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ON COUPLING OF BROWNIAN BRIDGES

Пусть $\{B(t), 0 \leq t \leq 1\}$ — броуновский мост и $Y(t) = \int_0^t f(u) dB(u)$, где $f: [0, 1] \rightarrow \{+1, -1\}$, — неслучайная измеримая функция. Тогда задача: «Существует ли броуновский мост B^* такой, что $|Y(t)| \geq |B^*(t)|$ п.н., $0 \leq t \leq 1$?» имеет положительное решение. В статье доказывается, что в качестве B^* можно взять

$$B^*(t) = \begin{cases} Y(t), & 0 \leq t \leq \tau, \\ B(t), & \tau \leq t \leq 1, Y(\tau) = +B(\tau), \\ -B(t), & \tau \leq t \leq 1, Y(\tau) = -B(\tau), \end{cases}$$

где $\tau = \max\{t \geq 0: |Y(t)| = |B(t)|\}$.

Обсуждается также положительный ответ на вопрос о существовании броуновского моста B_* такого, что $\max_{0 \leq t \leq 1} |B_*(t)| = \max_{0 \leq t \leq 1} \{|X_+(t)| \vee |X_-(t)|\}$, где $X_+(t) = \int_0^t 1_{\{f=+1\}}(u) dB(u)$, $X_-(t) = \int_0^t 1_{\{f=-1\}}(u) dB(u)$, $0 \leq t \leq 1$.

В качестве следствия этих построений мы получаем строгое неравенство, сравнивающее распределения величин $\max_{0 \leq t \leq 1} |B(t)|$ и $\max_{0 \leq t \leq 1} |Y(t)|$.

Ключевые слова и фразы: броуновский мост, каплинг (coupling), перестановочные случайные величины.

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1. Introduction and main results. Let $B(t)$, $0 \leq t \leq 1$, be a standard Brownian bridge, and let $f: [0, 1] \rightarrow \{+1, -1\}$ be a nonrandom, measurable function. We will construct two Brownian bridges associated with f .

F i r s t c o n s t r u c t i o n . Let

$$Y(t) = \int_0^t f(u) dB(u), \quad (1)$$

$$\tau = \max \{t \geq 0: |Y(t)| = |B(t)|\}. \quad (2)$$

Observe that τ is well defined because the subset of numbers that is maximized in (2) is not empty: $|Y(0)| = |B(0)| = 0$, so 0 is included. Also τ is a random time which is not a stopping time (with respect to the natural filtration of $\{B(t)\}$).

The idea of the construction is simple. The constructed process, to be denoted by $B^*(t)$, will follow $Y(t)$ up to the random time τ . After τ it will follow $B(t)$ if $Y(\tau) = B(\tau)$, or it will follow $-B(t)$ if $Y(\tau) = -B(\tau)$. Formally we define

$$B^*(t) = \begin{cases} Y(t), & 0 \leq t \leq \tau, \\ B(t), & \tau \leq t \leq 1, Y(\tau) = +B(\tau), \\ -B(t), & \tau \leq t \leq 1, Y(\tau) = -B(\tau). \end{cases} \quad (3)$$

From the construction it follows immediately that $B^*(t)$, $0 \leq t \leq 1$, is continuous a.s. and also it is a «bridge», namely $B^*(0) = B^*(1) = 0$. It turns out that $B^*(t)$ is actually a Brownian bridge. So our first result is the following:

Theorem 1. $B^*(t)$, $0 \leq t \leq 1$, is a standard Brownian bridge.

The idea of the proof is to establish a discrete version of the result first. This is done in Lemma 1, where the interchangeable role between the discrete counterparts of $\{B^*(t)\}$ and $\{B(t)\}$ is exploited. Then weak convergence techniques allow us to get the continuous-time result. A more detailed explanation of the steps of the proof appears in the paragraph that precedes the beginning of the formal proof.

