establish sharp results of this kind. But for our purpose, an inequality like (4.3) is enough.

Proposition 4.1. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an isotropic, operator-self-similar Gaussian random field in $\mathbf{R}^d$ with exponent D and stationary increments. If D is of the real canonical form, then there exist positive and*

finite constants $1 \leq p_j \leq d$ ($j = 1, \dots, d$) and K_4 , depending on D , N and d only, such that for every closed cube $T \subset \mathbf{R}^N$ and every $j = 1, 2, \dots, d$

$$\limsup_{h \downarrow 0} \frac{\omega_j(h)}{h^{\lambda_j} (\ln 1/h)^{p_j-1/2}} \leq K_4 \quad \text{a.s.}, \quad (4.3)$$

where

$$\omega_j(h) = \sup_{t, t+s \in T, |s| \leq h} |X_j(t+s) - X_j(t)|$$

denotes the uniform modulus of continuity of $X_j(t)$ on T . In particular, $X(t)$ ($t \in T$) satisfies almost surely a uniform Hölder condition of any order smaller than $\lambda = (\lambda_1, \dots, \lambda_d)$ on T , that is, for every $\varepsilon > 0$, almost surely

$$|X_j(t) - X_j(s)| \leq K_5 |t - s|^{\lambda_j - \varepsilon} \quad \text{for all } t, s \in T \quad (4.4)$$

and for every $j = 1, 2, \dots, d$.

P r o o f. Let $J_1, \dots, J_m$ be the diagonal blocks of D , where each J_j is either a Jordan cell matrix or a block of the form (4.2). For a fixed $1 \leq j \leq d$, we now show that $X_j(t)$ satisfies (4.3) almost surely. Suppose that the block corresponding to the eigenvalue $\alpha_j = \lambda_j + i\beta_j$ is J_l . For simplicity of notation, we will suppress the subscript l .

Denote the standard basis of $\mathbf{R}^d$ by $(e_1, \dots, e_d)$. Since X is isotropic, o.s.s. and has stationary increments, we have for all $s, t \in \mathbf{R}^N$

$$\begin{aligned} \mathbf{E}(X_j(t) - X_j(s))^2 &= \mathbf{E}(\langle X(t) - X(s), e_j \rangle^2) = \mathbf{E}(\langle |t-s|^D X(\mathbf{1}), e_j \rangle^2) \\ &= \mathbf{E} \left(\sum_{k=1}^d X_k(\mathbf{1}) \langle |t-s|^D e_k, e_j \rangle \right)^2. \end{aligned} \quad (4.5)$$

It is easy to see that $\langle |t-s|^D e_k, e_j \rangle$ is the (j, k) th entry of the matrix $|t-s|^D$. So $\langle |t-s|^D e_k, e_j \rangle \neq 0$ only if $j \leq k$ and is also an entry of $|t-s|^J$. Now we distinguish two cases: (i) J is a $p \times p$ Jordan cell matrix with eigenvalue λ_j or (ii) J is of the form (4.2) with $\lambda = \lambda_j$. In the first case, with some simple calculations we see that the nonzero entries in the j th row of $|t-s|^D$ are contained in

$$|t-s|^{\lambda_j}, \quad |t-s|^{\lambda_j} \ln |t-s|, \dots, |t-s|^{\lambda_j} \frac{(\ln |t-s|)^{p-1}}{(p-1)!}. \quad (4.6)$$

It follows from (4.5) and (4.6) that

$$\mathbf{E}(X_j(t) - X_j(s))^2 \leq K |t-s|^{2\lambda_j} (\ln |t-s|)^{2(p-1)}, \quad (4.7)$$

where $K > 0$ is a constant depending on p (or D) and $\mathbf{E}(X_k^2(\mathbf{1}))$, $k = 1, 2, \dots, d$.

In the second case, we assume J is a $(2p \times 2p)$ -matrix of the form (4.2). Let A_l ($1 \leq l \leq p$) be real (2×2) -matrices. We define $(2p \times 2p)$ -matrix $M(A_1, A_2, \dots, A_p)$ by

$$M(A_1, A_2, \dots, A_p) = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_{p-1} & A_p \\ 0 & A_1 & A_2 & \cdots & A_{p-2} & A_{p-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_1 & A_2 \\ 0 & 0 & 0 & \cdots & 0 & A_1 \end{pmatrix}.$$

Then we can write $J = M(D_2, I_2, 0, \dots, 0)$, and with some calculations we have

$$\begin{aligned} |t-s|^J &= M \left(|t-s|^{D_2}, |t-s|^{D_2} \ln |t-s|, \dots, |t-s|^{D_2} \frac{(\ln |t-s|)^{p-1}}{(p-1)!} \right) \\ &= M(|t-s|^{D_2}, 0, \dots, 0) \\ &\quad \times M \left(I_2, I_2 \ln |t-s|, \dots, I_2 \frac{(\ln |t-s|)^{p-1}}{(p-1)!} \right). \end{aligned} \quad (4.8)$$

By using the obvious fact that for every $r > 0$ $|r^{D_2}x| = r^{\lambda_j}|x|$ for every $x \in \mathbf{R}^2$ and (4.8) we have that for every $x \in \mathbf{R}^{2p}$

$$\begin{aligned} ||t-s|^J x| &= \left| M(|t-s|^{D_2}, 0, \dots, 0) \right. \\ &\quad \times M \left(I_2, I_2 \ln |t-s|, \dots, I_2 \frac{(\ln |t-s|)^{p-1}}{(p-1)!} \right) x \left. \right| \\ &= \left| M(I_2 |t-s|^{\lambda_j}, 0, \dots, 0) \right. \\ &\quad \times M \left(I_2, I_2 \ln |t-s|, \dots, I_2 \frac{(\ln |t-s|)^{p-1}}{(p-1)!} \right) x \left. \right|. \end{aligned} \quad (4.9)$$

By first reducing the dimension in the inner product and then using the Cauchy-Schwarz inequality, (4.9), (2.1) and (2.2), we have

$$\begin{aligned} \mathbf{E}(X_j(t) - X_j(s))^2 &= \mathbf{E}(\langle |t-s|^D X(1), e_j \rangle^2) \\ &\leq |t-s|^{2\lambda_j} \left\| M \left(I_2, I_2 \ln |t-s|, \dots, I_2 \frac{(\ln |t-s|)^{p-1}}{(p-1)!} \right) \right\|^2 \mathbf{E}(|X(1)|^2) \\ &\leq \frac{\mathbf{E}(|X(1)|^2)}{((p-1)!)^2} |t-s|^{2\lambda_j} |\ln |t-s||^{2(p-1)}. \end{aligned} \quad (4.10)$$

Now it is clear that (4.3) with $p_j = p$ above follows from (4.7), (4.10) and the tail probability estimates for Gaussian random fields (see, e.g., [25]) in a standard way. See, e.g., [1] or [2].

