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**TEMPERED INFINITELY DIVISIBLE DISTRIBUTIONS
AND PROCESSES¹⁾**

В настоящей статье строится новый класс темперированных безгранично делимых (ТБД) распределений. С учетом класса темперированных устойчивых распределений, введенного Я. Розинским, модификация темперировующей функции позволяет получить подходящие свойства. В частности, ТБД-распределения могут иметь экспоненциальные моменты любого порядка и сохраняют все свойства из постановки Розинского. Более того, мы доказываем, что модификация темперированного устойчивого распределения является ТБД-распределением, и приводим дальнейшие параметрические примеры.

Ключевые слова и фразы: устойчивые распределения, темперированные устойчивые распределения, темперированные безгранично делимые распределения.

1. Introduction. The formal and elegant definition of tempered stable distributions and processes has been proposed in the work of Rosiński [11] where a completely monotone function is chosen to transform the Lévy measure of a stable distribution. Tempered stable distributions may have all moments finite and exponential moments of some order. The idea of selecting a different tempering function has been already considered in the literature, see [5]. In this paper, by following the approach of Rosiński [11] and considering a particular family of tempering functions, a new class of

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distributions is introduced with the same suitable properties of the tempered stable class, but with the advantage that it may admit exponential moments of any order. By multiplying the Lévy measure of a stable distribution with a positive definite radial function, see [14], instead of with a completely monotone function as in [11], we obtain the class of tempered infinitely divisible (TID) distributions. In some cases, the characteristic function of a TID random variable is extendible to an entire function on $\mathbf{C}$, that is, it admits any exponential moment.

Some practical problems in the field of mathematical finance have motivated our study. Specifically, we sought an option pricing model that accounts for well-known stylized facts observed about real-world stock market returns: asymmetry, time-varying variance, and heavy tails.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{N}}, \mathbb{P})$ which is constructed as follows. Consider a sequence $(\varepsilon_t)_{t \in \mathbf{N}}$ of independent and identically distributed (i.i.d.) real random variables on a sequence of probability spaces $(\Omega_t, \mathbf{P}_t)_{t \in \mathbf{N}}$, such that ε_t is an infinitely divisible random variable with zero mean and unit variance on $(\Omega_t, \mathbf{P}_t)$, and assume that $\mathbb{E} e^{x\varepsilon_t} < \infty$, where $x \in I$ for some real interval I containing zero. Now we define $\Omega := \prod_{t \in \mathbf{N}} \Omega_t$, $\mathcal{F}_t := \bigotimes_{k=1}^t \sigma(\varepsilon_k) \otimes \mathcal{F}_0 \otimes \mathcal{F}_0 \cdots$, $\mathcal{F} := \sigma(\bigcup_{t \in \mathbf{N}} \mathcal{F}_t)$, and $\mathbb{P} := \bigotimes_{t \in \mathbf{N}} \mathbf{P}_t$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\sigma(\varepsilon_k)$ means the σ -algebra generated by ε_k on Ω_k .

On this probability space, we assume that the stock logarithmic returns have the following dynamics:

$$\ln \frac{S_t}{S_{t-1}} = r_t - d_t + \lambda_t \sigma_t - L(\sigma_t) + \sigma_t \varepsilon_t, \quad t \in \mathbf{N}, \quad (1.1)$$

where S_t is the stock price at time t , r_t and d_t denote the risk-free and dividend rate for the period $[t-1, t]$, respectively, and λ_t is an $\mathcal{F}_{t-1}$ -measurable random variable. The function $L(x)$ is the logarithm of the Laplace transform of ε_t , i.e., $L(x) = \ln \mathbb{E} e^{x\varepsilon_t}$. If $L(x)$ is defined only on a closed interval $[-a, b]$ with $a, b > 0$, then σ_t^2 follows a GARCH(1,1) process with a restriction $0 < \sigma_t \leq b$, i.e.,

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho, \quad t \in \mathbf{N}, \quad \varepsilon_0 = 0, \quad (1.2)$$

where $0 < \rho \leq b^2$ (see [7] for more details), and the symbol $\wedge$ refers to the minimum function (respectively, the symbol $\vee$ will refer to the maximum function). We propose a family of random variables that have finite exponential moments of any order so that we can (i) use the stock price model defined in (1.1), (ii) remove the restriction on σ_t in equation (1.2), and (iii) consider a random variable as close as possible to the α -stable, but with a Laplace transform defined on the entire real line.

Furthermore, we fill a gap in the literature. Specifically, the modified tempered stable (MTS) distribution is not a tempered stable distribution

of the Rosiński type [9] even though its properties are very close to the properties of that class. We will prove that the MTS distribution is in the TID class.

Although this distributional family is constructed by tempering the Lévy measure of a stable distribution, any stability property is lost. We will proceed as follows. In Section 2, basic definitions and distributional properties are given. Working with the Lévy measure may be a difficult task, therefore a spectral measure R is needed to figure out all characteristics of this class. This measure describes all distributional properties and allows one to obtain a close formula for the characteristic function. Since TID distributions are by construction infinitely divisible, a TID Lévy process can be considered. In Section 3, TID processes are analyzed. If the time scale increases, the TID process looks like a Gaussian process; conversely, if the time scale decreases, it looks like a stable process. Furthermore, under some condition on the tempering function, the change-of-measure problem between stable and TID processes can be solved. In Section 4, a view toward simulation is given. Taking into consideration [10], [11], a series representation is derived in terms of a measure Q , as already proved for the tempered stable class.

Similarly to the tempered stable framework, this class of distributions has an infinite dimensional parametrization by a family of measures [15], making it difficult to use. For this reason, in Section 5 some parametric examples in one dimension are proposed, and characteristic functions are derived.

2. Tempered infinitely divisible distribution. It is well known that the Lévy measure M_0 of an α -stable distribution on $\mathbf{R}^d$ can be written in polar coordinates in the form

$$M_0(dr, du) = r^{-\alpha-1} dr \sigma(du), \quad (2.1)$$

where $\alpha \in (0, 2)$ and σ is a finite measure on the unit sphere S^{d-1} .

Theorem 2.1. *If μ_0 is an α -stable distribution, then its characteristic function has the form*

$$\hat{\mu}_0(y) = \begin{cases} \exp \left\{ -c_\alpha \int_{S^{d-1}} |(y, u)|^\alpha \right. \\ \quad \left. \times \left(1 - i \operatorname{tg} \frac{\pi\alpha}{2} \operatorname{sign}(y, u) \right) \sigma(du) + i(y, a) \right\}, & \alpha \neq 1, \\ \exp \left\{ -c_1 \int_{S^{d-1}} \left(|(y, u)| \right. \right. \\ \quad \left. \left. + i \frac{2}{\pi} (y, u) \ln |(y, u)| \right) \sigma(du) + i(y, a) \right\}, & \alpha = 1, \end{cases} \quad (2.2)$$

where $a \in \mathbf{R}^d$ and

$$c_\alpha = \begin{cases} \left| \Gamma(-\alpha) \cos \frac{\pi\alpha}{2} \right|, & \alpha \neq 1, \\ \frac{\pi}{2}, & \alpha = 1. \end{cases}$$

P r o o f. See [13, Theorem 14.10].

D e f i n i t i o n 2.1. If Y is an α -stable random vector with characteristic function (2.2), we will write $Y \sim S_\alpha(\sigma, a)$.

Taking into account the approach of [11], we want to modify the radial component of M_0 and obtain a probability distribution with lighter tails than stable ones. A TID distribution is defined by tempering the radial term of M_0 as follows.

D e f i n i t i o n 2.2. Let μ be a infinitely divisible probability measure on $\mathbf{R}^d$ without Gaussian part. We call μ tempered infinitely divisible (TID) if its Lévy measure M can be written in polar coordinates as

$$M(dr, du) = r^{-\alpha-1} q(r, u) dr \sigma(du), \quad (2.3)$$

where α is a real number, $\alpha \in [0, 2)$, σ a finite measure on the unit sphere S^{d-1} , and $q: (0, \infty) \times S^{d-1} \rightarrow (0, \infty)$ is a Borel function defined by

$$q(r, u) := \int_0^\infty e^{-r^2 s^2/2} Q(ds | u), \quad (2.4)$$

with $\{Q(\cdot | u)\}_{u \in S^{d-1}}$ a measurable family of Borel measure on $(0, \infty)$. If $q(0+, u) = 1$ for each $u \in S^{d-1}$, then μ is referred to as a proper TID. The function q is called a tempering function.

In the case where $\{Q(\cdot | u)\}_{u \in S^{d-1}}$ are finite nonnegative Borel measures on $(0, \infty)$, $q(\cdot, u)$ are positive definite radial functions on $\mathbf{R}^d$. By [14], the following result holds.

Theorem 2.2. A continuous function $\varphi: (0, \infty) \rightarrow \mathbf{R}$ is positive definite and radial on $\mathbf{R}^d$ for all d if and only if it is of the form

$$\varphi(r) = \int_0^\infty e^{-r^2 s^2} \mu(ds),$$

where μ is a finite nonnegative Borel measure on $(0, \infty)$.