S e c o n d c o n s t r u c t i o n . Again, the idea of the construction is simple. The constructed process, to be denoted now by $B_*(t)$, can be described informally as follows. We cut the sample path of $B(t)$ according to the partition of its domain $[0, 1] = A_+ \cup A_-$, where we define

$$A_{\pm} = \{0 \leq u \leq 1: f(u) = \pm 1\}. \quad (4)$$

Those parts of the sample path that correspond to A_+ will be pasted together in their natural order and will make the first part of $B_*(t)$, while those that correspond to A_- will also be pasted together, but then reversed in time and those will make the second part of $B_*(t)$. Formally this is done as follows. First we define

$$\varphi_{\pm}(s) = \int_0^s 1_{A_{\pm}}(u) du, \quad 0 \leq s \leq 1. \quad (5)$$

Then we denote by $\psi_{\pm}(t)$ the inverse functions of $\varphi_{\pm}(t)$, respectively, i.e.,

$$\psi_{\pm}(t) = \inf\{0 \leq s: \varphi_{\pm}(s) = t\}, \quad 0 \leq t \leq \varphi_{\pm}(1). \quad (6)$$

Now we let

$$X_{\pm}(t) = \int_0^t 1_{A_{\pm}}(u) dB(u), \quad 0 \leq t \leq 1. \quad (7)$$

Finally we define

$$B_*(t) = \begin{cases} X_+(\psi_+(t)), & 0 \leq t \leq \varphi_+(1), \\ -X_-(\psi_-(1-t)), & \varphi_+(1) < t \leq 1. \end{cases} \quad (8)$$

And the result is

Theorem 2. $B_*(t)$, $0 \leq t \leq 1$, is a standard Brownian bridge.

The proof is based on the well-known fact that a Gaussian process, and that is what B_* is, is characterized by its covariance function.

We present here two corollaries that follow from our constructions.

Corollary 1. *There are Brownian bridges B^* and B_* so that a.s.:*

$$|Y(t)| \geq |B^*(t)|, \quad 0 \leq t \leq 1, \quad (9)$$

$$\max_{0 \leq t \leq 1} |B_*(t)| = \max_{0 \leq t \leq 1} \{|X_+(t)| \vee |X_-(t)|\}. \quad (10)$$

P r o o f. First we show (9). Indeed, $Y(t) = B^*(t)$, $0 \leq t \leq \tau$, and $|Y(t)| > |B^*(t)|$, $\tau < t \leq 1$. This follows from the definitions (2) and (3) and the continuity of the processes that are involved here. More precisely, if there is $\tau < t \leq 1$ for which $|Y(t)| < |B^*(t)|$, we get that there is $t < s \leq 1$ for which $|Y(s)| = |B(s)|$ because $|B^*(t)| = |B(t)|$ and $|Y(1)| \geq |B(1)| = 0$. The existence of such s contradicts definition (2) of τ .

Now we go to (10). It follows immediately from the representation of B_* in (8) and definitions (5) and (6).

Corollary 2. *The following inequality holds:*

$$\mathbf{P} \left\{ \max_{0 \leq s \leq 1} |B(s)| \geq t \right\} \leq \mathbf{P} \left\{ \max_{0 \leq s \leq 1} |Y(s)| \geq t \right\} \leq \mathbf{P} \left\{ \max_{0 \leq s \leq 1} |B(s)| \geq \frac{t}{2} \right\}. \quad (11)$$

P r o o f. The left-hand side is proved using $B^*(t)$ and the right-hand side is proved using $B_*(t)$.

P r o o f o f t h e l e f t - h a n d s i d e. It follows from (9) since $B^*(t)$ is a Brownian bridge.

P r o o f o f t h e r i g h t - h a n d s i d e. By the triangle inequality we get

$$\max_{0 \leq t \leq 1} |Y(t)| \leq \max_{0 \leq t \leq 1} |X_+(t)| + \max_{0 \leq t \leq 1} |X_-(t)|. \quad (12)$$

Using (10) and (12) we get

$$\max_{0 \leq t \leq 1} |Y(t)| \leq 2 \max_{0 \leq t \leq 1} |B_*(t)|. \quad (13)$$

And we are done since $B_*(t)$ is a Brownian bridge.