The proof of Proposition 4.1 also implies the following result about the law of iterated logarithm for the components of o.s.s. Gaussian random fields.

Proposition 4.2. *Under the conditions of Proposition 4.1, there exists a positive and finite constant K_6 , depending on D , N and d only, such that for every $j = 1, 2, \dots, d$*

$$\limsup_{t \rightarrow 0} \frac{|X_j(t)|}{|t|^{\lambda_j} (\ln 1/|t|)^{p_j-1} (\ln \ln 1/|t|)^{1/2}} \leq K_6 \quad \text{a.s.} \quad (4.11)$$

Now we prove the main result of this section.

Theorem 4.1. *If $\det \text{Cov}(X(1)) > 0$, then for every Borel set $F \subseteq \mathbf{R}^N$ almost surely*

$$\dim X(F) = \min \left\{ d; \left(\dim F + \sum_{i=1}^j (\lambda_j - \lambda_i) \right) \lambda_j^{-1}, \quad j = 1, \dots, d \right\} \quad (4.12)$$

$$= \begin{cases} \left(\dim F + \sum_{i=1}^k (\lambda_k - \lambda_i) \right) \lambda_k^{-1} & \text{if } \sum_{i=0}^{k-1} \lambda_i < \dim F \leq \sum_{i=0}^k \lambda_i, \\ d & \text{if } \dim F > \sum_{i=1}^d \lambda_i, \end{cases} \quad (4.13)$$

$$\dim Gr X(F) = \begin{cases} \dim X(F) & \text{if } \dim F \leq \sum_{i=1}^d \lambda_i, \\ \dim F + \sum_{i=1}^d (1 - \lambda_i) & \text{if } \dim F > \sum_{i=1}^d \lambda_i. \end{cases} \quad (4.14)$$

P r o o f. Let B be a real invertible $d \times d$ matrix such that $\tilde{D} = B^{-1}DB$ is of the real canonical form. Consider the Gaussian random field Y defined by

$$Y(t) = B^{-1}X(t), \quad t \in \mathbf{R}^N.$$

Then Y is an isotropic, o.s.s. Gaussian random field in $\mathbf{R}^d$ with exponent $\tilde{D}$ and stationary increments. Since B is invertible, the linear operator in $\mathbf{R}^d$ defined by $x \rightarrow Bx$ is bi-Lipschitz, and hence preserves the Hausdorff dimension. So without loss of generality, we will assume that D itself is of the real canonical form.

Since X is o.s.s. and isotropic with stationary increments we have for all $t, s \in \mathbf{R}^N$, $X(t) - X(s) \stackrel{d}{\sim} |t - s|^D X(1)$. This implies that for all $t, s \in \mathbf{R}^N$

$$\begin{aligned} \det \text{Cov}(X(t) - X(s)) &= (\det |t - s|^D)^2 \det \text{Cov}(X(1)) \\ &= \prod_{j=1}^d |t - s|^{2\alpha_j} \det \text{Cov}(X(1)) = \prod_{j=1}^d |t - s|^{2\lambda_j} \det \text{Cov}(X(1)). \end{aligned}$$

Then by the proof of the lower bound in Theorem 2.1 of [30] we see immediately that the right-hand sides of (4.12) and (4.14) serve as the lower bound for $\dim X(F)$ and $\dim GrX(F)$, respectively.

The reverse inequalities in (4.12) and (4.14) follow from Lemma 2.1 in [30] and (4.4) in Proposition 4.1. Finally, (4.13) is an easily verified fact, see Lemma 2.3 in [30]. This finishes the proof of Theorem 4.1.

R e m a r k. The condition $\det \text{Cov}(X(1)) > 0$ is only used in Theorem 4.1 in obtaining the lower bound. It is not determined by the self-similarity exponent D . This is easily seen from the following example. Let $X = \{X(t), t \in \mathbf{R}^N\}$ be the usual fractional Brownian motion with index α in $\mathbf{R}^d$. Then for every $B \in \text{End}(\mathbf{R}^d)$, $Y = BX$ is an o.s.s. Gaussian random field with exponent $D = \alpha I_d$. We see that $\det \text{Cov}(X(1)) > 0$ if and only if $B \in \text{Aut}(\mathbf{R}^d)$. For multiparameter operator fractional Brownian motion we have the following sufficient condition.

Proposition 4.3. *Let D be a linear operator on $\mathbf{R}^d$ with $0 < \lambda_D$, $\Lambda_D < 1$ and let X be the operator fractional Brownian motion with exponent D . If there exists a unitary matrix B in $\mathbf{C}$ such that BDB^{-1} is diagonal, then $\det \text{Cov}(X(1)) > 0$.*

P r o o f. The conclusion follows easily from (3.8) and some elementary calculations.