Define a measure Q on $\mathbf{R}^d$ by

$$Q(A) := \int_{S^{d-1}} \int_0^\infty I_A(ru) Q(dr | u) \sigma(du), \quad A \in \mathcal{B}(\mathbf{R}^d). \quad (2.5)$$

It is easy to check that $Q(\{0\}) = 0$. We also define a measure R on $\mathbf{R}^d$ by

$$R(A) := \int_{\mathbf{R}^d} I_A\left(\frac{x}{\|x\|^2}\right) \|x\|^\alpha Q(dx), \quad A \in \mathcal{B}(\mathbf{R}^d). \quad (2.6)$$

The measure R is equivalent to the measure Q and clearly $R(\{0\}) = 0$. By definition of R , for each Borel function F , the following equality is satisfied:

$$\int_{\mathbf{R}^d} F(x) R(dx) = \int_{\mathbf{R}^d} F\left(\frac{x}{\|x\|^2}\right) \|x\|^\alpha Q(dx), \quad (2.7)$$

in the sense that if one side exists, then the other exists and they are equal. If we choose $F(x) = I_A(x/\|x\|^2)\|x\|^\alpha$, then Q can be written as

$$Q(A) := \int_{\mathbf{R}^d} I_A\left(\frac{x}{\|x\|^2}\right) \|x\|^\alpha R(dx), \quad A \in \mathcal{B}(\mathbf{R}^d). \quad (2.8)$$

Sometimes the only knowledge of the Lévy measure cannot be enough to obtain analytical properties of tempered infinitely divisible distributions. Therefore, the definitions of measures Q and R allow one to overcome this problem and to obtain explicit analytic formulas and more explicit calculations. The following result allows one to figure out relations between the Lévy measure M and the measure R above defined.

Proposition 2.1. *Let μ be a TID distribution, and let the corresponding Lévy measure M be defined as in (2.3) and R as in (2.6). Then M can be written in the form*

$$M(A) = \int_{\mathbf{R}^d} \int_0^\infty I_A(tx) t^{-\alpha-1} e^{-t^2/2} dt R(dx), \quad A \in \mathcal{B}(\mathbf{R}^d), \quad (2.9)$$

if and only if the measure R on $\mathbf{R}^d$ satisfies the conditions $R(\{0\}) = 0$ and

$$\begin{cases} \int_{\mathbf{R}^d} (\|x\|^2 \wedge \|x\|^\alpha) R(dx) < \infty, & 0 < \alpha < 2, \\ \int_{\mathbf{R}^d} (\ln(1 + \|x\|) + 1) R(dx) < \infty, & \alpha = 0. \end{cases} \quad (2.10)$$

P r o o f. Let μ be a TID distribution with Lévy measure M . First we will prove that there exists at least one measure R , defined in (2.6), such that M , defined in (2.3), can be written in the form (2.9). To show this, we take the measure R as in (2.6) and for each $A \in \mathcal{B}(\mathbf{R}^d)$, by using (2.4)–(2.7) and the Fubini theorem, we have

$$\begin{aligned} M(A) &= \int_{S^{d-1}} \int_0^\infty I_A(ru) r^{-\alpha-1} q(r, u) dr \sigma(du) \\ &= \int_{S^{d-1}} \int_0^\infty \int_0^\infty I_A(ru) r^{-\alpha-1} e^{-r^2 s^2/2} Q(ds | u) dr \sigma(du) \\ &= \int_{S^{d-1}} \int_0^\infty \left(\int_0^\infty I_A(ru) r^{-\alpha-1} e^{-r^2 s^2/2} dr \right) Q(ds | u) \sigma(du) \\ &= \int_{S^{d-1}} \int_0^\infty \left(\int_0^\infty I_A\left(\frac{t}{s}u\right) t^{-\alpha-1} e^{-t^2/2} dt \right) s^\alpha Q(ds | u) \sigma(du) \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \left(\int_{S^{d-1}} \int_0^\infty I_A\left(t \frac{u}{s}\right) s^\alpha Q(ds | u) \sigma(du) \right) t^{-\alpha-1} e^{-t^2/2} dt \\
&= \int_0^\infty \left(\int_{\mathbf{R}^d} I_A\left(t \frac{x}{\|x\|^2}\right) \|x\|^\alpha Q(dx) \right) t^{-\alpha-1} e^{-t^2/2} dt \\
&= \int_0^\infty \left(\int_{\mathbf{R}^d} I_A(ty) R(dy) \right) t^{-\alpha-1} e^{-t^2/2} dt \\
&= \int_{\mathbf{R}^d} \int_0^\infty I_A(ty) t^{-\alpha-1} e^{-t^2/2} dt R(dy). \tag{2.11}
\end{aligned}$$

Conversely, given a measure R , let Q be the measure defined by (2.8) and let us consider the decomposition $Q(dr, du) = Q(dr | u) \sigma(du)$, where σ is a finite measure on S^{d-1} . Thus, we can define $q(r, u)$ by (2.4) and the computation (2.11) proves that M can be written in the form (2.3).

Now we want to prove that M is a Lévy measure if and only if (2.10) holds. Suppose M is a Lévy measure, then we have

$$\int_{\|x\| \leq 1} \|x\|^2 M(dx) < \infty$$

and by considering (2.9) and $\alpha \neq 0$, we obtain

$$\begin{aligned}
\infty &> \int_{\|x\| \leq 1} \|x\|^2 M(dx) = \int_{\mathbf{R}^d} \|x\|^2 \int_0^{1/\|x\|} t^{1-\alpha} e^{-t^2/2} dt R(dx) \\
&\geq \int_{\|x\| \leq 1} \|x\|^2 \int_0^1 t^{1-\alpha} e^{-t^2/2} dt R(dx) \\
&\quad + \int_{\|x\| > 1} \|x\|^2 \int_0^{1/\|x\|} t^{1-\alpha} e^{-t^2/2} dt R(dx) \\
&\geq e^{-1/2} (2 - \alpha)^{-1} \int_{\|x\| \leq 1} \|x\|^2 R(dx) + e^{-1/2} (2 - \alpha)^{-1} \int_{\|x\| > 1} \|x\|^\alpha R(dx),
\end{aligned}$$

thus, we obtain the desired inequality $\int_{\mathbf{R}^d} (\|x\|^2 \wedge \|x\|^\alpha) R(dx) < \infty$.

Now, let us consider the case $\alpha = 0$. By definition of the Lévy measure we have $\int_{\|x\| \geq 1} M(dx) < \infty$ and by considering (2.9) with $\alpha = 0$, the following inequalities are satisfied:

$$\begin{aligned}
\infty &> \int_{\|x\| > 1} M(dx) = \int_{\mathbf{R}^d} \int_{\frac{1}{\|x\|}}^\infty t^{-1} e^{-t^2/2} dt R(dx) \\
&= \int_{\|x\| \leq 1} \int_{1/\|x\|}^\infty t^{-1} e^{-t^2/2} dt R(dx) + \int_{\|x\| > 1} \int_{\frac{1}{\|x\|}}^\infty t^{-1} e^{-t^2/2} dt R(dx) \\
&\geq \int_{\|x\| \leq 1} K R(dx) + e^{-1/2} \int_{\|x\| > 1} \ln(\|x\|) R(dx)
\end{aligned}$$

where K is a finite constant. Then, also when $\alpha = 0$, condition (2.10) is a necessary condition. Conversely, now we prove that (2.10) is also sufficient.

Suppose that there is a measure R satisfying (2.10). Then the measure M can be written in the form (2.9). If $\alpha \neq 0$, we can write

$$\begin{aligned} \int_{\|x\| \leq 1} \|x\|^2 M(dx) &= \int_{\mathbf{R}^d} \|x\|^2 \int_0^{1/\|x\|} t^{1-\alpha} e^{-t^2/2} dt R(dx) \\ &\leq \int_{\|x\| \leq 1} \|x\|^2 \int_0^\infty t^{1-\alpha} e^{-t^2/2} dt R(dx) + \frac{1}{2-\alpha} \int_{\|x\| > 1} \|x\|^\alpha R(dx) \\ &= 2^{-\alpha/2} \Gamma\left(1 - \frac{\alpha}{2}\right) \int_{\|x\| \leq 1} \|x\|^2 R(dx) + \frac{1}{2-\alpha} \int_{\|x\| > 1} \|x\|^\alpha R(dx) < \infty \end{aligned}$$

and

$$\begin{aligned} \int_{\|x\| > 1} M(dx) &= \int_{\mathbf{R}^d} \int_{1/\|x\|}^\infty t^{-1-\alpha} e^{-t^2/2} dt R(dx) \\ &\leq C \int_{\|x\| \leq 1} \int_{1/\|x\|}^\infty t^{-3} dt R(dx) + \int_{\|x\| > 1} \int_{1/\|x\|}^\infty t^{-\alpha-1} dt R(dx) \\ &= \frac{C}{2} \int_{\|x\| \leq 1} \|x\|^2 R(dx) + \frac{1}{\alpha} \int_{\|x\| > 1} \|x\|^\alpha R(dx), \end{aligned}$$

where $C := \sup_{t \geq 1} t^{2-\alpha} e^{-t^2/2}$. Thus M is a Lévy measure. A similar argument also holds for $\alpha = 0$. In order to show that (2.9) is well defined, one can prove by contradiction that R is uniquely determined. This argument completes the proof.

R e m a r k 2.1. The case $\alpha = 0$ is considered only for completeness and the theory will not be completely extended to this limiting case. It may be an interesting case in some applications.

R e m a r k 2.2. If $\alpha \in (0, 2)$ and R satisfies the additional inequality

$$\int_{\mathbf{R}^d} \|x\|^\alpha R(dx) < \infty, \quad 0 < \alpha < 2, \quad (2.12)$$

we call μ a proper TID distribution. Furthermore, the following equality holds:

$$Q(\mathbf{R}^d) = \int_{\mathbf{R}^d} \|x\|^\alpha R(dx), \quad 0 < \alpha < 2. \quad (2.13)$$

In this case Q is a finite measure and it can be represented in polar coordinates as $Q(dr, du) = Q(dr | u) \sigma(du)$, where $Q(\cdot | u)$ are finite measures and σ is a finite measure on S^{d-1} .