R e m a r k s. 1. A discrete-time version of the second construction appears in [1].

2. In [4] we got a discrete-time version of inequality (11) and then extended it. In this paper, as was demonstrated above, our coupling methods allows us to get (11) directly.

2. Proof of the theorems. We will present first the proof of Theorem 2 because the method of its proof is mentioned at some point in the proof of Theorem 1. As explained in the introduction, this proof amounts mainly to verifying that the covariance function of $B_*(t)$ has the correct form.

P r o o f o f T h e o r e m 2. We need to show that $B_*(t)$ is a Brownian bridge.

First we will check that $B_*(t)$ is continuous at $t = \varphi_+(1)$. Indeed $\psi_+(\varphi_+(1)) = 1$ and $\psi_-(1 - \varphi_+(1)) = \psi_-(\varphi_-(1)) = 1$, so we need to prove that $X_+(1) = -X_-(1)$. But this follows from $X_+(1) + X_-(1) = B(1) = 0$.

Since obviously $B_*(t)$ is a centered Gaussian process all we need is to calculate its covariance function. Namely we have to show that

$$\mathbf{E}(B_*(s) B_*(t)) = s(1-t), \quad 0 \leq s \leq t \leq 1. \quad (14)$$

We will use the representation $B(t) = W(t) - tW(1)$, $0 \leq t \leq 1$, where $W(t)$ is a standard Brownian motion. With this presentation we get

$$X_+(t) = \int_0^t 1_{A_+}(u) dW(u) - \varphi_+(t) W(1), \quad 0 \leq t \leq 1, \quad (15)$$

and similarly

$$X_-(t) = \int_0^t 1_{A_-}(u) dW(u) - \varphi_-(t) W(1), \quad 0 \leq t \leq 1. \quad (16)$$

We separate the proof of (14), in a natural way, to 3 cases which are based on the location of s and t with respect to $\varphi_+(1)$.

C a s e I. Let $0 \leq s \leq t \leq \varphi_+(1)$. We calculate

$$\begin{aligned} \mathbf{E}(B_+(s) B_+(t)) &= \mathbf{E}(X_+(\psi_+(s)) X_+(\psi_+(t))) \\ &= \mathbf{E}\left(\int_0^{\psi_+(s)} 1_{A_+}(u) dW(u) - s W(1)\right) \\ &\quad \times \left(\int_0^{\psi_+(t)} 1_{A_+}(u) dW(u) - t W(1)\right) = s(1-t), \end{aligned}$$

where we used the following

$$\mathbf{E}\left[\int_0^{\psi_+(s)} 1_{A_+}(u) dW(u) \int_0^{\psi_+(t)} 1_{A_+}(u) dW(u)\right] = \varphi_+(\psi_+(s)) = s, \quad (17)$$

$$\mathbf{E}\left[W(1) \int_0^{\psi_+(s)} 1_{A_+}(u) dW(u)\right] = s. \quad (18)$$

C a s e II. $\varphi_+(1) \leq s \leq t \leq 1$. This is similar to Case I.

C a s e III. $0 \leq s \leq \varphi_+(1) \leq t \leq 1$. Here we use, in addition to the equations (17) and (18), also the following

$$\mathbf{E}\left[\int_0^{\psi_+(s)} 1_{A_+}(u) dW(u) \int_0^{\psi_-(1-t)} 1_{A_-}(u) dW(u)\right] = 0, \quad (19)$$

and the result follows.

This ends the proof of Theorem 2.