R e m a r k. Another problem of interest is the Hausdorff dimension of the level set $X^{-1}(x) = \{t \in \mathbf{R}^N: X(t) = x\}$, where $x \in \mathbf{R}^d$. By using Proposition 4.1 and an argument in Cuzick [3], it is easy to prove that if $N < \sum_{j=1}^d \lambda_j$, then for all $x \in \mathbf{R}^d$, $X^{-1}(x) = \emptyset$ a.s. Combining (4.14) in Theorem 4.1 and a cross sectional argument (Cuzick [3], [11]) we can prove that if $N > \sum_{j=1}^d \lambda_j$, then for almost all $x \in \mathbf{R}^d$

$$\dim X^{-1}(x) = N - \sum_{j=1}^d \lambda_j \quad \text{a.s.} \quad (4.15)$$

But (4.15) does not give the Hausdorff dimension of the level set $X^{-1}(x)$ for any given $x \in \mathbf{R}^d$. For fractional Brownian motion, the Hausdorff dimension of the level set $X^{-1}(x)$ were obtained by Kahane [11] for every fixed $x \in \mathbf{R}^d$. In fact, much more have been proven for the level sets of fractional Brownian motion and Gaussian random fields with fractional Brownian motion components. We mention that Monrad and Pitt [20] proved a uniform Hausdorff dimension result for the level sets of fractional Brownian motion, and Xiao [33] obtained the exact Hausdorff measure function for $X^{-1}(x)$, where X is a Gaussian random field with fractional Brownian motion components. In all these papers, the concept of «local nondeterminism» was applied. It is natural to ask whether o.s.s. Gaussian random fields are locally nondeterministic. An affirmative answer will be useful in studying the local times and the Hausdorff dimension of $X^{-1}(x)$ for each fixed $x \in \mathbf{R}^d$.

Talagrand [25], [26] and Xiao [32], [33] considered the exact Hausdorff measure of the image and graph of fractional Brownian motion and Gaussian fields with fractional Brownian motion components. Similar problems for o.s.s. Gaussian random fields are also open.

5. Hitting probabilities and applications. In this section, we study polar sets and transience for isotropic o.s.s. Gaussian random fields $X = \{X(t), t \in \mathbf{R}^N\}$ in $\mathbf{R}^d$ with stationary increments. For this purpose, we need first to prove some estimates for the probabilities that X hits a ball in $\mathbf{R}^d$.

By the proof of Proposition 4.1 we see that for each $j = 1, \dots, d$ and any $s, t \in \mathbf{R}^N$

$$\mathbf{E}\left((X_j(s) - X_j(t))^2\right) \leq K\sigma_j^2(|t - s|),$$

where for $x > 0$

$$\sigma_j(x) = x^{\lambda_j} |\ln x|^{p_j-1}.$$

It is easy to see that for $x > 0$ small (or large) enough, $\sigma(x)$ has an inverse function $\sigma_j^{-1}(y)$ with $\sigma_j^{-1}(y) \sim Ky^{1/\lambda_j} |\ln y|^{-(p_j-1)/\lambda_j}$ for $y \rightarrow 0^+$ or ∞ for some constant $K > 0$ depending on λ_j and p_j only. We denote $\gamma = \min\{(N + \sum_{i=1}^j (\lambda_j - \lambda_i))/\lambda_j, 1 \leq j \leq d\}$. If k satisfies

$$\sum_{i=1}^{k-1} \lambda_i < N \leq \sum_{i=1}^k \lambda_i, \quad (5.1)$$

then it is easy to verify that (cf. Lemma 2.3 in [30]) $\gamma = (N + \sum_{i=1}^k (\lambda_k - \lambda_i))/\lambda_k$ and we notice that γ is the Hausdorff dimension of $X([0, 1]^N)$ or $X(\mathbf{R}^N)$.

The proof of the following proposition is reminiscent to those of Lemma 3.1 in [31], Proposition 2.1 in [35] and Lemma 4 in [12].

Proposition 5.1. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an isotropic operator-self-similar Gaussian random field in $\mathbf{R}^d$ with exponent D and stationary increments. We assume that D is of the real canonical form and $S = \{t \in \mathbf{R}^N: a \leq |t| \leq 2a\}$ ($a > 0$). If $\det \text{Cov}(X(1)) > 0$ and $N < \sum_{j=1}^d \lambda_j$, then there exist positive and finite constants K_7, K_8 and $\rho \geq 0$, depending on N, d, D and $\text{Cov}(X(1))$ only, such that for any $0 < r < 1$ with $\sigma_k^{-1}(r) \leq a$, where k is as in (5.1), and for any $y \in \mathbf{R}^d$*

$$\begin{aligned} & \mathbf{P} \left\{ \inf_{t \in S} |X(t) - y| \leq r \right\} \\ & \leq K_7 \exp \left(-K_8 \frac{|y|^2}{\|a^D\|^2} \right) \cdot a^{N - \sum_{j=1}^d \lambda_j} r^{d-\gamma} \left(\ln \frac{1}{r} \right)^\rho. \end{aligned} \quad (5.2)$$

In particular, if D is diagonal, then $\rho = 0$.

P r o o f. For any bounded set $F \subset \mathbf{R}^N$, we use $N(F, r)$ to denote the smallest number of open balls of radius r that are needed to cover F . Then we have

$$N(S, \sigma_k^{-1}(r)) \leq K a^N (\sigma_k^{-1}(r))^{-N}. \quad (5.3)$$

Let $\{S_m, 1 \leq m \leq N(S, \sigma_k^{-1}(r))\}$ be a family of balls of radius $\sigma_k^{-1}(r)$ that cover S and let

$$A = \left\{ \inf_{t \in S} |X(t) - y| < r \right\}, \quad A_m = \left\{ \inf_{t \in S_m} |X(t) - y| < r \right\}.$$

Then

$$A \subseteq \bigcup_{m=1}^{N(S, \sigma_k^{-1}(r))} A_m. \quad (5.4)$$

Now fix a $1 \leq m \leq N(S, \sigma_k^{-1}(r))$. Let $\varepsilon_n = \sigma_k^{-1}(r) \exp(-2^{n+1})$, $n = 0, 1, 2, \dots$, and let $\{t_i^{(n)}, 1 \leq i \leq N(S_m, \varepsilon_n)\}$ be a set of the centers of open balls with radius ε_n that cover S_m . For each $n \geq 0$, let $r_{n,j} = \beta \sigma_j(\varepsilon_n) 2^{(p_j-1/2)(n+1)}$, $j = 1, 2, \dots, d$, where p_j ($j = 1, \dots, d$) are the constants in Proposition 4.1 and where $\beta \geq 2K_4$ is a constant to be determined later. We notice that

$$\sum_{n=0}^{\infty} r_{n,j} \leq K \beta r^{\lambda_j/\lambda_k} \left(\ln \frac{1}{r} \right)^{(1-\lambda_j/\lambda_k)(p_j-1)}. \quad (5.5)$$