D e f i n i t i o n 2.3. The unique measure R in (2.9) is called a spectral measure of the corresponding TID distribution. We call R the Rosiński measure [15].

We focus on the following result.

R e m a r k 2.3. If μ is a proper TID distribution, then Q is a finite measure and $\{Q(\cdot | u)\}_{u \in S^{d-1}}$ is a measurable family of finite Borel measures

on $(0, \infty)$. Since the equation $q(0+, u) = 1$ holds, they are probability measures. Furthermore, for any fixed $u \in S^{d-1}$, function $q(\cdot, u)$ is a positive definite radial function.

Taking into consideration Lemma 2.14 of [11], we want to figure out the relation between parameters of the proper TID distribution and stable ones.

Proposition 2.2. *Let M be a Lévy measure of a proper TID distribution, as in (2.3), with corresponding spectral measure R . Then, the Lévy measure M_0 of an α -stable distribution, given in (2.1), can be written in the following form:*

$$M_0(A) = \int_{\mathbf{R}^d} \int_0^\infty I_A(tx) t^{-\alpha-1} dt R(dx), \quad A \in \mathcal{B}(\mathbf{R}^d), \quad (2.14)$$

and additionally

$$\sigma(B) = \int_{\mathbf{R}^d} I_B\left(\frac{x}{\|x\|}\right) \|x\|^\alpha R(dx), \quad B \in \mathcal{B}(S^{d-1}). \quad (2.15)$$

P r o o f. For the proof, readers are referred to [2].

2.1. Distributional properties. A TID distribution may have moments and also exponential moments of any order. The behavior of the tails depends on the measure R .

Proposition 2.3. *Let μ be a TID distribution with Lévy measure M given by (2.9) and $\alpha \in (0, 2)$. Then*

(a) *For $p \in (0, \alpha)$*

$$\int_{\mathbf{R}^d} \|x\|^p \mu(dx) < \infty;$$

(b) *$\int_{\mathbf{R}^d} \|x\|^\alpha \mu(dx) < \infty$ if and only if $\int_{\|x\|>1} \|x\|^\alpha \ln \|x\| R(dx) < \infty$;*

(c) *if $p > \alpha$, then*

$$\int_{\mathbf{R}^d} \|x\|^p \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^p R(dx) < \infty;$$

(d) *for each $\theta > 0$, we have*

$$\int_{\mathbf{R}^d} e^{\theta\|x\|} \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^{-(\alpha+1)} e^{\theta^2\|x\|^2/2} R(dx) < \infty;$$

(e) *if $\alpha = 0$ and $p > 0$, then*

$$\int_{\mathbf{R}^d} \|x\|^p \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^p R(dx) < \infty.$$

P r o o f. The proofs of (a), (b), and (c) follow the approach of [11], and it can be found in [2]. It is well known that moment conditions for μ are related to the corresponding conditions for $M_{\{\|x\|>1\}}$, see [13].

In order to prove (d), we consider the integral

$$\int_{\|x\|>1} e^{\theta\|x\|} M(dx) = \int_{\mathbf{R}^d} \int_{1/\|x\|}^{\infty} e^{\theta t\|x\|} t^{-(\alpha+1)} e^{-t^2/2} dt R(dx)$$

and we define

$$J_{\theta}(x) := \int_{1/\|x\|}^{\infty} e^{\theta t\|x\|} t^{-(\alpha+1)} e^{-t^2/2} dt.$$

It is easy to check that $I_{\theta}(x)$ goes to 0 exponentially fast as $\|x\| \rightarrow 0$. Now, let us consider the case $\|x\| \rightarrow \infty$. We have

$$J_{\theta}(x) = e^{\theta^2\|x\|^2/2} \int_{1/\|x\|}^{\infty} t^{-(\alpha+1)} e^{-(t-\theta\|x\|)^2/2} dt =: e^{\theta^2\|x\|^2/2} = K_{\theta}(x).$$

By changing variables in the integral we obtain

$$\begin{aligned} K_{\theta}(x) &= \int_{1/\|x\|-\theta\|x\|}^{\infty} (t + \theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &= \int_{1/\|x\|-\theta\|x\|}^{-\theta\|x\|/2} (t + \theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &\quad + \int_{-\theta\|x\|/2}^{\infty} (t + \theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &= K_{\theta}^{(1)}(x) + K_{\theta}^{(2)}(x). \end{aligned}$$

Furthermore,

$$\begin{aligned} K_{\theta}^{(1)}(x) &\leq e^{-\theta^2\|x\|^2/8} \int_{1/\|x\|-\theta\|x\|}^{-\theta\|x\|/2} (t + \theta\|x\|)^{-(\alpha+1)} dt \\ &\leq C\|x\|^{\alpha+2} e^{-\theta^2\|x\|^2/8}, \end{aligned}$$

and for $\|x\| \rightarrow \infty$,

$$\begin{aligned} K_{\theta}^{(2)}(x) &\leq \int_{-\theta\|x\|/2}^{\infty} (t + \theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &\sim (\theta\|x\|)^{-(\alpha+1)} \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}(\theta\|x\|)^{-(\alpha+1)}. \end{aligned}$$

It follows that, as $\|x\| \rightarrow \infty$,

$$K_{\theta}(x) \sim \sqrt{2\pi}(\theta\|x\|)^{-(\alpha+1)}, \quad J_{\theta}(x) \sim \sqrt{2\pi}(\theta\|x\|)^{-(\alpha+1)} e^{\theta^2\|x\|^2/2},$$

therefore condition (d) holds. Part (e) can be proved with a similar argument to (c).

R e m a r k 2.4. If the measure R relative to a tempered stable random vector X has a bounded support, then $\mathbb{E} e^{\theta\|X\|} < \infty$ for all $\theta > 0$; that is, there exist finite exponential moments of any order.

As we said before, sometimes it is more convenient to work with the measure R , in order to find some distributional property of a TID distribution. Taking into account Proposition 2.8 of [11], we will show a result about finite variation.

Proposition 2.4. *Let M be the Lévy measure of a TID distribution and R a measure as in (2.9) and (2.6). The following conditions are equivalent:*

- (i) $\int_{\|x\| \leq 1} \|x\| M(dx) < \infty$;
- (ii) $\int_{\|x\| \leq 1} \|x\| R(dx) < \infty$ and $\alpha \in (0, 1)$.

P r o o f. The proof follows the approach of [11], and it can be found in [2].

2.2. Characteristic function of a TID distribution. It is well known that given a Lévy measure of an infinitely divisible distribution, we have an explicit formula for the characteristic function, see [13]. Sometimes, working with a Lévy measure of the form (2.3) may be difficult and, as a consequence, we will provide an expression for the characteristic function of a TID distribution with respect to the measure R . The measure R allows one to obtain explicit analytic formulas and more explicit calculations. Let us now define functions

$$\psi_\alpha(s) = \int_0^\infty (e^{ist} - 1 - ist)t^{-\alpha-1} e^{-t^2/2} dt, \quad (2.16)$$

$$\psi_\alpha^0(s) = \int_0^\infty (e^{ist} - 1)t^{-\alpha-1} e^{-t^2/2} dt. \quad (2.17)$$

In order to find a more useful form for the characteristic function of a TID distribution, we will need the following results.

Lemma 2.1. *The following limits are verified:*

$$\begin{aligned} \lim_{s \rightarrow 0} s^{-2} \psi_\alpha(s) &= -2^{-\alpha/2-1} \Gamma\left(1 - \frac{\alpha}{2}\right), & \alpha \in (0, 2), \\ \lim_{s \rightarrow \infty} s^{-1} \psi_0(s) &= -i \sqrt{\frac{\pi}{2}}, & \alpha = 0, \\ \lim_{s \rightarrow \infty} s^{-1} \psi_\alpha(s) &= -2^{-\alpha/2-1/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) i, & \alpha \in (0, 1), \\ \lim_{s \rightarrow \infty} (s^{-1} \psi_1(s) + i \ln s) &= -\frac{\pi}{2} + i, & \alpha = 1, \\ \lim_{s \rightarrow \infty} s^{-\alpha} \psi_\alpha(s) &= \Gamma(-\alpha) e^{-i\alpha\pi/2}, & \alpha \in (1, 2). \end{aligned} \quad (2.18)$$

Furthermore, if $\alpha \in (0, 1)$ we have

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{\psi_\alpha^0(s)}{s} &= 2^{-\alpha/2-1/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) i, \\ \lim_{s \rightarrow \infty} \frac{\psi_\alpha^0(s)}{s^\alpha} &= \Gamma(-\alpha) e^{-i\alpha\pi/2}, \end{aligned} \quad (2.19)$$

then, there exists for each α a finite positive constant C_α such that for all $s \in \mathbf{R}$ the following inequalities are fulfilled:

$$\begin{aligned} C_\alpha^{-1}(s^2 \wedge |s|^{\alpha \vee 1}) &\leq |\psi_\alpha(s)| \leq C_\alpha(s^2 \wedge |s|^{\alpha \vee 1}), & \alpha \neq 1, \\ C_1^{-1}[s^2 \wedge |s|(1 + \ln^+ |s|)] &\leq |\psi_1(s)| \leq C_1[s^2 \wedge |s|(1 + \ln^+ |s|)], & \alpha = 1, \\ C_\alpha^{-1}(s^2 \wedge |s|^\alpha) &\leq |\psi_\alpha^0(s)| \leq C_\alpha(s^2 \wedge |s|^\alpha), & \alpha \in (0, 1), \\ C_0^{-1}[1 + \ln(1 + s)] &\leq |\psi_0(s)| \leq C_0[1 + \ln(1 + s)], & \alpha = 0. \end{aligned} \quad (2.20)$$

P r o o f. By solving the limit and using [13, Lemma 14.11], (2.18) and (2.20) are verified.