We will now present the proof of Theorem 1. The proof is divided into 4 steps. Step 1 establishes the result in a discrete-time setup. This is formulated in Lemma 1 in terms of step processes, namely: the step process analogue of $B^*(t)$ is equal in distribution to the step process analogue of $B(t)$. Step 2 deals with a step function approximation of $f(t)$. Step 3 establishes a weak convergence result that is formulated in Lemma 2. The result is that a sequence of step process analogue of $(B(t), Y(t))$ converges in distribution to $(B(t), Y(t))$. Step 4 exploits this weak convergence result to establish that the sequence of step process analogue of $B^*(t)$, does indeed converge in distribution to $B^*(t)$. The theorem now follows from the combination of Steps 1, 3 and 4.

P r o o f o f T h e o r e m 1. S t e p 1. Here we deal with a discrete version of the construction. Let n be a positive integer and let $\{\xi_k: k = 1, 2, \dots, 2n\}$ be a collection of random variables that is distributed uniformly on $\Delta_n = \{(\varepsilon_1, \dots, \varepsilon_{2n}): \varepsilon_k = \pm 1, \sum_{k=1}^{2n} \varepsilon_k = 0\}$. Namely, for every $(\varepsilon_1, \dots, \varepsilon_{2n}) \in \Delta_n$ we have

$$\mathbf{P}\{\xi_1 = \varepsilon_1, \dots, \xi_{2n} = \varepsilon_{2n}\} = \frac{(n!)^2}{(2n)!}. \quad (20)$$

Let $(\theta_1, \dots, \theta_{2n}) \in \{-1, +1\}^{2n}$. We now define $\{\xi_k^*: k = 1, 2, \dots, 2n\}$ by

$$\xi_k^* = \begin{cases} \theta_k \xi_k, & 1 \leq k \leq \tau, \\ \xi_k, & \tau < k \leq 2n, \quad \sum_{l=1}^{\tau} \theta_l \xi_l = \sum_{l=1}^{\tau} \xi_l, \\ -\xi_k, & \tau < k \leq 2n, \quad \sum_{l=1}^{\tau} \theta_l \xi_l = -\sum_{l=1}^{\tau} \xi_l, \end{cases} \quad (21)$$

where τ is a random time defined by

$$\tau = \max \left\{ 1 \leq j \leq 2n: \left| \sum_{i=1}^j \theta_i \xi_i \right| = \left| \sum_{i=1}^j \xi_i \right| \right\}. \quad (22)$$

Observe that τ is well defined because the set of integers, where there is equality in (22) is not empty, it contains $j = 1$.

Claim. $\{\xi_k^*: k = 1, 2, \dots, 2n\}$ is equal in distribution to $\{\xi_k: k = 1, 2, \dots, 2n\}$.

P r o o f o f t h e c l a i m. It is not difficult to observe that in fact there is a symmetry between $\{\xi_k: k = 1, 2, \dots, 2n\}$ and $\{\xi_k^*: k = 1, 2, \dots, 2n\}$. This means that we can switch the roles of ξ_k and ξ_k^* in formulas (21) and (22). Namely, we can write

$$\tau = \max \left\{ 1 \leq j \leq 2n: \left| \sum_{i=1}^j \theta_i \xi_i^* \right| = \left| \sum_{i=1}^j \xi_i^* \right| \right\}, \quad (23)$$

$$\xi_k = \begin{cases} \theta_k \xi_k^*, & 1 \leq k \leq \tau, \\ \xi_k^*, & \tau < k \leq 2n, \quad \sum_{l=1}^{\tau} \theta_l \xi_l^* = \sum_{l=1}^{\tau} \xi_l^*, \\ -\xi_k^*, & \tau < k \leq 2n, \quad \sum_{l=1}^{\tau} \theta_l \xi_l^* = -\sum_{l=1}^{\tau} \xi_l^*. \end{cases} \quad (24)$$

It follows from (23) and (24) that the mapping defined in (21) from $\{\xi_k: k = 1, 2, \dots, 2n\}$ into $\{\xi_k^*: k = 1, 2, \dots, 2n\}$ is one to one and the claim is proved.