For ease of notation, we denote the right-hand side of (5.5) by $\beta \rho_j$, in which the logarithmic factor disappears when $p_j = 1$ or $\lambda_j = \lambda_k$. Let

$$A_i^{(l)} = \left\{ |X_j(t_i^{(l)}) - y_j| \leq r + \sum_{q=l}^{\infty} r_{q,j} \text{ for all } 1 \leq j \leq d \right\},$$

$$A^{(n)} = \bigcup_{l=1}^n \bigcup_{i=1}^{N(S_m, \varepsilon_l)} A_i^{(l)} = A^{(n-1)} \cup \bigcup_{i=1}^{N(S_m, \varepsilon_n)} A_i^{(n)}. \quad (5.6)$$

Then by (4.3) in Proposition 4.1, we have

$$\mathbf{P}(A_m) \leq \lim_{n \rightarrow \infty} \mathbf{P}(A^{(n)}). \quad (5.7)$$

By (5.6), we have

$$\mathbf{P}(A^{(n)}) \leq \mathbf{P}(A^{(n-1)}) + \mathbf{P}(A^{(n)} \setminus A^{(n-1)}), \quad (5.8)$$

$$\mathbf{P}(A^{(n)} \setminus A^{(n-1)}) \leq \sum_{i=1}^{N(S_m, \varepsilon_n)} \mathbf{P}(A_i^{(n)} \setminus A_i^{(n-1)}), \quad (5.9)$$

where i' is chosen so that $|t_i^{(n)} - t_{i'}^{(n-1)}| < \varepsilon_{n-1}$. To get an upper bound for the probabilities in (5.9), we use (5.5) and the triangle inequality to derive

$$\begin{aligned} \mathbf{P}(A_i^{(n)} \setminus A_{i'}^{(n-1)}) &= \mathbf{P} \left\{ \bigcap_{j=1}^d \left\{ |X_j(t_i^{(n)}) - y_j| < r + \sum_{q=n}^{\infty} r_{q,j} \right\} \right. \\ &\quad \left. \bigcap \left(\bigcup_{j=1}^d \left\{ |X_j(t_{i'}^{(n-1)}) - y_j| > r + \sum_{q=n-1}^{\infty} r_{q,j} \right\} \right) \right\} \\ &\leq \mathbf{P} \left\{ \bigcap_{j=1}^d \left\{ |X_j(t_i^{(n)}) - y_j| < r + \beta \rho_j \right\} \right. \\ &\quad \left. \bigcap \left(\bigcup_{j=1}^d \left\{ |X_j(t_i^{(n)}) - X_j(t_{i'}^{(n-1)})| \geq r_{n-1,j} \right\} \right) \right\}. \quad (5.10) \end{aligned}$$

In order to estimate the probability in (5.10), we need to create independence among the events. Since $\text{Cov}(X(1))$ is invertible, there exists a real invertible $d \times d$ matrix U_1 such that $\text{Cov}(X(1)) = U_1 U_1'$. By the operator self-similarity, for every $t \in \mathbf{R}^N \setminus \{0\}$,

$$\text{Cov}(X(t)) = \text{Cov}(|t|^D X(1)) = |t|^D U_1 (|t|^D U_1)'. \quad (5.11)$$

We denote the matrix $|t|^D U_1$ by U_t , and we will break the rest of proof into several lemmas.

Lemma 5.1. *Under the conditions of Proposition 5.1, for $t \in \mathbf{R}^N \setminus \{0\}$ and $s \neq t$ both $U_t^{-1} X(t)$ and $U_{t-s}^{-1}(X(t) - X(s))$ are normal random variables in $\mathbf{R}^d$ with mean 0 and covariance matrix I_d .*

P r o o f. This follows from (5.11) immediately.

Lemma 5.2. *Assume that X and Y are two normal random variables in $\mathbf{R}^d$ with mean 0 and covariance matrix I_d , and they are also jointly normal. Then there exist a $d \times d$ matrix $\tilde{U}$ and a mean 0 normal random variable Ξ in $\mathbf{R}^d$ which is independent of X such that*

$$Y = \tilde{U} X + \Xi. \quad (5.12)$$

Furthermore, there is a positive constant K_9 such that for any $u > 0$,

$$\mathbf{P}\{|\Xi| \geq u\} \leq \exp(-K_9 u^2). \quad (5.13)$$

P r o o f. Let $\tilde{U} = (u_{ij})$ with $u_{ij} = \mathbf{E}(Y_i X_j)$. Then it is easy to verify that $\Xi = Y - \tilde{U} X$ is a normal random variable in $\mathbf{R}^d$ with mean 0 and covariance matrix $\text{Cov}(\Xi) = I_d - \tilde{U} \tilde{U}'$, and Ξ is independent of X .

Finally since $|u_{ij}| \leq 1$, by (2.2) we have $\|\tilde{U}\| \leq d^{3/2}$ and $|\Xi| \leq |Y| + \|\tilde{U}\| |X| \leq |Y| + d^{3/2} |X|$. Hence

$$\mathbf{P}\{|\Xi| \geq u\} \leq \mathbf{P}\left\{|Y| \geq \frac{u}{2}\right\} + \mathbf{P}\left\{|X| \geq \frac{u}{2d^{3/2}}\right\}$$

and (5.13) follows from the well-known estimates for the tail probability of normal random variables in $\mathbf{R}^d$.

