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Claim. U is continuous a.s. with respect to the law of the process $\{(B(t), Y(t)): 0 \leq t \leq 1\}$.

P r o o f. We start by the following observation. Let $(x, y) \in C_{0,0}[0, 1] \times C_0[0, 1]$, $x(T) - y(T) = 0$, and every neighborhood of T contains both negative and positive values of $x(t) - y(t)$ (on the left of T !). Let $(x_n, y_n) \in C_{0,0}[0, 1] \times C_0[0, 1]$ be a sequence that converges to (x, y) in the uniform metric. Then by standard arguments of continuity we get both $T(x_n, y_n) \rightarrow T(x, y)$ and $U(x_n, y_n) \rightarrow U(x, y)$. A similar result holds, of course, when we replace $x(T) + y(T) = 0$ by $x(T) - y(T) = 0$. We want to prove the claim by using that observation. To do that we need to show that this situation happens a.s. with respect to the process $\{B(t), Y(t): 0 \leq t \leq 1\}$. Indeed, we have

$$\begin{aligned} T(B, Y) &= \max \left\{ 0 \leq t \leq 1: B(t) - Y(t) = 0 \text{ or } B(t) + Y(t) = 0 \right\} \\ &= \max \left\{ 0 \leq t \leq 1: X_+(t) X_-(t) = 0 \right\}. \end{aligned}$$

The zero-set of $X_+ X_-$, namely the set $\{0 \leq t \leq 1: X_+(t) X_-(t) = 0\}$, behaves like the zero-set of a Brownian motion (see [2, p. 392], for the latter). From this we conclude that if a zero is isolated on the right then it is a limit of increasing zeros from the left. Also, every neighborhood of $T(B, Y)$ contains (on the left of T) both negative and positive values of $B(t) - Y(t)$ or of $B(t) + Y(t)$. This proves the claim.

To finish the proof of Theorem 1 we combine the claim that was just proved and Lemma 2 and we get by standard weak convergence arguments that:

$$U(Z_n, Y_n) \rightarrow U(B, Y), \quad (35)$$

where the convergence is in distribution on the space $C_{0,0}[0, 1] \times C_0[0, 1]$. Next we observe that, in fact, $U(Z_n, Y_n) = Z_n^*$ and $U(B, Y) = B^*$. So we get

$$Z_n^* \rightarrow B^*. \quad (36)$$

It follows now from Lemma 1 that $Z_n \rightarrow B^*$ and by (27) we finally get $B = B^*$ in distribution. This completes the proof of Theorem 1.

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ON MEASURABLE MODIFICATION OF STOCHASTIC FUNCTIONS

Рассматривается ряд критериев существования измеримой модификации.

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Ключевые слова и фразы: стохастическая квазинепрерывность, измеримая корреляция, измеримые вероятности, согласованная аппроксимация.

1. Introduction. Let ξ_t , $t \in T$, be a stochastic function on a measurable bounded set $T \subseteq \mathbf{R}$, say, a question is whether finite dimensional probability distributions

$$P_{t_1, \dots, t_n}(B_1 \times \dots \times B_n) = \mathbf{P}\{\xi_{t_1} \in B_1, \dots, \xi_{t_n} \in B_n\}, \quad B_1 \times \dots \times B_n \subseteq \mathbf{R}^n, \quad (1.1)$$

are consistent with the existence of a *measurable modification* $\tilde{\xi}_t$, $t \in T$, on a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$: $\mathbf{P}\{\tilde{\xi}_t = \xi_t\} = 1$, $t \in T$, and the corresponding function

$$\tilde{\xi}_t(\omega), \quad (\omega, t) \in \Omega \times T, \quad (1.2)$$

on the product space $\Omega \times T$, is *measurable* with respect to the product measure $\mathbf{P}(d\omega) \times dt$.

Up to now, the following answer for a common use: if the function ξ_t , $t \in T$, is stochastically continuous, i.e., for all $\varepsilon > 0$,

$$\mathbf{P}\{|\xi_t - \xi_{t_0}| \geq \varepsilon\} = \int \int_{|x-y| \geq \varepsilon} P_{t, t_0}(dx dy) \rightarrow 0, \quad t \rightarrow t_0 \in T, \quad (1.3)$$

then a measurable modification does exist (cf. [2], [3] or [5], for example).

Of course condition (1.3) is not necessary, and, in its line, we would like to present the following result, announced in [1] — cf. the famous Egorov's and Lusin's theorems in Analysis.