Lemma 2.2. *Let $\alpha \in (0, 2)$, $\alpha \neq 1$. Then*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{z^n}{n!} \Gamma\left(\frac{1}{2}(n - \alpha)\right) &= \Gamma\left(-\frac{\alpha}{2}\right) {}_1F_1\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{z}{2}\right)^2\right) \\ &\quad + z\Gamma\left(\frac{1-\alpha}{2}\right) {}_1F_1\left(\frac{1}{2} - \frac{\alpha}{2}, \frac{3}{2}; \left(\frac{z}{2}\right)^2\right), \end{aligned} \quad (2.21)$$

where ${}_1F_1$ is the Kummer or confluent hypergeometric function of first kind.

P r o o f. The proof of (2.21) can be found in [2]. For an overview of the confluent hypergeometric functions, see [16] or [1].

Theorem 2.3 (characteristic function). *Let μ be a TID distribution with Lévy measure given by (2.9), $\alpha \in [0, 2)$ and $\alpha \neq 1$. If the distribution has finite mean, i.e., $\int_{\mathbf{R}^d} \|x\| \mu(dx) < \infty$, then*

$$\hat{\mu}(y) = \exp \left\{ \int_{\mathbf{R}^d} \psi_\alpha((y, x)) R(dx) + i(y, m) \right\}, \quad (2.22)$$

where

$$\begin{aligned} \psi_\alpha(s) &= 2^{-\alpha/2-1} \left[\Gamma\left(-\frac{\alpha}{2}\right) \left({}_1F_1\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{i\sqrt{2}s}{2}\right)^2\right) - 1 \right) \right. \\ &\quad \left. + i\sqrt{2}s \Gamma\left(\frac{1-\alpha}{2}\right) \left({}_1F_1\left(\frac{1}{2} - \frac{\alpha}{2}, \frac{3}{2}; \left(\frac{i\sqrt{2}s}{2}\right)^2\right) - 1 \right) \right]. \end{aligned} \quad (2.23)$$

and $m = \int_{\mathbf{R}^d} x \mu(dx)$. Furthermore, if $0 < \alpha < 1$, then the characteristic function can be written in an alternative form

$$\hat{\mu}(y) = \exp \left\{ \int_{\mathbf{R}^d} \psi_\alpha^0((y, x)) R(dx) + i(y, m_0) \right\}, \quad (2.24)$$

where

$$\begin{aligned} \psi_\alpha(s) &= 2^{-\alpha/2-1} \left[\Gamma\left(-\frac{\alpha}{2}\right) \left({}_1F_1\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{i\sqrt{2}s}{2}\right)^2\right) - 1 \right) \right. \\ &\quad \left. + i\sqrt{2}s \Gamma\left(\frac{1-\alpha}{2}\right) {}_1F_1\left(\frac{1}{2} - \frac{\alpha}{2}, \frac{3}{2}; \left(\frac{i\sqrt{2}s}{2}\right)^2\right) \right]. \end{aligned} \quad (2.25)$$

P r o o f. First, integrals (2.22) and (2.24) are well defined due to conditions (2.10) and (2.20) of Lemma 2.1. It is well known that if the mean is finite, that is, if the first absolute moment exists, i.e., $\int_{\mathbf{R}^d} \|x\| \mu(dx) < \infty$, then $\hat{\mu}$ can be written as

$$\hat{\mu} = \exp \left(\int_{\mathbf{R}^d} (e^{i(y,x)} - 1 - i(y,x)) \nu(dx) + i(y, m) \right),$$

where $m = \int_{\mathbf{R}^d} x \mu(dx)$. By (2.9), we obtain the equality (2.22), where, if $\alpha \in [0, 2)$,

$$\psi_\alpha(s) = \int_0^\infty (e^{ist} - 1 - ist) t^{-\alpha-1} e^{-t^2/2} dt, \quad (2.26)$$

If $\alpha \in [0, 1)$ and $\int_{\|x\| \leq 1} \|x\| R(dx) < \infty$, then by Proposition 2.4 $\int_{\|x\| \leq 1} \|x\| \nu(dx) < \infty$, in which case $\hat{\mu}$ can be written as

$$\exp \left(\int_{\mathbf{R}^d} (e^{i(y,x)} - 1) \nu(dx) + i(y, m_0) \right),$$

where m_0 is the drift as defined in [13]. By (2.9), we obtain equality (2.24), where

$$\psi_\alpha^0(s) = \int_0^\infty (e^{ist} - 1) t^{-\alpha-1} e^{-t^2/2} dt, \quad (2.27)$$

and, furthermore, the equality

$$\psi_\alpha(s) = \psi_\alpha^0(s) - is \int_0^\infty t^{-\alpha} e^{-t^2/2} dt \quad (2.28)$$

holds. Now we will prove (2.23) and (2.25). If $\alpha \in (0, 1)$, then we obtain by equality (2.21)

$$\begin{aligned} \int_0^\infty (e^{ist} - 1) t^{-\alpha-1} e^{-t^2/2} dt &= \sum_{n=1}^\infty \frac{(is)^n}{n!} \int_0^\infty t^{n-\alpha-1} e^{-t^2/2} dt \\ &= \sum_{n=1}^\infty \frac{(is)^n}{n!} \cdot 2^{(n-\alpha-2)/2} \Gamma\left(\frac{1}{2}(n-\alpha)\right) \\ &= 2^{-\alpha/2-1} \sum_{n=1}^\infty \frac{(i\sqrt{2}s)^n}{n!} \Gamma\left(\frac{1}{2}(n-\alpha)\right) \\ &= 2^{-\alpha/2-1} \left(\Gamma\left(-\frac{\alpha}{2}\right) {}_1F_1\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{i\sqrt{2}s}{2}\right)^2\right) \right. \\ &\quad \left. + i\sqrt{2}s \Gamma\left(\frac{1-\alpha}{2}\right) {}_1F_1\left(\frac{1-\alpha}{2}, \frac{3}{2}; \left(\frac{i\sqrt{2}s}{2}\right)^2\right) - \Gamma\left(-\frac{\alpha}{2}\right) \right). \end{aligned}$$

A similar calculus proves the result for $\alpha \in (0, 2)$ and $\alpha \neq 1$.

R e m a r k 2.5. With a similar technique, the characteristic exponent can be calculated also for both cases $\alpha = 0$ and $\alpha = 1$.

Definition 2.4. We will write $X \sim \text{TID}_\alpha(R, m)$ to indicate that X is a TID random variable with characteristic function (2.22) and $X \sim \text{TID}_\alpha^0(R, m_0)$ to indicate that X is a TID random variable with characteristic function (2.24). The constant m is exactly the mean: $m = \mathbb{E} X$.

3. TID processes. In this section, we will introduce TID processes. By Definition 2.2, if μ is a TID distribution, it is infinitely divisible and therefore there exists a Lévy process $\{X(t): t \geq 0\}$ such that μ is the distribution of $X(1)$.

3.1. Short and long time behavior. The following theorems will show the different behavior of a TID process for different time scale. If one decreases the time scales, a TID process looks like a stable process; otherwise, if one increases the time scale, it looks like a Brownian motion. To figure out this different time behavior, we consider the time rescaled process

$$\{X_h(t): t \geq 0\} = \{X(ht): t \geq 0\}, \quad (3.1)$$

where $h > 0$.

Theorem 3.1 (short time behavior). *Let $\{X(t): t \geq 0\}$ be a TID Lévy process in $\mathbf{R}^d$ such that the distribution of $X(1)$ has spectral measure R .*

(a) *Let us consider a TID process with $X(1) \sim \text{TID}_\alpha^0(R, 0)$ if $\alpha \in (0, 1)$ and with $X(1) \sim \text{TID}_\alpha(R, 0)$ if $\alpha \in (1, 2)$. Assume that*

$$\int_{\mathbf{R}^d} \|x\|^\alpha R(dx) < \infty \quad (3.2)$$

and let σ be the finite measure on S^{d-1} defined in (2.15). Then

$$h^{-1/\alpha} X_h \xrightarrow{d} Y \quad (3.3)$$

as $h \rightarrow 0$, where $\{Y(t): t \geq 0\}$ is a strictly α -stable Lévy process with $Y(1) \sim S_\alpha(\sigma, 0)$.

(b) *Let us consider a TID process with $X(1) \sim \text{TID}_\alpha(R, 0)$, if $\alpha = 1$. Assume that*

$$\int_{\mathbf{R}^d} \|x\| |\ln \|x\|| R(dx) < \infty. \quad (3.4)$$

Then

$$h^{-1/\alpha} X_h - a_h \xrightarrow{d} Y,$$

where $a_h(t) = t \ln h \int_{\mathbf{R}^d} x R(dx)$ and $\{Y(t): t \geq 0\}$ is an α -stable Lévy process with $Y(1) \sim S_1(\sigma, b)$, $b = \int_{\mathbf{R}^d} x(1 - \ln \|x\|) R(dx)$.