Lemma 5.3. *Let $t \in \mathbf{R}^N \setminus \{0\}$ and $s \neq t$ with $|t - s| < 1$. Then there exists a constant $K_{10} > 0$ such that*

$$\left|U_{t-s}^{-1}(X(t) - X(s))\right| \geq K_{10} \max_{1 \leq j \leq d} \frac{|X_j(t) - X_j(s)|}{\sigma_j(|t - s|)}. \quad (5.14)$$

P r o o f. First we note that $U_{t-s}^{-1} = U_1^{-1}(|t - s|^{-1})^D$ which implies that for every $x \in \mathbf{R}^d$

$$|U_{t-s}^{-1} x| \geq K \left|(|t - s|^{-1})^D x\right|. \quad (5.15)$$

Suppose that D is of the real canonical form with diagonal blocks $J_1, \dots, J_m$, where each J_l is a $(p_l \times p_l)$ -matrix and is either a Jordan cell matrix or a block of the form (4.2). Then we can write $\mathbf{R}^d$ as a direct sum of $\mathbf{R}^{p_l}$ ($l = 1, \dots, m$). Now we fix a block J which is a $p \times p$ matrix with eigenvalue $\alpha = \lambda + i\beta$. By using (2.1), (2.2) and an argument similar to that in the proof of Proposition 4.1, and distinguishing two cases (i) J is a Jordan cell matrix or (ii) J is of the form (4.2), we see that $\| |t - s|^J \| \leq K_{11} |t - s|^\lambda |\ln |t - s||^{p-1}$, where $K_{11} > 0$ is a constant depending on J only. Hence for every $x \in \mathbf{R}^p$

$$\begin{aligned} \left| \left(\frac{1}{|t - s|} \right)^J x \right| &\geq \frac{1}{\| |t - s|^J \|} \cdot |x| \geq K_{11}^{-1} \frac{|x|}{|t - s|^\lambda |\ln |t - s||^{p-1}} \\ &\geq K_{11}^{-1} \max_{1 \leq l \leq p} \frac{|x_l|}{|t - s|^\lambda |\ln |t - s||^{p-1}}. \end{aligned}$$

It follows that for every $x \in \mathbf{R}^d$,

$$\left| \left(\frac{1}{|t - s|} \right)^D x \right| \geq K_{11}^{-1} \max_{1 \leq j \leq d} \frac{|x_j|}{|t - s|^{\lambda_j} |\ln |t - s||^{p_j-1}}. \quad (5.16)$$

Combining (5.15) and (5.16) we get (5.14).

Lemma 5.4. *Under the conditions of Proposition 5.1, we have that for any $t \in S$, $y \in \mathbf{R}^d$ and $\mathbf{r} = (r_1, \dots, r_d) > 0$ with $\sigma_k^{-1}(|\mathbf{r}|) \leq a$*

$$\begin{aligned} &\mathbf{P}\left\{|X_j(t) - y_j| \leq r_j \text{ for all } 1 \leq j \leq d\right\} \\ &\leq K_{12} \exp\left(-K_{13} \frac{|y|^2}{\| |t|^D \|^2}\right) \prod_{j=1}^d \frac{r_j}{|t|^{\lambda_j}}. \end{aligned} \quad (5.17)$$

P r o o f. Let $V_1 = \{v \in \mathbf{R}^d: |v_j| \leq r_j \text{ for all } 1 \leq j \leq d\}$. If $|y| \geq 4|\mathbf{r}|$, then for every $v \in V_1$, we have $|v - y|^2 \geq |y|^2/2$. It follows that the probability in the left-hand side of (5.17) is

$$\begin{aligned} & \int_{V_1} \frac{1}{(2\pi)^{d/2} \sqrt{\det \text{Cov}(X(t))}} \exp\left(-\frac{1}{2} |U_t^{-1}(v - y)|^2\right) dv \\ & \leq \int_{V_1} \frac{1}{(2\pi)^{d/2} \sqrt{\det \text{Cov}(X(t))}} \exp\left(-K \frac{|y|^2}{\|U_t^{-1}\|^2}\right) dv \\ & \leq K_{12} \exp\left(-K_{13} \frac{|y|^2}{\|t\|^D}\right) \prod_{j=1}^d \frac{r_j}{|t|^{\lambda_j}}. \end{aligned} \quad (5.18)$$

On the other hand, if $|y| < 4|\mathbf{r}|$, then it is easy to see that $\exp(-K_{13} |y|^2 / \|t\|^D) \geq K$. Hence an inequality similar to (5.18) still holds. This proves Lemma 5.4.

Now we can put things together to finish the proof of Proposition 5.1.

P r o o f o f P r o p o s i t i o n 5.1 (c o n t i n u e d). By using Lemmas 5.1 and 5.2 we can write

$$U_{t_i^{(n)} - t_i^{(n-1)}}^{-1} (X(t_i^{(n)}) - X(t_i^{(n-1)})) = \tilde{U} U_{t_i^{(n)}}^{-1} X(t_i^{(n)}) + \Xi, \quad (5.19)$$

where $X(t_i^{(n)})$ and Ξ are independent. Recall that $|t_i^{(n)} - t_i^{(n-1)}| < \varepsilon_{n-1}$ and by Lemma 5.3 we see that if $|X_j(t_i^{(n)}) - X_j(t_i^{(n-1)})| \geq r_{n-1,j}$ for some $1 \leq j \leq d$, then

$$\left| U_{t_i^{(n)} - t_i^{(n-1)}}^{-1} (X(t_i^{(n)}) - X(t_i^{(n-1)})) \right| \geq K_{14} \beta 2^{n/2}. \quad (5.20)$$

It follows from (5.19), (5.20) that the probability in (5.10) is at most

$$\begin{aligned} & \mathbf{P} \left\{ \left(\bigcap_{j=1}^d \{ |X_j(t_i^{(n)}) - y_j| < r + \beta \rho_j \} \right) \right. \\ & \quad \cap \left\{ |\Xi| \geq K_{14} \beta 2^{n/2} - \left| \tilde{U} U_{t_i^{(n)}}^{-1} X(t_i^{(n)}) \right| \right\} \\ & \leq \mathbf{P} \left\{ \left(\bigcap_{j=1}^d \{ |X_j(t_i^{(n)}) - y_j| < r + \beta \rho_j \} \right) \cap \left\{ |\Xi| \geq \frac{K_{14} \beta}{2} 2^{n/2} \right\} \right\} \\ & \quad + \mathbf{P} \left\{ \left(\bigcap_{j=1}^d \{ |X_j(t_i^{(n)}) - y_j| < r + \beta \rho_j \} \right) \right. \\ & \quad \cap \left\{ \left| \tilde{U} U_{t_i^{(n)}}^{-1} X(t_i^{(n)}) \right| \geq \frac{K_{14} \beta}{2} 2^{n/2} \right\} \right\} =: Q_1 + Q_2. \end{aligned} \quad (5.21)$$