Theorem 1. *For the measurable modification existence, it is necessary and sufficient that ξ_t , $t \in T$, is stochastically quasi-continuous, i.e., stochastic continuity (1.3) holds for $t, t_0 \in T^m$ on appropriate sets $T^m \subseteq T$:*

$$\text{mes}(T - T^m) \rightarrow 0, \quad m \rightarrow \infty. \quad (1.4)$$

In particular this criterion shows that *there is a set $T_0 \subseteq T$, with $\text{mes}(T - T_0) = 0$, where the function ξ_t , $t \in T$, generates a separable σ -algebra $\mathfrak{A}_{T_0}$ of events equipped with the metric*

$$\rho(A_1, A_2) = \mathbf{P}\{(A_1 - A_2) \cup (A_2 - A_1)\}, \quad A_1, A_2 \in \mathfrak{A}_{T_0}.$$

Accordingly, it seems worthy to stress on the *necessary* existence of the above type set $T_0 \subseteq T$ in such a known result as the following: *suppose that the set T_0 does exist, then, for the measurable modification existence, it is necessary and sufficient that the probabilities*

$$\mathbf{P}\{\xi_t \in B, \xi_{t_0} \in B_0\} = P_{t, t_0}(B \times B_0), \quad t \in T, \quad (1.5)$$

are measurable, with any Borel sets $B, B_0 \subseteq \mathbf{R}$ and $t_0 \in T$ (cf. [4], [6]).

For illustration, one can take into account the *nondeterministic* function ξ_t , $t \in T$, with independent, identically distributed values, say. Obviously, on any noncountable set $T_0 \subseteq T$, this function is *not* stochastically continuous and ξ_t , $t \in T_0$, generates the *nonseparable* σ -algebra of events. Hence, the function ξ_t , $t \in T$, has *no* measurable modification, though all probabilities

$$P_{t, t_0}(B \times B_0) = \begin{cases} P_{t_0}(B) P_{t_0}(B_0), & t \neq t_0, \\ P_{t_0}(B \cap B_0), & t = t_0, \end{cases}$$

are very simple (measurable) functions of $t \in T$, *constant* for all $t \neq t_0$.

On the above lines, for the stochastic function ξ_t , $t \in T$, in the standard L_2 -space

$$H = L_2(\Omega, \mathfrak{A}, \mathbf{P}), \quad (1.6)$$

with the L_2 -norm $\|\cdot\| = (\mathbf{E}|\cdot|^2)^{1/2}$, we suggest the following criterion.

Theorem 2. *The measurable modification does exist if and only if, for some set T_0 with $\text{mes}(T - T_0) = 0$, a linear L_2 -closure H_{T_0} of ξ_t , $t \in T_0$, is separable and the covariance function*

$$\mathbf{E}\xi_t \xi_{t_0} = \int \int_{\mathbf{R}^2} xy P_{t, t_0}(dx dy), \quad t \in T, \quad (1.7)$$

is measurable, with any $t_0 \in T$.

Apart from that, in between, we actually show that, in the L_p -space ($p \geq 1$), for the measurable stochastic function ξ_t , $t \in T$, such that

$$\int_T \mathbf{E}|\xi_t|^p dt < \infty, \quad (1.8)$$

a path integral $\int_T \xi_t dt = \int_T \xi_t(\omega) dt$, $\omega \in \Omega$, represents a standard L_p -space integral $\int_T \xi_t dt = \lim_{n \rightarrow \infty} \int_T \xi_t^n dt$, with

$$\lim_{n \rightarrow \infty} \int_T \mathbf{E}|\xi_t - \xi_t^n|^p dt = 0, \quad (1.9)$$

for some *simple functions* ξ_t^n , $t \in T$, of a form

$$\xi_t^n = \sum_{k=1}^n \xi_{s_{kn}} 1_{(s_{kn}, t_{kn}]}(t), \quad t \in T, \quad (1.10)$$

with indicators

$$1_{(s_{kn}, t_{kn}]}(t) = \begin{cases} 1, & s_{kn} < t \leq t_{kn}, \\ 0, & \text{otherwise,} \end{cases}$$

of some *disjoint intervals* $(s_{kn}, t_{kn}]$, where the corresponding ξ_t^n , $t \in T$, preserves the permanent values $\xi_{s_{kn}}$ ($k = 1, \dots, n$). Hence, with the appropriate subsequence of simple functions (1.9), (1.10) in hands, the certain measurable modification $\tilde{\xi}_t$, $t \in T$, can be defined (for almost all t) as

$$\tilde{\xi}_t(\omega) := \lim_{n \rightarrow \infty} \sum_{k=1}^n \xi_{s_{kn}}(\omega) 1_{(s_{kn}, t_{kn}]}(t) \quad (1.11)$$

for all $(\omega, t) \in \Omega \times T$, where the above limit does exist, with $\tilde{\xi}_t(\omega) = 0$ for other (ω, t) which set is of zero $\mathbf{P}(d\omega) \times dt$ measure.