P r o o f. The proof follows the approach of [11], and it can be found in [2].

Now, we will prove that if one increases the time scale, a TID process looks like a Brownian motion.

Theorem 3.2 (long time behavior). *Let $\{X(t): t \geq 0\}$ be a TID Lévy process in $\mathbf{R}^d$ such that $X(1) \sim \text{TID}_\alpha(R, 0)$ and $\alpha \in (0, 2)$. Assume that*

$$\int_{\mathbf{R}^d} \|x\|^2 R(dx) < \infty. \quad (3.5)$$

Then

$$h^{-1/2} X_h \xrightarrow{d} B$$

as $h \rightarrow \infty$, where $\{B(t): t \geq 0\}$ is a Brownian motion with characteristic function

$$\mathbb{E} e^{i(y, B(t))} = \exp \left\{ -t 2^{-\alpha/2-1} \Gamma\left(1 - \frac{\alpha}{2}\right) \int_{\mathbf{R}^d} (y, x)^2 R(dx) \right\}. \quad (3.6)$$

P r o o f. The proof follows the approach of [11], and it can be found in [2].

3.2. Change of measure. In this section, a result on density transformations between stable and TID processes is considered.

Theorem 3.3. *Let $\mathbf{P}_0$ and $\mathbf{P}$ be probability measures on $(\Omega, \mathcal{F})$ such that the canonical process $\{X(t): t \geq 0\}$ is a Lévy α -stable process $S_\alpha(\sigma, a)$ under $\mathbf{P}_0$, while it is a proper TID process $\text{TID}_\alpha(R, b)$ under $\mathbf{P}$, where σ is given by (2.15). Then*

(i) $\mathbf{P}_{0|\mathcal{F}_t}$ and $\mathbf{P}_{|\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if

$$\int_{S^{d-1}} \int_0^1 (1 - q(r, u))^2 r^{-\alpha-1} dr \sigma(du) < \infty \quad (3.7)$$

and

$$b - a = \begin{cases} 0, & 0 < \alpha < 1, \\ \int_{\mathbf{R}^d} x \left(\ln \|x\| + \frac{\ln 2}{2} - 1 + \frac{\gamma}{2} \right) R(dx), & \alpha = 1, \\ 2^{-1/2-\alpha/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) \int_{\mathbf{R}^d} x R(dx), & 1 < \alpha < 2. \end{cases} \quad (3.8)$$

Condition (3.7) implies that the integral exist. Furthermore, if either (3.7) or (3.8) fails, then $\mathbf{P}_{0|\mathcal{F}_t}$ and $\mathbf{P}_{|\mathcal{F}_t}$ are singular for all $t > 0$.

(ii) If (3.7) and (3.8) hold, then for each $t > 0$

$$\frac{d\mathbf{P}}{d\mathbf{P}_{0|\mathcal{F}_t}} = e^{Z_t}, \quad (3.9)$$

where $\{Z_t: t \geq 0\}$ is a Lévy process on $(\Omega, \mathcal{F}, \mathbf{P}_0)$ given by

$$Z_t = \lim_{\delta \downarrow 0} \left\{ \sum_{\{s \leq t: \|\Delta X_s\| > \delta\}} \ln q\left(\|\Delta X_s\|, \frac{\Delta X_s}{\|\Delta X_s\|}\right) + t \int_{S^{d-1}} \int_\delta^\infty (1 - q(r, u)) r^{-\alpha-1} dr \sigma(du) \right\}.$$

The convergence is uniform in t on any bounded interval, $\mathbf{P}_0$ -a.s.

P r o o f. First, we will prove part (i). By (2.1) and (2.3), we have

$$\frac{dM}{dM_0}(x) = q\left(\|x\|, \frac{x}{\|x\|}\right), \quad x \in \mathbf{R}^d \setminus \{0\}, \quad (3.10)$$

and for each $A \in \mathcal{B}(\mathbf{R}^d)$

$$\int_A q\left(\|x\|, \frac{x}{\|x\|}\right) M_0(dx) = \int_{S^{d-1}} \int_0^\infty I_A(ru) q(r, u) r^{-\alpha-1} dr \sigma(du) = M(A).$$

By [13, Theorem 33.1], define the function $\phi(x)$ by

$$\frac{dM}{dM_0}(x) = e^{\phi(x)}.$$

Thus, we have

$$\phi(x) = \ln q\left(\|x\|, \frac{x}{\|x\|}\right),$$

and $\mathbf{P}_0|_{\mathcal{F}_t}$ and $\mathbf{P}|_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if

$$\int_{\mathbf{R}^d} \left(e^{\phi(x)/2-1} - 1\right)^2 M_0(dx) < \infty \quad (3.11)$$

and

$$B_\alpha = 0, \quad (3.12)$$

where B_α is defined by

$$B_\alpha = \begin{cases} b + \int_{\|x\| \leq 1} x M(dx) - \left(a + \int_{\|x\| \leq 1} x M_0(dx)\right) \\ \quad - \int_{\|x\| \leq 1} x (M - M_0)(dx), & 0 < \alpha < 1, \\ b - \int_{\|x\| > 1} x M(dx) - \left(a - c \int_{S^{d-1}} u \sigma(du)\right) \\ \quad - \int_{\|x\| \leq 1} x (M - M_0)(dx), & \alpha = 1, \\ b - \int_{\|x\| > 1} x M(dx) - \left(a - \int_{\|x\| > 1} x M_0(dx)\right) \\ \quad - \int_{\|x\| \leq 1} x (M - M_0)(dx), & 1 < \alpha < 2. \end{cases}$$

If $\alpha = 1$, then $c = 1 - \gamma$, where γ is the Euler constant. Inequality (3.11) can be written as

$$\int_{\mathbf{R}^d} \left(1 - q^{1/2}\left(\|x\|, \frac{x}{\|x\|}\right)\right)^2 M_0(dx) < \infty. \quad (3.13)$$

Since the integrand is bounded and M_0 is a Lévy measure, we may focus our attention only on integration over $\{\|x\| \leq 1\}$. Applying elementary inequalities $(1-y)^2/4 \leq (1-\sqrt{y})^2 \leq (1-y)^2$ for $y \in [0, 1]$, inequality (3.13) becomes

$$\int_{\{\|x\| \leq 1\}} \left(1 - q\left(\|x\|, \frac{x}{\|x\|}\right)\right)^2 M_0(dx),$$

and writing the above integral in polar coordinates, we obtain (3.7). Now, we will prove the equivalence between conditions (3.8) and (3.12). By finiteness of the integral above and the Hölder inequality, we have

$$\begin{aligned} & \int_{\|x\| \leq 1} \|x\| (M_0 - M)(dx) \\ &= \int_{\|x\| \leq 1} \|x\| \left(1 - q\left(\|x\|, \frac{x}{\|x\|}\right)\right) M_0(dx) \leq \left(\int_{\|x\| \leq 1} \|x\|^2 M_0(dx)\right)^{1/2} \\ & \quad \times \left(\int_{\|x\| \leq 1} \left(1 - q\left(\|x\|, \frac{x}{\|x\|}\right)\right)^2 M_0(dx)\right)^{1/2} < \infty. \end{aligned} \quad (3.14)$$

If $0 < \alpha < 1$, then $B_\alpha = b - a = 0$ by (3.8). Suppose $1 < \alpha < 2$, then we have

$$\int_{\|x\| > 1} \|x\| M(dx) = \int_{\|x\| > 1} \|x\| q\left(\|x\|, \frac{x}{\|x\|}\right) M_0(dx),$$

and, since we are considering a proper TID process $q(\|x\|, x/\|x\|) \leq 1$, and M_0 is the Lévy measure of an α -stable distribution with $1 < \alpha < 2$, we obtain

$$\int_{\|x\| > 1} \|x\| M(dx) \leq \int_{\|x\| > 1} \|x\| M_0(dx) < \infty.$$

Furthermore, by (3.14) $\int_{\mathbf{R}^d} \|x\| (M_0 - M)(dx) < \infty$. By using (2.9) and (2.14) and integrating by parts, the following result is obtained:

$$\begin{aligned} \int_{\mathbf{R}^d} \|x\| (M_0 - M)(dx) &= \int_{\mathbf{R}^d} \int_0^\infty \|x\| t^{-\alpha} (1 - e^{-t^2/2}) dt R(dx) \\ &= -2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) \int_{\mathbf{R}^d} \|x\| R(dx) < \infty. \end{aligned}$$

With a similar calculus, we can write

$$\begin{aligned} B_\alpha &= \int_{\mathbf{R}^d} x (M_0 - M)(dx) + b - a \\ &= -2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) \int_{\mathbf{R}^d} x R(dx) + b - a = 0, \end{aligned}$$

where the latter equality follows by (3.8), proving (3.12). It remains to verify (3.12) in the case $\alpha = 1$. By taking into account (3.14),

$$\infty > \int_{\|x\| \leq 1} \|x\| (M_0 - M)(dx) = \int_{\mathbf{R}^d} \|x\| \int_0^{1/\|x\|} t^{-1} (1 - e^{-t^2/2}) dt R(dx)$$

$$\geq \frac{1}{4} \int_{\|x\| \leq 1} \|x\| \int_1^{1/\|x\|} t^{-1} dt R(dx) = \frac{1}{4} \int_{\|x\| \leq 1} \|x\| (-\ln \|x\|) R(dx).$$