The following lemma is now obvious. It describes a discrete version of our construction. We will use triangular array notation: for any n , $\{\xi_{n,k}: k = 1, 2, \dots, 2n\}$ will satisfy (20) and $\{\theta_{n,k}: k = 1, \dots, 2n\}$ will denote a vector of nonrandom signs.

Lemma 1. Let $Z_n(t) = (2n)^{-1/2} \sum_{k < 2nt} \xi_{n,k}$, $0 \leq t \leq 1$, and let $Y_n(t) = (2n)^{-1/2} \sum_{k < 2nt} \theta_{n,k} \xi_{n,k}$, $0 \leq t \leq 1$, and finally let $\tau_n = \max\{0 \leq t \leq 1: |Y_n(t)| = |Z_n(t)|\}$. Then the process $Z_n(t)$ and the process $Z_n^*(t)$, defined by

$$Z_n^*(t) = \begin{cases} Y_n(t), & 0 \leq t \leq \tau_n, \\ Z_n(t), & \tau_n \leq t \leq 1 \quad \text{and} \quad Y_n(\tau_n) = Z_n(\tau_n), \\ -Z_n(t), & \tau_n \leq t \leq 1 \quad \text{and} \quad Y_n(\tau_n) = -Z_n(\tau_n), \end{cases} \quad (25)$$

are equal in distribution.

P r o o f. The processes $Z_n(t)$ and $Z_n^*(t)$ are merely a scaled version of the partial sum process of $\{\xi_{n,k}\}$ and $\{\xi_{n,k}^*\}$, respectively. Since those two are equal in distribution the result follows.

S t e p 2. There exists a triangular array of signs $(\theta_{n,k})$ so that

$$f_n(t) = \sum_{k=0}^{2n-1} \theta_{n,k} 1_{[k/(2n), (k+1)/(2n)]}(t) \quad (26)$$

satisfies $\lambda(f_n \neq f) \rightarrow 0$, where λ denotes the Lebesgue measure.

P r o o f o f s t e p 2. The result follows easily from the fact that $A_{\pm} = \{0 \leq u \leq 1: f(u) = \pm 1\}$ are measurable subsets of $[0, 1]$ and can be approximated by a finite number of intervals to a desired level. Those intervals in turn, can also be approximated, again to a desired level, by intervals of the type of (26) if n is large enough.

S t e p 3. Weak convergence. We use the notation of Lemma 1, and the signs $(\theta_{n,k})$ are those from the approximation of $f(t)$ in (26). We claim that

Lemma 2. The following convergence holds:

$$\{(Z_n(t), Y_n(t)): 0 \leq t \leq 1\} \longrightarrow \{(B(t), Y(t)): 0 \leq t \leq 1\}, \quad (27)$$

where the convergence is that of processes in distribution with respect to the uniform metric on the space $C_{0,0}[0, 1] \times C_0[0, 1]$, where $C_0[0, 1] = \{x \in C[0, 1]: x(0) = 0\}$, $C_{0,0}[0, 1] = \{x \in C_0[0, 1]: x(1) = 0\}$.

R e m a r k. The processes of the left-hand side of (27) are formally step processes, but as is well known, without loss of generality we can take them as continuous by making them linear between each of the consecutive points $k/(2n)$, $(k+1)/(2n)$. With this agreement Lemma 1 is still valid.

P r o o f o f L e m m a 2. a) The convergence $Z_n \rightarrow B$. This follows from Theorem 3.5 in [3, p. 186]:

Theorem. $(\sum_{i=1}^N (c_i - \bar{c}) a_N(R_i, t)) (\sum_{i=1}^N (c_i - \bar{c})^2)^{-1/2}$, $0 \leq t \leq 1$, converges in distribution to a Brownian bridge if $\sum_{i=1}^N (c_i - \bar{c})^2 (\max_{1 \leq i \leq N} (c_i - \bar{c})^2)^{-1} \rightarrow \infty$, where $c_i = c_{N,i}$ stands for a triangular array of scalars, $\bar{c} = \sum_{i=1}^N c_i / N$, $a_N(i, t) = 1_{\{iN+1 \leq i\}} + (i - tN) 1_{\{iN \leq i < N+1\}}$, and $(R_1, \dots, R_n)$ represents a uniformly distributed random permutation on $\{1, \dots, N\}$.