2. Hilbert space set up. Let us consider the Hilbert space (1.6), with its scalar product $\langle \xi, \eta \rangle = \mathbf{E}\xi\eta$, $\xi, \eta \in H$. A function ξ_t , $t \in T$, in H is *simple* if it admits representation

$$\xi_t = \sum_j \xi_j 1_{\Delta_j}(t), \quad t \in T, \quad (2.1)$$

with the constant values $\xi_j \in H$ on the measurable sets Δ_j , for some *finite* partition $\sum_j \Delta_j = T$. And a natural class of *H-measurable* functions ξ_t , $t \in T$, appears by the limits

$$\xi_t = \lim_{n \rightarrow \infty} \xi_t^n, \quad t \in T, \quad (2.2)$$

of simple functions in H , for almost all $t \in T$ (of course, for *any* H -measurable functions ξ_t^n , $t \in T$, their (2.2) limit represents the H -measurable function ξ_t , $t \in T$).

Taking into account that the limit (2.2) implies uniform convergence $\|\xi_t - \xi_t^n\| \rightarrow 0$, $t \in T^m$, on some increasing approximating sets $T^m \subseteq T$: $\text{mes}(T - T^m) \rightarrow 0$, $m \rightarrow \infty$, one can apply the *new* simple functions

$$\xi_t^n = \begin{cases} \xi_j^n, & t \in \Delta_{j,n} \subseteq T^n, \\ 0, & \text{otherwise,} \end{cases}$$

on the *new increasing* approximating sets $T^n \subseteq T$: $\text{mes}(T - T^n) \rightarrow 0$, as *compacts*, with their partitions $\sum_j \Delta_{j,n} = T^n$ by a *finite* number of the indicated *disjoint compacts* $\Delta_{j,n}$, such that the uniform ε -approximations

$$\|\xi_t - \xi_t^n\| \leq \varepsilon_n \rightarrow 0, \quad t \in T^n, \quad (2.3)$$

hold true. Here, the piecewise constant functions ξ_t^n , $t \in T^m$, $n \geq m$, are continuous on every approximating compact T^m , and their uniform limit ξ_t , $t \in T^m$, is *continuous* as well. This is a way to arrive at the following result in the line of the well-known *Egorov's* and *Lusin's theorems*.

Lemma 1. *The function ξ_t , $t \in T$, is H -measurable if and only if it is quasi-continuous in H .*

Obviously, for the quasi-continuous function ξ_t , $t \in T$, in H , there is some set T_0 , with $\text{mes}(T - T_0) = 0$, such that the linear closure H_{T_0} of ξ_t , $t \in T_0$ is separable and the scalar function (1.7):

$$\langle \xi_t, \xi_{t_0} \rangle = E \xi_t \xi_{t_0}, \quad t \in T,$$

is measurable, with any $t_0 \in T$ (here one can take into account that this scalar function is quasi-continuous). On the other hand, if the above conditions hold, then we can take some orthonormal basis η_k , $k = 1, 2, \dots$, in the separable linear closure H_{T_0} of ξ_t , $t \in T_0$, and obtain ξ_t , $t \in T_0$ on the set T_0 with $\text{mes}(T - T_0) = 0$, as a limit

$$\xi_t = \lim_{n \rightarrow \infty} \sum_{k=1}^n \langle \xi_t, \eta_k \rangle \eta_k \quad (2.4)$$

of H -measurable functions $\xi_t^n = \sum_{k=1}^n \langle \xi_t, \eta_k \rangle \eta_k$, $t \in T_0$; note that any scalar function $\langle \xi_t, \eta \rangle$, $t \in T$, with $\eta \in H_{T_0}$, is measurable: $\langle \xi_t, \eta \rangle = \lim \sum_j c_j \langle \xi_t, \xi_{t_j} \rangle$ for $\eta = \lim \sum_j c_j \xi_{t_j} \in H_{T_0}$ as a limit of appropriate linear combination $\sum_j c_j \xi_{t_j}$, say. Hence, the function ξ_t , $t \in T$, is H -measurable. And it obviously leads to the following result.