By the independence of $X(t_i^{(n)})$ and Ξ , Lemmas 5.2 and 5.4 we obtain

$$Q_1 \leq K_{12} \exp\left(-K_{13} \frac{|y|^2}{\|a^D\|^2}\right) \prod_{j=1}^d \frac{r + \beta \rho_j}{a^{\lambda_j}} \cdot \exp(-K_{14} \beta^2 2^n). \quad (5.22)$$

To estimate Q_2 , let

$$V_2 = \left\{ v \in \mathbf{R}^d: |v_j| < r + \beta \rho_j \text{ for all } 1 \leq j \leq d \right\}.$$

Then

$$Q_2 \leq \mathbf{P} \left\{ \left\{ X(t_i^{(n)}) - y \in V_2 \right\} \cap \left\{ \left| U_{t_i^{(n)}}^{-1} X(t_i^{(n)}) \right| \geq \frac{K_{14}\beta}{2} 2^{n/2} \right\} \right\}.$$

Since $U_{t_i^{(n)}}^{-1} X(t_i^{(n)})$ is normal with mean 0 and covariance matrix I_d , we have

$$\begin{aligned} Q_2 &\leq \int_{\{U_{t_i^{(n)}}^{-1} X(t_i^{(n)}) - y \in V_2, |u| \geq K_{14}\beta 2^{n/2}\}} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|u|^2}{2}\right) du \\ &\leq \int_{\{U_{t_i^{(n)}}^{-1} X(t_i^{(n)}) - y \in V_2\}} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|u|^2}{4}\right) du \cdot \exp\left(-\frac{K_{14}\beta^2}{8} 2^n\right) \\ &\leq \mathbf{P} \left\{ \left| X_j(t_i^{(n)}) - y_j \right| \leq r + \beta \rho_j \text{ for all } 1 \leq j \leq d \right\} \cdot \exp\left(-\frac{K_{14}\beta^2}{8} 2^n\right) \\ &\leq K_{12} \exp\left(-K_{13} \frac{|y|^2}{\|a^D\|^2}\right) \prod_{j=1}^d \frac{r + \beta \rho_j}{a^{\lambda_j}} \cdot \exp\left(-\frac{K_{14}\beta^2}{8} 2^n\right), \end{aligned} \quad (5.23)$$

where the last inequality follows from Lemma 5.4. Recall that

$$\rho_j = K r^{\lambda_j/\lambda_k} \left(\ln \frac{1}{r} \right)^{(1-\lambda_j/\lambda_k)(p_j-1)}.$$

Combining (5.21)–(5.23) and the fact that $r < 1$ we see that the probability in (5.10) is at most

$$\begin{aligned} &K_{15} \exp\left(-K_{13} \frac{|y|^2}{\|a^D\|^2}\right) r^{d-k+\sum_{i=1}^k \lambda_i/\lambda_k} \\ &\times \left(\ln \frac{1}{r} \right)^\rho a^{-\sum_{j=1}^d \lambda_j} \exp\left(-\frac{K_{14}\beta^2}{8} 2^n\right), \end{aligned} \quad (5.24)$$

where $\rho \geq 0$ is a constant, and $\rho = 0$ if D is diagonalizable (this is because $p_j = 1$ for all $j = 1, 2, \dots, d$). We denote (5.24) by Q_3 . Now we take β large, say $K_{14}\beta^2 > 32N$, then by (5.8)–(5.10) and (5.24) we obtain

$$\begin{aligned} \mathbf{P}(A^{(n)}) &\leq \mathbf{P}(A^{(n-1)}) + Q_3 \\ &\leq K_{16} \left[N(S_m, \varepsilon_0) + \sum_{k=1}^{\infty} N(S_m, \varepsilon_k) \exp\left(-\frac{K_{14}\beta^2}{8} 2^k\right) \right] \\ &\quad \times \exp\left(-K_{13} \frac{|y|^2}{\|a^D\|^2}\right) r^{d-k+\sum_{i=1}^k \lambda_i/\lambda_k} \left(\ln \frac{1}{r} \right)^\rho a^{-\sum_{j=1}^d \lambda_j} \\ &\leq K_{17} \exp\left(-K_{13} \frac{|y|^2}{\|a^D\|^2}\right) r^{d-k+\sum_{i=1}^k \lambda_i/\lambda_k} \left(\ln \frac{1}{r} \right)^\rho a^{-\sum_{j=1}^d \lambda_j}. \end{aligned} \quad (5.25)$$

Therefore, by (5.4), (5.7) and (5.25), we obtain

$$\begin{aligned} \mathbf{P}(A) &\leq K_{18} \exp\left(-K_{13} \frac{|y|^2}{\|a^D\|^2}\right) a^{N-\sum_{j=1}^d \lambda_j} r^{-N/\lambda_k+d-k+\sum_{i=1}^k \lambda_i/\lambda_k} \left(\ln \frac{1}{r}\right)^\rho \\ &= K_{18} \exp\left(-K_{13} \frac{|y|^2}{\|a^D\|^2}\right) a^{N-\sum_{j=1}^d \lambda_j} r^{d-\gamma} \left(\ln \frac{1}{r}\right)^\rho. \end{aligned}$$

This completes the proof of Proposition 5.1 with $K_7 = K_{18}$ and $K_8 = K_{13}$.

By applying Proposition 5.1, we can prove the following theorems.