To apply the quoted theorem in our setup we select $N = 2n$, and $c_i = +1$, if $1 \leq i \leq n$, and $c_i = -1$, if $n < i \leq 2n$. With the above notation we get that $Z_n(t)$ is equal in distribution to $[\sum_{i=1}^N (c_i - \bar{c}) a_N(R_i, t)] [\sum_{i=1}^N (c_i - \bar{c})^2]^{-1/2}$ and the result follows.

b) *Uniform tightness of (Z_n, Y_n) .* We need to verify that for any $\varepsilon > 0$

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{|s-t| < h} |Y_n(t) - Y_n(s)| > \varepsilon \right\} \rightarrow 0, \quad h \rightarrow 0. \quad (28)$$

A similar result holds, of course, for Z_n , because $Z_n \rightarrow B$. But we claim that the following holds:

$$\mathbf{P} \left\{ \sup_{|s-t| < h} |Y_n(t) - Y_n(s)| > \varepsilon \right\} \leq \mathbf{P} \left\{ \sup_{|s-t| < h} |Z_n(t) - Z_n(s)| > \frac{\varepsilon}{2} \right\}. \quad (29)$$

And this will give us (28). To see (29), we produce a permutation π on $(1, 2, \dots, 2n)$. This permutation will place first $\{k: \theta_{n,k} = +1\}$ in their natural order and then will place $\{k: \theta_{n,k} = -1\}$ also in their natural order. We denote $\tilde{Z}_n(t) = (2n)^{-1/2} \sum_{k < 2nt} \xi_{n,\pi(k)}$, $0 \leq t \leq 1$. We observe that $Z_n = \tilde{Z}_n$ in distribution, because of the exchangeability of $\{\xi_{n,k}: k = 1, 2, \dots, 2n\}$. By using the triangle inequality we get

$$\left\{ \sup_{|s-t| < h} |Y_n(t) - Y_n(s)| > \varepsilon \right\} \subset \left\{ \sup_{|s-t| < h} |\tilde{Z}_n(t) - \tilde{Z}_n(s)| > \frac{\varepsilon}{2} \right\},$$

and (29) follows. By that we verified the uniform tightness of $\{Y_n\}$ which together with that of $\{Z_n\}$ implies the uniform tightness of the pair (Z_n, Y_n) .

c) *Finite dimension convergence of (Z_n, Y_n) .* Let $0 \leq t_1 < \dots < t_k \leq 1$. We need to show that $\{(Z_n(t_i), Y_n(t_i)): 1 \leq i \leq k\}$ converges in distribution to $\{(B(t_i), Y(t_i)): 1 \leq i \leq k\}$. Again because of the exchangeability of $\{\xi_{n,k}: k = 1, 2, \dots, 2n\}$, we can assume without loss of generality that in fact,

$$\forall 1 \leq i \leq k, \exists t_i < s_{n,i} < t_{i+1} \quad \text{such that} \quad \begin{cases} \theta_{n,j} = +1 & \text{if } 2nt_i < j < 2ns_{n,i}, \\ \theta_{n,j} = -1 & \text{if } 2ns_{n,i} < j < 2nt_{i+1}. \end{cases}$$

Step 2 implies that:

$$s_{n,i} \rightarrow s_i = t_i + \lambda(t_i < u < t_{i+1}: f(u) = +1), \quad (30)$$

where λ denotes the Lebesgue measure. The convergence of $\{(Z_n(t_i), Y_n(t_i)): 1 \leq i \leq k\}$ in distribution follows now in a straightforward manner from the convergence $Z_n \rightarrow B$. We get

$$Y_n(t_i) = 2 \left(\sum_{j=1}^{i-1} Z_n(s_{n,j}) - Z_n(t_j) \right) - Z_n(t_i) \rightarrow 2 \left(\sum_{j=1}^{i-1} B(s_j) - B(t_j) \right) - B(t_i). \quad (31)$$

And now we can show by the method that was used in the proof of Theorem 2, that the limit in (31) has the right distribution, namely

$$Y(t_i) = 2 \left(\sum_{j=1}^{i-1} B(s_j) - B(t_j) \right) - B(t_i), \quad (32)$$

in distribution. This ends the proof of Lemma 2.