Lemma 2. *The function ξ_t , $t \in T$, is H -measurable if and only if the linear closure H_{T_0} of ξ_t , $t \in T_0$, is separable, for some set T_0 with $\text{mes}(T - T_0) = 0$, and the scalar function $E \xi_t \xi_{t_0}$, $t \in T$, is measurable, with any $t_0 \in T$.*

Now, let us return to the uniform approximations of ξ_t , $t \in T$, by the simple functions ξ_t^n , $t \in T$, see (2.3). Recall, the H -measurable function is quasi-continuous in H ; hence, for some increasing compacts T^n : $\text{mes}(T - T^n) \rightarrow 0$, we have

$$\|\xi_s - \xi_t\| \leq \varepsilon_n \rightarrow 0, \quad s, t \in T^n: |s - t| \leq h_n \rightarrow 0.$$

Accordingly, we can apply some finite partition intervals $(t_{j-1}, t_j]$: $\sum_j (t_{j-1}, t_j] \supseteq T$, $t_j - t_{j-1} \leq h_n$, and define the simple functions

$$\xi_t^n = \sum_j \xi_{s_j} 1_{(s_j, t_j]}(t), \quad t \in T, \quad (2.5)$$

where $s_j = \min\{s: s \in [t_{j-1}, t_j] \cap T^n\}$, for the indicated nonempty intersection. Then (not counting the above partition points $t = s_j, t_j$) we obtain the uniform approximations of the type (2.3), since $\|\xi_t - \xi_t^n\| = \|\xi_t - \xi_{s_j}\| \leq \varepsilon_n$, $t \in T^n$ ($s_j < t < t_j$).

For the H -measurable function ξ_t , $t \in T$, with the bounded norm $\|\xi_t\| \leq C$, the uniform approximations (2.3) by the simple functions of the type (2.5) give the integral approximations (1.9) (for $p = 2$), since $\|\xi_t^n\| \leq C$ as well and

$$\int_T \|\xi_t - \xi_t^n\|^2 dt \leq \text{mes}(T) \varepsilon_n^2 + 2C \text{mes}(T - T^n).$$

With the condition (1.8) for $p = 2$, the general H -measurable function ξ_t , $t \in T$, can be approximated in the sense of (1.9) by its truncations

$$\xi_t^n := \begin{cases} \xi_t, & \|\xi_t\| \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

which have (if not zero) the same values as the very function ξ_t , $t \in T$, namely,

$$\lim_{n \rightarrow \infty} \int_T \|\xi_t - \xi_t^n\|^2 dt = 0. \quad (2.6)$$

Hence, all the truncations can be approximated (in the sense considered) by the corresponding simple functions of the form (2.5) — with the involved values of the very function ξ_t , $t \in T$. Therefore, this function admits approximation (2.6) by the functions of the type (2.5) as well. And, of course, the above arguments in a similar way can be applied for the approximation (1.9) in a case of any particular $p \geq 1$.

For the $p = 2$ we are dealing with, the corresponding approximation (2.6) implies $\lim_{m, n \rightarrow \infty} \int_T \|\xi_t^m - \xi_t^n\|^2 dt = 0$, with $\int_T \|\xi_t^m - \xi_t^n\|^2 dt = \iint_{\Omega \times T} |\xi_t^m(\omega) - \xi_t^n(\omega)|^2 P(d\omega) dt$, for the measurable functions $\xi_t^n(\omega)$, $(\omega, t) \in \Omega \times T$, in the corresponding

L_2 -space with respect to the product measure $\mathbf{P}(d\omega) \times dt$, thanks to the Fubini's theorem. Hence, there is a limit measurable function

$$\tilde{\xi}_t(\omega), \quad (\omega, t) \in \Omega \times T: \lim_{n \rightarrow \infty} \int_{\Omega \times T} |\tilde{\xi}_t(\omega) - \xi_t^n(\omega)|^2 \mathbf{P}(d\omega) dt = 0 \quad (2.7)$$

with $\int_{\Omega \times T} |\tilde{\xi}_t(\omega) - \xi_t^n(\omega)|^2 \mathbf{P}(d\omega) dt = \int_T \|\tilde{\xi}_t - \xi_t^n\|^2 dt$, where $\tilde{\xi}_t = \tilde{\xi}_t(\omega)$, $\omega \in \Omega$, are random variables, such that