Theorem 5.1. *Let $X(t)$ ($t \in \mathbf{R}^N$) be an isotropic, operator-self-similar Gaussian random field in $\mathbf{R}^d$ with exponent D and stationary increments. If $\det \text{Cov}(X(1)) > 0$ and $N < \sum_{j=1}^d \lambda_j$, then for any $0 < r < 1$ and $y \in \mathbf{R}^d$*

$$\begin{aligned} &\mathbf{P}\left\{\exists t \in \mathbf{R}^N \text{ such that } |X(t) - y| \leq r\right\} \\ &\leq K_{19} \int_0^\infty x^{N-\sum_{j=1}^d \lambda_j-1} \exp\left(-K_{20} \frac{|y|^2}{\|x^D\|^2}\right) dx \cdot r^{d-\gamma} \left(\ln \frac{1}{r}\right)^\rho. \end{aligned} \quad (5.26)$$

In particular, if D is diagonalizable, then $\rho = 0$.

Proof. Let B be a real invertible $(d \times d)$ -matrix such that $\tilde{D} = B^{-1}DB$ is of the real canonical form. We define an isotropic o.s.s. Gaussian random field $Y = \{Y(t), t \in \mathbf{R}^N\}$ with exponent $\tilde{D}$ as in the proof of Theorem 4.1. Then $|X(t) - y| \leq r$ implies $|Y(t) - B^{-1}y| \leq \|B^{-1}\| r$, and a similar inverse is also true. Therefore we may and will assume that D itself is of the real canonical form.

The rest of the proof is similar to that of Theorem 3.1 in Xiao [31]. For any integer $n \in \mathbf{Z}$, let $S_n = \{t \in \mathbf{R}^N: 2^n \leq |t| \leq 2^{n+1}\}$. Then $\mathbf{R}^N \setminus \{0\} = \cup_{n \in \mathbf{Z}} S_n$. By using Proposition 5.1 for $n \geq 0$ and its proof for $n < 0$, we have

$$\begin{aligned} \mathbf{P}\left\{\exists t \in \mathbf{R}^N \text{ such that } |X(t) - y| \leq r\right\} &\leq \sum_{n \in \mathbf{Z}} \mathbf{P}\left\{\inf_{t \in S_n} |X(t) - y| \leq r\right\} \\ &\leq K_7 \sum_{n \in \mathbf{Z}} \exp\left(-K_8 \frac{|y|^2}{\|2^{nD}\|^2}\right) \cdot 2^{n(N-\sum_{j=1}^d \lambda_j)} r^{d-\gamma} \left(\ln \frac{1}{r}\right)^\rho \\ &\leq K_{19} \int_0^\infty x^{N-\sum_{j=1}^d \lambda_j-1} \exp\left(-K_{20} \frac{|y|^2}{\|x^D\|^2}\right) dx \cdot r^{d-\gamma} \left(\ln \frac{1}{r}\right)^\rho. \end{aligned}$$

This proves (5.26).

Theorem 5.2. *Under the conditions of Theorem 5.1, we have that then for any $T > 0$ and any $0 < r < \min\{1, \sigma_k(T)\}$, where k is as in (5.1),*

$$\begin{aligned} &\mathbf{P}\left\{\exists t \in \mathbf{R}^N \text{ such that } |t| \geq T \text{ and } |X(t)| \leq r\right\} \\ &\leq K_{21} T^{N-\sum_{j=1}^d \lambda_j} r^{d-\gamma} \left(\ln \frac{1}{r}\right)^\rho. \end{aligned} \quad (5.27)$$

In particular, if D is diagonalizable, then $\rho = 0$.

P r o o f. Let S_n be as in the proof of Theorem 5.1 and let n_0 be the integer satisfying $2^{n_0} \leq T < 2^{n_0+1}$. It follows from Proposition 5.1 that

$$\begin{aligned} & \mathbf{P} \left\{ \exists t \in \mathbf{R}^N \text{ such that } |t| \geq T \text{ and } |X(t)| \leq r \right\} \\ & \leq \sum_{n=n_0+1}^{\infty} K_7 (2^n)^{N-\sum_{j=1}^d \lambda_j} r^{d-\gamma} \left(\ln \frac{1}{r} \right)^\rho \leq K_{21} T^{N-\sum_{j=1}^d \lambda_j} r^{d-\gamma} \left(\ln \frac{1}{r} \right)^\rho. \end{aligned}$$

R e m a r k. Theorems 5.1 and 5.2 partially extend the hitting and delayed hitting probability estimates of [4] for Brownian motion and [31], [35] for fractional Brownian motion. Hitting probability estimates have played crucial roles in the studies of various sample path properties such as escape rates, polar sets and packing measure of the image set. We note that the proof of (5.27) does not use conditioning and is much simpler than those of Dvoretzky and Erdős [4] and Xiao [31]. If we further assume that D is diagonal, then an argument similar to that in Xiao [35] can be applied to prove the lower bounds in (5.26) and (5.27).

It is interesting to notice that $d-\gamma$ in (5.26) and (5.27) is $d-\dim X(\mathbf{R}^N)$. We can interpret it as the co-dimension of the sample paths of X . The following result, which gives a sufficient condition for a compact $F \subset \mathbf{R}^d \setminus \{0\}$ to be a polar set for an o.s.s. Gaussian random field X , offers more support for this interpretation.

Theorem 5.3. *Let $X(t)$ ($t \in \mathbf{R}^N$) be an isotropic, operator-self-similar Gaussian random field in $\mathbf{R}^d$ with exponent D and stationary increments. Assume that $\det \text{Cov}(X(\mathbf{1})) > 0$ and $N < \sum_{j=1}^d \lambda_j$. If $F \subseteq \mathbf{R}^d \setminus \{0\}$ is a compact set and $s^{d-\gamma} ((\ln 1/s)/s)^{\rho-m(F)} = 0$, then $X^{-1}(F) = \emptyset$ almost surely. In particular, if $\dim F < d - \dim X(\mathbf{R}^N)$, then $X^{-1}(F) = \emptyset$ a.s.*

P r o o f. Let Y be the o.s.s. Gaussian random field in the proof of Theorem 5.1. Then $X^{-1}(F) = \emptyset$ if and only if $Y^{-1}(B^{-1}F) = \emptyset$. Without loss of generality, we will assume that D is of the real canonical form.