Since the latter integral is finite, the second integral in (3.8) is well defined and we can calculate

$$\begin{aligned} \int_{\|x\| \leq 1} x (M_0 - M)(dx) &= \int_{\mathbf{R}^d} x \int_0^{1/\|x\|} t^{-1} (1 - e^{-t^2/2}) dt R(dx) \\ &= \frac{1}{2} \int_{\mathbf{R}^d} x \left(E_1 \left(\frac{1}{2\|x\|^2} \right) - 2 \ln \|x\| - \ln 2 + \gamma \right) R(dx) \end{aligned}$$

by changing variable and using equation 5.1.39 in [1], where the function $E_1(x)$ is the exponential integral function defined by

$$E_1(x) = \int_x^\infty t^{-1} e^{-t} dt.$$

By assertion (b) of Proposition 2.3, the first moment is finite, thus $\int_{\|x\| > 1} x M(dx) < \infty$ and

$$\begin{aligned} \int_{\|x\| > 1} x M(dx) &= \int_{\mathbf{R}^d} x \int_{1/\|x\|}^\infty t^{-1} e^{-t^2/2} dt R(dx) \\ &= \frac{1}{2} \int_{\mathbf{R}^d} x E_1 \left(\frac{1}{2\|x\|^2} \right) R(dx). \end{aligned}$$

Adding together the above results, we have

$$\begin{aligned} B_1 &= b - \frac{1}{2} \int_{\mathbf{R}^d} x E_1 \left(\frac{1}{2\|x\|^2} \right) R(dx) - a + (1 - \gamma) \int_{S^{d-1}} u \sigma(du) \\ &\quad + \frac{1}{2} \int_{\mathbf{R}^d} x \left(E_1 \left(\frac{1}{2\|x\|^2} \right) - 2 \ln \|x\| - \ln 2 + \gamma \right) R(dx) \\ &= b - a + (1 - \gamma) \int_{\mathbf{R}^d} x R(dx) - \int_{\mathbf{R}^d} x \left(\ln \|x\| + \frac{\ln 2}{2} \right) R(dx) \\ &\quad + \frac{\gamma}{2} \int_{\mathbf{R}^d} x R(dx) = 0. \end{aligned}$$

By considering the remark in [13, Notes, p. 236], we can complete the proof of part (i). Indeed, since M and M_0 are mutually absolutely continuous by (3.10), $\mathbf{P}_0|_{\mathcal{F}_t}$ and $\mathbf{P}|_{\mathcal{F}_t}$ are mutually absolutely continuous or singular for all $t > 0$.

Part (ii) is an application of Theorem 33.2 in [13], where the form of Radon–Nikodým derivative is specified for two mutually absolutely continuous Lévy processes.

4. Simulation of proper TID laws and processes. There are different methods to simulate Lévy processes, but most of these methods are not suitable for the simulation of TID processes due to the complicated structure of their Lévy measures. The usual method of the inverse of the Lévy measure is difficult to implement, even if the spectral measure R has a simple form, readers are referred to [11]. To overcome this problem, we will find a shot noise representation for proper TID distributions, and consequently also TID processes, without constructing any inverse. The representation, we will show, is based on results in [10] and [11].

Let M be the Lévy measure of a proper TID distribution on $\mathbf{R}^d$, given by (2.3), and Q and R corresponding measures defined in (2.5) and (2.6). Let us define $\|\sigma\|$ as

$$\|\sigma\| := \sigma(S^{d-1}), \quad (4.1)$$

and by equalities (2.13) and (2.15), we obtain

$$\|\sigma\| = Q(\mathbf{R}^d) = \int_{\mathbf{R}^d} \|x\|^\alpha R(dx) < \infty.$$

Let $\{v_j\}$ be an i.i.d. sequence of random vectors in $\mathbf{R}^d$ with distribution $Q/\|\sigma\|$. Let $\{u_j\}$ be an i.i.d. sequence of uniform random variables on $(0, 1)$, and let $\{e_j\}$ and $\{e'_j\}$ be i.i.d. sequences of exponential random variables with parameter 1. Furthermore, we assume that $\{v_j\}$, $\{u_j\}$, $\{e_j\}$, and $\{e'_j\}$ are independent. We consider $\gamma_j = e'_1 + \dots + e'_j$ and, by definition of $\{e'_j\}$, $\{\gamma_j\}$ is a Poisson point process on $(0, \infty)$ with Lebesgue intensity measure. Now, we will prove a useful lemma.

Lemma 4.1. *Under the above definitions, let us define the function*

$$H(\gamma_j, (v_j, e_j, u_j)) := \left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_j^{1/2} u_j^{1/\alpha}}{\|v_j\|} \right) \frac{v_j}{\|v_j\|}. \quad (4.2)$$

Then, for every nonempty set $A \in \mathcal{B}(\mathbf{R}^d)$, the equality

$$\int_0^\infty \mathbf{P}(H(s, (v_1, e_1, u_1)) \in A) ds = M(A)$$

is verified.

P r o o f. By conditioning and by the definition of the measure M , the computation of the integral follows. The complete proof can be found in [2].

First, we consider a simple case.

Theorem 4.1 ($\alpha \in (0, 1)$ and symmetric case). *Suppose that all the above assumptions are fulfilled. If $\alpha \in (0, 1)$, or if $\alpha \in [1, 2)$ and Q is symmetric, then the series*

$$S_0 = \sum_{j=1}^\infty \left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_j^{1/2} u_j^{1/\alpha}}{\|v_j\|} \right) \frac{v_j}{\|v_j\|}. \quad (4.3)$$

converges a.s. Furthermore, $S_0 \sim \text{TID}_\alpha^0(R, 0)$ for $\alpha \in (0, 1)$ and $S_0 \sim \text{TID}_\alpha(R, 0)$ for $\alpha \in [1, 2)$.

P r o o f. To prove this theorem, we are going to use [10, Theorem 4.1] and [11, Theorem 5.1]. If H is defined as in (4.2), we can apply Lemma 4.1. In the case $\alpha \in (0, 1)$ Theorem 4.1(A) in [10] proves the theorem, while if $\alpha \in [1, 2)$ then one can consider Theorem 4.1(B) in [10].

Now we consider the nonsymmetric case.

Theorem 4.2 (nonsymmetric case). *Under the above notation, suppose that $\alpha \in [1, 2)$, Q is nonsymmetric and additionally that*

$$\int_{\mathbf{R}^d} \|x\| \ln \|x\| R(dx) < \infty \quad \text{when } \alpha = 1 \quad (4.4)$$

$$\int_{\mathbf{R}^d} \|x\| R(dx) < \infty, \quad \text{when } \alpha \in (1, 2). \quad (4.5)$$

Then, the series

$$S_1 = \sum_{j=1}^{\infty} \left[\left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_j^{1/2} u_j^{1/\alpha}}{\|v_j\|} \right) \frac{v_j}{\|v_j\|} - \left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + b, \quad (4.6)$$

where

$$\begin{aligned} x_0 &= \mathbb{E} \frac{v_j}{\|v_j\|} = \|\sigma\|^{-1} \int_{S^{d-1}} u \sigma(du), \quad x_1 = \int_{\mathbf{R}^d} x R(dx), \\ b &= \begin{cases} \zeta\left(\frac{1}{\alpha}\right) \alpha^{-1/\alpha} \|\sigma\|^{1/\alpha} x_0 - 2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) x_1, & 1 < \alpha < 2, \\ \left(\frac{3}{2}\gamma - \frac{\ln 2}{2} + \ln \|\sigma\|\right) x_1 - \int_{\mathbf{R}^d} x \ln \|x\| R(dx), & \alpha = 1, \end{cases} \end{aligned} \quad (4.7)$$

ζ denotes the Riemann zeta function, and γ is the Euler constant, converges almost surely. Furthermore, we have that $S_1 \sim \text{TID}_\alpha(R, 0)$.

P r o o f. To prove this theorem, we are going to use [10, Theorem 4.1] and [11, Theorem 5.1]. If H is defined as in (4.2), we can apply Lemma 4.1. If $\alpha \in [1, 2)$, then by Proposition 2.3 we have

$$\begin{aligned} & \int_0^\infty \mathbb{E} \left(\|H(s, (v_1, e_1, u_1))\| I(\|H(s, (v_1, e_1, u_1))\| > 1) \right) ds \\ &= \int_{\|x\| > 1} \|x\| M(dx) < \infty \end{aligned}$$

and, by [10, Theorem 4.1(B)], we can consider a series

$$S_1 = \sum_{j=1}^{\infty} \left[\left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_j^{1/2} u_j^{1/\alpha}}{\|v_j\|} \right) \frac{v_j}{\|v_j\|} - c_j \right]$$

which converges a.s. and $S_1 \sim \text{TID}_\alpha(R, 0)$, where

$$c_j = \int_{j-1}^j \mathbb{E} \left[\left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_j^{1/2} u_j^{1/\alpha}}{\|v_j\|} \right) \frac{v_j}{\|v_j\|} \right] ds.$$

We have to prove that the equality

$$\sum_{j=1}^{\infty} \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j \right] = b \quad (4.8)$$

holds, where b is given by (4.7).