Step 4. Proving Theorem 1. Let $T: C_0[0,1] \times C_0[0,1] \rightarrow [0,1]$ be defined by

$$T(x(\cdot), y(\cdot)) = \max \left\{ 0 \leq t \leq 1: |x(t)| = |y(t)| \right\}. \quad (33)$$

Let $U: C_0[0,1] \times C_0[0,1] \rightarrow C_0[0,1]$ be defined by

$$U(x(\cdot), y(\cdot)) = \begin{cases} y(t), & 0 \leq t \leq T, \\ x(t), & T \leq t \leq 1 \text{ and } y(T) = x(T), \\ -x(t), & T \leq t \leq 1 \text{ and } y(T) = -x(T). \end{cases} \quad (34)$$

Claim. U is continuous a.s. with respect to the law of the process $\{(B(t), Y(t)): 0 \leq t \leq 1\}$.

P r o o f. We start by the following observation. Let $(x, y) \in C_{0,0}[0, 1] \times C_0[0, 1]$, $x(T) - y(T) = 0$, and every neighborhood of T contains both negative and positive values of $x(t) - y(t)$ (on the left of T !). Let $(x_n, y_n) \in C_{0,0}[0, 1] \times C_0[0, 1]$ be a sequence that converges to (x, y) in the uniform metric. Then by standard arguments of continuity we get both $T(x_n, y_n) \rightarrow T(x, y)$ and $U(x_n, y_n) \rightarrow U(x, y)$. A similar result holds, of course, when we replace $x(T) + y(T) = 0$ by $x(T) - y(T) = 0$. We want to prove the claim by using that observation. To do that we need to show that this situation happens a.s. with respect to the process $\{B(t), Y(t): 0 \leq t \leq 1\}$. Indeed, we have

$$\begin{aligned} T(B, Y) &= \max \left\{ 0 \leq t \leq 1: B(t) - Y(t) = 0 \text{ or } B(t) + Y(t) = 0 \right\} \\ &= \max \left\{ 0 \leq t \leq 1: X_+(t) X_-(t) = 0 \right\}. \end{aligned}$$

The zero-set of $X_+ X_-$, namely the set $\{0 \leq t \leq 1: X_+(t) X_-(t) = 0\}$, behaves like the zero-set of a Brownian motion (see [2, p. 392], for the latter). From this we conclude that if a zero is isolated on the right then it is a limit of increasing zeros from the left. Also, every neighborhood of $T(B, Y)$ contains (on the left of T) both negative and positive values of $B(t) - Y(t)$ or of $B(t) + Y(t)$. This proves the claim.

To finish the proof of Theorem 1 we combine the claim that was just proved and Lemma 2 and we get by standard weak convergence arguments that:

$$U(Z_n, Y_n) \rightarrow U(B, Y), \quad (35)$$

where the convergence is in distribution on the space $C_{0,0}[0, 1] \times C_0[0, 1]$. Next we observe that, in fact, $U(Z_n, Y_n) = Z_n^*$ and $U(B, Y) = B^*$. So we get

$$Z_n^* \rightarrow B^*. \quad (36)$$

It follows now from Lemma 1 that $Z_n \rightarrow B^*$ and by (27) we finally get $B = B^*$ in distribution. This completes the proof of Theorem 1.

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ON MEASURABLE MODIFICATION OF STOCHASTIC FUNCTIONS

Рассматривается ряд критериев существования измеримой модификации.

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