It is sufficient to show that for $T = [1, 2]^N$, $X(T) \cap F = \emptyset$ almost surely. Since $s^{d-\gamma} (\ln(1/s))^{\rho-m(F)} = 0$, it follows from Theorem 32 in [21] that there exists a sequence of open balls $\{B(y_n, r_n)\}$ such that

$$F \subseteq \limsup_{n \rightarrow \infty} B(y_n, r_n) \quad \text{and} \quad \sum_{n=1}^{\infty} r_n^{d-\gamma} \left(\ln \frac{1}{r_n} \right)^\rho < \infty. \quad (5.28)$$

Consider the events $E_n = \{X(T) \cap B(y_n, r_n) \neq \emptyset\}$. Then we have

$$\{X(T) \cap F \neq \emptyset\} \subseteq \limsup_{n \rightarrow \infty} E_n. \quad (5.29)$$

By using Proposition 5.1, we have that for n large enough

$$\mathbf{P}(E_n) \leq K r_n^{d-\gamma} \left(\ln \frac{1}{r_n} \right)^\rho. \quad (5.30)$$

By (5.28)–(5.30) and the Borel–Cantelli lemma, we have $X(T) \cap F = \emptyset$ a.s. This finishes the proof of Theorem 5.3.

If $X = \{X(t), t \in \mathbf{R}^N\}$ is an ordinary fractional Brownian motion or a Gaussian random field with fractional Brownian motion components, some partial inverse results to Theorem 5.3 can also be proved. See [28] and [35]. It would be interesting to prove such results for general o.s.s. Gaussian random fields.

Notice that we have assumed $0 < r < 1$ in Proposition 5.1. If $r > 1$, the hitting probability estimate is different. Let

$$\gamma' = \max \left\{ \frac{N + \sum_{i=j+1}^d (\lambda_j - \lambda_i)}{\lambda_j}, j = 1, \dots, d \right\}.$$

It is easy to verify that if k' satisfies

$$\sum_{i=k'+1}^d \lambda_i < N \leq \sum_{i=k'}^d \lambda_i, \quad (5.31)$$

then $\gamma' = [N + \sum_{i=k'+1}^d (\lambda_{k'} - \lambda_i)] / \lambda_{k'}$.

Proposition 5.2. *Let $X = \{X(t), t \in \mathbf{R}^N\}$ be an isotropic operator-self-similar Gaussian random field in $\mathbf{R}^d$ with exponent D and stationary increments. We assume that D is of the real canonical form and $S = \{t \in \mathbf{R}^N: a \leq |t| \leq 2a\}$ ($a > 1$). If $\det \text{Cov}(X(1)) > 0$ and $N < \sum_{j=1}^d \lambda_j$, then there exist positive and finite constants K_{22} , K_{23} and $\rho' \geq 0$, depending on N, d, D and $\text{Cov}(X(1))$ only, such that for any $r > 1$ with $\sigma_{k'}^{-1}(r) \leq a$, where k' is as in (5.31), and $y \in \mathbf{R}^d$*

$$\begin{aligned} \mathbf{P} \left\{ \inf_{t \in S} |X(t) - y| \leq r \right\} &\leq K_{22} \exp \left(-K_{23} \frac{|y|^2}{\|a^D\|^2} \right) \\ &\times a^{N - \sum_{j=1}^d \lambda_j} r^{d - \gamma'} (\ln r)^{\rho'}. \end{aligned} \quad (5.32)$$

If D is diagonalizable, then $\rho' = 0$.

P r o o f. The proof is very similar to that of Proposition 5.1 with k' replacing k . Since $r > 1$, the probability in (5.10) is not bounded above by (5.25), but by

$$\begin{aligned} K_{24} \exp \left(-K_{13} \frac{|y|^2}{\|a^D\|^2} \right) r^{k' + \sum_{i=k'+1}^d \lambda_i / \lambda_{k'}} (\ln r)^{\rho'} \\ \times a^{-\sum_{j=1}^d \lambda_j} \exp \left(-\frac{K_{14} \beta^2}{8} 2^n \right). \end{aligned}$$

We omit the details.

As an application of Proposition 5.2, we prove the following result about the transience of o.s.s. Gaussian random fields.

Theorem 5.4. *Let $X(t)$ ($t \in \mathbf{R}^N$) be an isotropic, operator-self-similar Gaussian random field in $\mathbf{R}^d$ with exponent D and stationary increments. If $\det \text{Cov}(X(1)) > 0$ and $N < \sum_{j=1}^d \lambda_j$, then X is transient in the sense that*

$$\mathbf{P} \left\{ \lim_{t \rightarrow \infty} |X(t)| = \infty \right\} = 1. \quad (5.33)$$

P r o o f. Similar to the proof of Theorem 5.1, without loss of generality we assume that the exponent D is of the real canonical form. Then by Proposition 5.2 we have for all $n \geq 1$

$$\mathbf{P} \left\{ \inf_{t \in S_n} |X(t)| \leq n \right\} \leq K 2^{n(N - \sum_{j=1}^d \lambda_j)} n^{d-\gamma'} (\ln n)^{\rho'}.$$

Therefore (5.33) follows immediately from the Borel–Cantelli lemma.

In light of Theorem 5.4, it is natural to study the lower functions (or escape rates) at infinity and near zero for general transient o.s.s. Gaussian random fields. For ordinary Brownian motion in $\mathbf{R}^d$, the problem was completely solved by Dvoretzky and Erdős [4] for $d \geq 3$, by Spitzer [24] for planar Brownian motion. Their results have been extended by many authors to Lévy stable processes, diffusions and certain self-similar Markov processes. See Xiao [36] for a list of related references. For fractional Brownian motion, the lower functions near zero in the transient case was characterized by Kôno [12] using an integral test. It is easy to see that an integral test of the same form also holds for the lower functions at infinity. We can establish integral tests for the lower functions at infinity and near zero for a special class of o.s.s. Gaussian random fields, and show that in general, the test at infinity differs significantly from the test near zero. But for general transient o.s.s. Gaussian random fields, the problems seem more difficult and depend on the real canonical form of the exponent D , we have not been able to solve the problems completely.

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