First consider the case $\alpha \in (1, 2)$. Define for $j \geq 1$ [11, equation (5.8)]

$$c'_j = \int_{j-1}^j \mathbb{E} \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \frac{v_1}{\|v_1\|} \right] ds = \frac{\alpha^{1-1/\alpha} \|\sigma\|^{1/\alpha}}{\alpha - 1} [j^{1-1/\alpha} - (j-1)^{1-1/\alpha}] x_0. \quad (4.9)$$

We have

$$\|c'_j - c_j\| \leq \int_{j-1}^j \mathbb{E} \left\{ \left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_1^{1/2} u_1^{1/\alpha}}{\|v_1\|} \right] \right\} ds.$$

Furthermore by [11, equation (5.9)], for every $\theta > 0$ the equality

$$\int_0^\infty \left\{ \left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \theta \right] \right\} ds = \frac{\|\sigma\|}{\alpha(\alpha-1)} \theta^{1-\alpha} \quad (4.10)$$

holds. Using this identity for $\theta = \sqrt{2} e_1^{1/2} u_1^{1/\alpha} \|v_1\|^{-1}$ pointwise, we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} \|c'_j - c_j\| &= \mathbb{E} \int_0^\infty \left\{ \left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_1^{1/2} u_1^{1/\alpha}}{\|v_1\|} \right] \right\} ds \\ &= \frac{\|\sigma\|}{\alpha(\alpha-1)} \mathbb{E} [2^{(1-\alpha)/2} e_1^{(1-\alpha)/2} u_1^{-1+1/\alpha} \|v_1\|^{\alpha-1}] \\ &= 2^{(1-\alpha)/2} \Gamma\left(\frac{3}{2} - \frac{\alpha}{2}\right) \frac{\|\sigma\|}{\alpha-1} \mathbb{E} \|v_1\|^{\alpha-1} \\ &= -2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) \int_{\mathbf{R}^d} \|x\| R(dx) < \infty. \end{aligned} \quad (4.11)$$

By using (4.10) we obtain

$$\begin{aligned} &\sum_{j=1}^{\infty} (c'_j - c_j) \\ &= \mathbb{E} \left\{ \int_0^\infty \left(\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_1^{1/2} u_1^{1/\alpha}}{\|v_1\|} \right] \right) ds \frac{v_1}{\|v_1\|} \right\} \\ &= \mathbb{E} \left[\frac{\|\sigma\|}{\alpha(\alpha-1)} 2^{(1-\alpha)/2} e_1^{(1-\alpha)/2} u_1^{-1+1/\alpha} \|v_1\|^{\alpha-1} \frac{v_1}{\|v_1\|} \right] \end{aligned}$$

$$\begin{aligned}
&= 2^{(1-\alpha)/2} \Gamma\left(\frac{3}{2} - \frac{\alpha}{2}\right) \frac{1}{\alpha-1} \int_{\mathbf{R}^d} x \|x\|^{\alpha-2} Q(dx) \\
&= -2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) \int_{\mathbf{R}^d} x R(dx) \\
&= -2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) x_1.
\end{aligned}$$

Then we have

$$\sum_{j=1}^n \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c'_j \right] = \left(\sum_{j=1}^n j^{-1/\alpha} - \frac{\alpha}{\alpha-1} n^{1-1/\alpha} \right) \alpha^{-1/\alpha} \|\sigma\|^{1/\alpha} x_0.$$

From a classical formula (see [1, 23.2.9])

$$\sum_{j=1}^n j^{-z} - \frac{n^{1-z}}{1-z} = \zeta(z) + z \int_n^\infty \frac{s - [s]}{s^{z+1}} ds, \quad \operatorname{Re} z > 0, \quad \operatorname{Re} z \neq 1, \quad (4.12)$$

we obtain $\sum_{j=1}^\infty [(\alpha j / \|\sigma\|)^{-1/\alpha} x_0 - c'_j] = \zeta(1/\alpha) \alpha^{-1/\alpha} \|\sigma\|^{1/\alpha} x_0$ and we can write

$$\begin{aligned}
\sum_{j=1}^\infty \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j \right] &= \sum_{j=1}^\infty \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c'_j \right] + \sum_{j=1}^\infty (c'_j - c_j) \\
&= \zeta\left(\frac{1}{\alpha}\right) \alpha^{-1/\alpha} \|\sigma\|^{1/\alpha} x_0 - 2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) x_1 = b
\end{aligned}$$

which proves (4.8). Now, we consider the case $\alpha = 1$. By the same computation as above, define for $j \leq 2$

$$c'_j = \int_{j-1}^j \mathbb{E} \left[\left(\frac{s}{\|\sigma\|} \right)^{-1/\alpha} \frac{v_1}{\|v_1\|} \right] ds = (\ln j - \ln(j-1)) \|\sigma\| x_0, \quad (4.13)$$

and put $c'_1 = 0$. For every $\theta > 0$ [11, equation (5.14)], we have

$$\begin{aligned}
&\int_1^\infty \left\{ \left(\frac{s}{\|\sigma\|} \right)^{-1} - \left[\left(\frac{s}{\|\sigma\|} \right)^{-1} \wedge \theta \right] \right\} ds \\
&= \{\theta - \|\sigma\| \ln \theta + \|\sigma\| \ln \|\sigma\| - \|\sigma\|\} I(\theta \leq \|\sigma\|) \leq \|\sigma\| \ln^+ \frac{\|\sigma\|}{\theta}. \quad (4.14)
\end{aligned}$$

By assumption (4.4), we can write

$$\begin{aligned}
\sum_{j=1}^\infty \|c'_j - c_j\| &= \mathbb{E} \int_1^\infty \left\{ \left(\frac{s}{\|\sigma\|} \right)^{-1} - \left[\left(\frac{s}{\|\sigma\|} \right)^{-1} \wedge \frac{\sqrt{2} e_1^{1/2} u_1}{\|v_1\|} \right] \right\} ds \\
&\leq \|\sigma\| \mathbb{E} \ln^+ \left(\frac{\|\sigma\| \|v_1\|}{\sqrt{2} e_1^{1/2} u_1} \right) \\
&\leq \|\sigma\| \left(|\ln \|\sigma\|| + \mathbb{E} |\ln \|v_1\|| + \mathbb{E} |\ln(\sqrt{2} e_1^{1/2} u_1)| \right) \\
&= \|\sigma\| |\ln \|\sigma\|| + \int_{\mathbf{R}^d} |\ln \|x\|| \|x\| R(dx) + K \|\sigma\| < \infty, \quad (4.15)
\end{aligned}$$

where $K = \mathbb{E} |\ln(\sqrt{2}e_1^{1/2}u_1)| < \infty$.

Before computing the series $\sum_{j=1}^{\infty} (c'_j - c_j)$, we recall some useful relations [11]. For every $\theta > 0$

$$\int_0^1 \left(\left(\frac{s}{\|\theta\|} \right)^{-1} \wedge \theta \right) ds = \theta I(\theta \leq \|\sigma\|) + \{\|\sigma\| - \|\sigma\| \ln \|\sigma\| + \|\sigma\| \ln \theta\} I(\theta > \|\sigma\|)$$

and by (4.14) we get

$$\begin{aligned} & - \int_0^1 \left(\left(\frac{s}{\|\theta\|} \right)^{-1} \wedge \theta \right) ds + \int_1^{\infty} \left\{ \left(\frac{s}{\|\theta\|} \right)^{-1} - \left[\left(\frac{s}{\|\theta\|} \right)^{-1} \wedge \theta \right] \right\} ds \\ & = \|\sigma\| (\ln \|\sigma\| - \ln \theta - 1). \end{aligned}$$

By using this formula for $\theta = \sqrt{2}e_1^{1/2}u_1\|v_1\|^{-1}$ we get

$$\begin{aligned} \sum_{j=1}^{\infty} (c'_j - c_j) &= \mathbb{E} \left\{ \left[- \int_0^1 \left(\left(\frac{s}{\|\theta\|} \right)^{-1} \wedge \frac{\sqrt{2}e_1^{1/2}u_1}{\|v_1\|} \right) ds \right. \right. \\ & \quad \left. \left. + \int_1^{\infty} \left(\left(\frac{s}{\|\theta\|} \right)^{-1} - \left[\left(\frac{s}{\|\theta\|} \right)^{-1} \wedge \frac{\sqrt{2}e_1^{1/2}u_1}{\|v_1\|} \right] \right) ds \right] \frac{v_1}{\|v_1\|} \right\} \\ &= \|\sigma\| \mathbb{E} \left\{ \left(\ln \|\sigma\| + \ln \|v_1\| - \ln(\sqrt{2}e_1^{1/2}u_1) - 1 \right) \frac{v_1}{\|v_1\|} \right\}. \end{aligned}$$

The following expectation can be calculated:

$$\mathbb{E} \ln(\sqrt{2}e_1^{1/2}u_1) = \frac{\ln 2}{2} - 1 - \frac{1}{2} \gamma,$$

where $\gamma = -\int_0^{\infty} \ln(x)e^{-x} dx$ is the Euler constant, see [1, 6.1.3]. By equation (2.7), the series above can be rewritten as

$$\begin{aligned} \sum_{j=1}^{\infty} (c'_j - c_j) &= \|\sigma\| \mathbb{E} \left\{ \frac{v_1}{\|v_1\|} \left(\ln \|\sigma\| + \ln \|v_1\| + \frac{1}{2} \gamma - \frac{\ln 2}{2} \right) \right\} \\ &= \int_{\mathbf{R}^d} \frac{x}{\|x\|} \left(\ln \|\sigma\| + \ln \|x\| + \frac{1}{2} \gamma - \frac{\ln 2}{2} \right) Q(dx) \\ &= \int_{\mathbf{R}^d} x \left(\ln \|\sigma\| - \ln \|x\| + \frac{1}{2} \gamma - \frac{\ln 2}{2} \right) R(dx) \\ &= \left(\frac{1}{2} \gamma - \frac{\ln 2}{2} + \ln \|\sigma\| \right) x_1 - \int_{\mathbf{R}^d} x \ln \|x\| R(dx). \end{aligned}$$

By [11, Theorem 5.1], the equality $\sum_{j=1}^{\infty} [(j/\|\sigma\|)^{-1}x_0 - c'_j] = \gamma x_1$ holds, where γ is the Euler constant, thus we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} \left[\left(\frac{j}{\|\sigma\|} \right)^{-1} x_0 - c_j \right] &= \sum_{j=1}^{\infty} \left[\left(\frac{j}{\|\sigma\|} \right)^{-1} x_0 - c'_j \right] + \sum_{j=1}^{\infty} (c'_j - c_j) \\ &= \left(\frac{3}{2} \gamma - \frac{\ln 2}{2} + \ln \|\sigma\| \right) x_1 - \int_{\mathbf{R}^d} x \ln \|x\| R(dx) = b, \end{aligned}$$

which completes the proof of Theorem 4.2.