$$\tilde{\xi}_t = \xi_t \quad (2.8)$$

in the L_2 -space H , for almost all $t \in T$, and by this we are getting the certain measurable modification $\tilde{\xi}_t$, $t \in T$. Now, for the general H -measurable function ξ_t , $t \in T$, in H , let us consider this function on the increasing sets $T^m = t: \|\xi_t\| \leq m$, $m = 1, 2, \dots$, where the corresponding measurable modifications $\tilde{\xi}_t$, $t \in T^m$, are selected as extension (for increasing n) of one to another. Clearly, in this way we are getting the well-defined measurable modification $\tilde{\xi}_t$, $t \in T_0$, on the set $T_0 = \cup_m T^m$, $\text{mes}(T - T_0) = 0$, and therefore, we can select the measurable modification $\tilde{\xi}_t$, $t \in T$. Thus, *any H -measurable function ξ_t , $t \in T$, admits its measurable modification*. On the other hand, if there is some measurable modification $\tilde{\xi}_t$, $t \in T$, for the function ξ_t , $t \in T$, in the L_2 -space H , then, on the increasing sets T^m , $\text{mes}(T - T^m) \rightarrow 0$, where the norm $\|\xi_t\|$, $t \in T^m$, is bounded, the limit (2.7) holds for some simple functions ξ_t^n , $t \in T$:

$$\begin{aligned} \xi_t^n(\omega) &= \sum_{k,j} x_{k,j}^n 1_{A_{k,n}}(\omega) 1_{\Delta_{j,n}}(t) = \sum_j \left[\sum_k x_{k,j}^n 1_{A_{k,n}}(\omega) \right] 1_{\Delta_{j,n}}(t) \\ &= \sum_j \xi_j^n 1_{\Delta_{j,n}}(t), \quad (\omega, t) \in \Omega \times T, \end{aligned} \quad (2.9)$$

— for some constants $x_{k,j}^n \in \mathbf{R}$ on the appropriate products $A_{k,n} \times \Delta_{j,n} \subseteq \Omega \times T$ with the values $\xi_j^n = \sum_k x_{k,j}^n 1_{A_{k,n}}(\omega)$, $\omega \in \Omega$, in H on the corresponding sets $\Delta_{j,n} \subseteq T$. Hence, the very stochastic function ξ_t , $t \in T$, is H -measurable on the sets T^m and, consequently, it is H -measurable on the set T . Thus we obtain the following result.

Lemma 3. *The measurable modification does exist if and only if the function ξ_t , $t \in T$, is H -measurable.*

Altogether, the above lemmas give a complete proof of Theorem 2.

3. Measurable modification of general stochastic functions. Now, let us turn on Theorem 1 proof. For the general function ξ_t , $t \in T$, we apply its approximations by

$$\xi_t^n = \varphi^n(\xi_t), \quad t \in T: \varphi^n(x) := \begin{cases} -n, & x < -n, \\ x, & -n \leq x \leq n, \\ n, & x > n. \end{cases} \quad x \in \mathbf{R}, \quad (3.1)$$

Let the function ξ_t , $t \in T$, be stochastically quasi-continuous; then its approximations (3.1) are of the same type, since $\{|\xi_t^n - \xi_{t_0}^n| \geq \varepsilon\}$ implies $\{|\xi_t - \xi_{t_0}| \geq \varepsilon\}$, say. Hence, the bounded functions ξ_t^n , $t \in T$, are quasi-continuous in the L_2 -space H , and there are measurable modifications $\tilde{\xi}_t^n$, $t \in T$, according to Lemmas 1, 3. Obviously, the corresponding measurable functions $\tilde{\xi}_t^n(\omega)$, $(\omega, t) \in \Omega \times T$, converge in measure (with respect to the product measure $\mathbf{P}(d\omega) \times dt$), and the appropriate limit measurable function $\tilde{\xi}_t(\omega)$, $(\omega, t) \in \Omega \times T$, represents the measurable modification $\tilde{\xi}_t$, $t \in T$, — cf. (2.7), etc.

On the other hand, if there is some measurable modification $\tilde{\xi}_t$, $t \in T$, then the corresponding approximations of the type (3.1) $\tilde{\xi}_t^n = \varphi^n(\tilde{\xi}_t)$, $t \in T$, represent measurable modifications for the functions (3.1); hence, these functions ξ_t^n , $t \in T$, are stochastically quasi-continuous. Consequently, the very function ξ_t , $t \in T$, is stochastically quasi-continuous as well. To explain, for some increasing approximating sets T^m with $\text{mes}(T - T^m) \rightarrow 0$, we have

$$\mathbf{P}\{|\xi_t| \leq n\} \geq 1 - \delta_n \rightarrow 1, \quad t \in T^m,$$

and

$$\begin{aligned} \mathbf{P}\{|\xi_t - \xi_{t_0}| > \varepsilon\} &\leq \mathbf{P}\{|\xi_t^n - \xi_{t_0}^n| > \varepsilon, \max\{|\xi_t|, |\xi_{t_0}|\} \leq n\} + \mathbf{P}\{\max\{|\xi_t|, |\xi_{t_0}|\} > n\} \\ &\leq \mathbf{P}\{|\xi_t^n - \xi_{t_0}^n| > \varepsilon\} + 2\delta_n, \quad t, t_0 \in T^m, \end{aligned} \quad (3.2)$$

with $\delta_n \rightarrow 0$; hence, on the new appropriate T^m , $\text{mes}(T - T^m) \rightarrow 0$, where the functions ξ_t^n , $t \in T^m$, are stochastically continuous, we have $\lim_{t \rightarrow t_0} \mathbf{P}\{|\xi_t - \xi_{t_0}| > \varepsilon\} = 0$, $t, t_0 \in T^m$. By this, we end the proof, with a stress that, for the measurable modification, *it is necessary the existence of some set T_0 , $\text{mes}(T - T_0) = 0$, where the function ξ_t , $t \in T_0$, generates the separable σ -algebra $\mathfrak{A}_{T_0} \subseteq \mathfrak{A}$* . To explain, if ξ_t , $t \in T$, is stochastically quasi-continuous and therefore, for some sets T^m with $\text{mes}(T - T^m) \rightarrow 0$, the function ξ_t , $t \in T^m$, generates separable σ -algebras of events, then ξ_t , $t \in T_0$, generates a separable σ -algebra on the union set $T_0 = \cup T^m$, as well.

To conclude, let us also show that, *whenever we have the above type $T_0 \subseteq T$, the measurable modification exists if and only if the probabilities $\mathbf{P}\{\xi_t \in B, \xi_{t_0} \in B_0\}$, $t \in T$, are measurable for any Borel set $B, B_0 \subseteq \mathbf{R}$ and $t_0 \in T$* . Obviously, with the measurable modification $\tilde{\xi}_t$, $t \in T$, we have measurable functions $\varphi_t(\omega) = 1_B(\tilde{\xi}_t(\omega)) 1_{B_0}(\tilde{\xi}_{t_0}(\omega))$, $(\omega, t) \in \Omega \times T$, with the indicators $1_B, 1_{B_0}$ of the Borel sets $B, B_0 \subseteq \mathbf{R}$ and $t_0 \in T$; hence, the corresponding integrals

$$\int_{\Omega} \varphi_t(\omega) \mathbf{P}(d\omega) = \mathbf{P}\{\xi_t \in B, \xi_{t_0} \in B_0\}, \quad t \in T,$$

represent measurable functions. On the other hand, if, for some set T_0 with $\text{mes}(T - T_0) = 0$, the function ξ_t , $t \in T_0$, generates the separable σ -algebra $\mathfrak{A}_{T_0}$, then the corresponding L_2 -space $L_2(\Omega, \mathfrak{A}_{T_0}, \mathbf{P})$ is separable. And, with the measurable probability distributions $P_{t, t_0}(B \times B_0) = \mathbf{P}\{\xi_t \in B, \xi_{t_0} \in B_0\}$, $B \times B_0 \subseteq \mathbf{R}$, as functions of $t \in T$, we have the measurable integral

$$\iint_{\mathbf{R}^2} \varphi^n(x) \varphi^n(y) P_{t, t_0}(dx dy) = \mathbf{E} \xi_t^n \xi_{t_0}^n, \quad t \in T,$$

for the functions (3.1) φ^n . Hence, according to Theorem 2, the corresponding functions (3.1) ξ_t^n , $t \in T$, admit the measurable modifications $\tilde{\xi}_t^n$, $t \in T$. And, clearly, with the measurable modifications $\tilde{\xi}_t^n$, $t \in T$, in hands, we can get a measurable modification for the very function ξ_t , $t \in T$, cf. (2.7), etc.

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