A series representation for TID processes can be obtained.

Theorem 4.3. *Under the above notation and assumptions, given a fixed $T > 0$, let $\{\tau_j\}$ be an i.i.d. sequence of uniform random variables in $[0, T]$. Assume $\{\tau_j\}$ independent of the random sequences $\{v_j\}$, $\{u_j\}$, $\{e_j\}$, and $\{\gamma_j\}$.*

(i) *If $\alpha \in (0, 1)$, or if $\alpha \in [1, 2)$ and Q is symmetric, set for every $t \in [0, T]$*

$$X_0(t) = \sum_{j=1}^{\infty} I_{(0,t]}(\tau_j) \left(\left(\frac{\alpha \gamma_j}{T \|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_j^{1/2} u_j^{1/\alpha}}{\|v_j\|} \right) \frac{v_j}{\|v_j\|}, \quad (4.16)$$

then the series converges a.s. uniformly in $t \in [0, T]$ to a Lévy process such that $X_0(t) \sim \text{TID}_\alpha^0(tR, 0)$ if $\alpha \in (0, 1)$ and $X_0(t) \sim \text{TID}_\alpha^0(tR, 0)$ if $\alpha \in [1, 2)$.

(ii) *If $\alpha \in [1, 2)$, Q is nonsymmetric, and additionally*

$$\begin{aligned} \int_{\mathbf{R}^d} \|x\| |\ln \|x\|| R(dx) &< \infty \quad \text{when } \alpha = 1, \\ \int_{\mathbf{R}^d} \|x\| R(dx) &< \infty \quad \text{when } \alpha \in (1, 2), \end{aligned}$$

then the series

$$\begin{aligned} X_1(t) = \sum_{j=1}^{\infty} \left[I_{(0,t]}(\tau_j) \left(\left(\frac{\alpha \gamma_j}{T \|\sigma\|} \right)^{-1/\alpha} \wedge \frac{\sqrt{2} e_j^{1/2} u_j^{1/\alpha}}{\|v_j\|} \right) \frac{v_j}{\|v_j\|} \right. \\ \left. - \frac{t}{T} \left(\frac{\alpha j}{T \|\sigma\|} \right)^{-1/\alpha} x_0 \right] + t b_T, \end{aligned} \quad (4.17)$$

where

$$b_T = \begin{cases} \zeta\left(\frac{1}{\alpha}\right) \alpha^{-1/\alpha} T^{-1} (T \|\sigma\|)^{1/\alpha} x_0 - 2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) x_1, & 1 < \alpha < 2, \\ \left(\frac{3}{2}\gamma - \frac{\ln 2}{2} + \ln(T \|\sigma\|)\right) x_1 - \int_{\mathbf{R}^d} x \ln \|x\| R(dx), & \alpha = 1, \end{cases} \quad (4.18)$$

converges a.s. uniformly in $t \in [0, T]$ to a Lévy process such that $X_1(t) \sim \text{TID}_\alpha(tR, 0)$.

P r o o f. The proof follows the approach of [11], and it can be found in [2].

R e m a r k 4.5. By removing the tempering part $\sqrt{2} e_j^{1/2} u_j^{1/\alpha} \|v_j\|^{-1}$ in the shot noise representation, a well-known result for α -stable processes can be found, see [11, Theorem 5.4] or [12].

5. Examples. A real TID law can be defined by fixing a positive definite radial function q with a measure σ on S^1 or alternatively by defining its spectral measure R . We are going to show in the following three parametric examples of TID laws in one dimension. In the first example, the measure R is the sum of two Dirac measures multiplied for opportune constants. The spectral measure R of the second example has a non-trivial bounded support and the derived TID distribution has exponential moments of any order. In the last example, the MTS distribution is considered, see [9, 7], the spectral measure is defined on an unbounded support and there exist exponential moments of some order.

5.1. Example 1: Simple TID distribution. The Lévy measure of the form

$$M(dx) = \left(C_+ e^{-\lambda_+^2 x^2/2} I_{\{x>0\}} + C_- e^{-\lambda_-^2 |x|^2/2} I_{\{x<0\}} \right) \frac{dx}{|x|^{\alpha+1}}$$

can be written in polar coordinates as

$$M(dr, du) = r^{-\alpha-1} q(r, u) dr \sigma(du),$$

where

$$q(r, \pm 1) = e^{-\lambda_{\pm}^2 r^2/2} \quad \text{and} \quad \sigma(\pm 1) = C_{\pm}.$$

The positive definite radial function q , by Theorem 2.2, has the form

$$q(r, u) = \int_0^\infty e^{-r^2 s^2/2} Q(ds | u), \quad \text{where} \quad Q(ds | \pm 1) = \delta_{\lambda_{\pm}}(s) ds,$$

and we have

$$Q(A) = C_+ \int_A \delta_{\lambda_+}(x) dx + C_- \int_A \delta_{-\lambda_-}(x) dx,$$

and hence the spectral measure R can be defined by

$$R(A) = C_+ \int_A \lambda_+^\alpha \delta_{1/\lambda_+}(x) dx + C_- \int_A \lambda_-^\alpha \delta_{-1/\lambda_-}(x) dx.$$

Definition 5.5. Let C_+ , C_- , λ_+ , λ_- be strictly positive constants, $\alpha \in (0, 2)$, $\alpha \neq 1$, and $\mu \in \mathbf{R}$. An infinitely divisible distribution is called the simple TID distribution with parameter $(\alpha, C_+, C_-, \lambda_+, \lambda_-)$ and mean μ , if its Lévy triplet is given by $(0, \mu, M)$ where

$$M(dx) = \left(C_+ e^{-\lambda_+^2 x^2/2} I_{\{x>0\}} + C_- e^{-\lambda_-^2 |x|^2/2} I_{\{x<0\}} \right) \frac{dx}{|x|^{\alpha+1}}.$$

Proposition 5.1. *The characteristic function of the simple TID distribution with parameter $(\alpha, C_+, C_-, \lambda_+, \lambda_-, \mu)$ becomes*

$$\phi(u) = \exp(iu\mu + G(iu; \alpha, C_+, \lambda_+) + G(-iu; \alpha, C_-, \lambda_-)), \quad (5.1)$$

where

$$\begin{aligned} G(x; \alpha, C, \lambda) = & 2^{-\alpha/2-1} C \lambda^\alpha \left[\Gamma\left(-\frac{\alpha}{2}\right) \left({}_1F_1\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{\sqrt{2}x}{2\lambda}\right)^2\right) - 1\right) \right. \\ & \left. + \frac{\sqrt{2}x}{\lambda} \Gamma\left(\frac{1-\alpha}{2}\right) \left({}_1F_1\left(\frac{1}{2} - \frac{\alpha}{2}, \frac{3}{2}; \left(\frac{\sqrt{2}x}{2\lambda}\right)^2\right) - \right) \right]. \end{aligned} \quad (5.2)$$

P r o o f. It follows by Theorem 2.3.

5.2. Example 2: nontrivial spectral measure. In the first example, the spectral measure R has nonzero mass only at two points. Now we will consider a spectral measure with power decay defined on a bounded support of $\mathbf{R}$. By taking into consideration the construction of the KR tempered stable distribution in [8], we can consider the same spectral measure R . Indeed, we have

$$R(dx) = (C_+ r_+^{-p_+} I_{(0, r_+)}(x) |x|^{p_+-1} + C_- r_-^{-p_-} I_{(-r_-, 0)}(x) |x|^{p_--1}) dx. \quad (5.3)$$

By Theorem 2.3, the characteristic function of this distribution can be written in close form.

Lemma 5.1. *Let $\alpha \in (0, 2)$ and $\alpha \neq 1$. Then, the following equality holds:*

$$\begin{aligned} & \int \left[\Gamma\left(-\frac{\alpha}{2}\right) {}_1F_1\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{i\sqrt{2}ux}{2}\right)^2\right) \right. \\ & \quad \left. + i\sqrt{2}ux \Gamma\left(\frac{1-\alpha}{2}\right) {}_1F_1\left(\frac{1}{2} - \frac{\alpha}{2}, \frac{3}{2}; \left(\frac{i\sqrt{2}ux}{2}\right)^2\right) \right] x^{p-1} dx \\ &= \frac{x^p}{p} \Gamma\left(-\frac{\alpha}{2}\right) {}_2F_2\left(\frac{p}{2}, -\frac{\alpha}{2}; 1 + \frac{p}{2}, \frac{1}{2}; \left(\frac{i\sqrt{2}ux}{2}\right)^2\right) \\ & \quad + i\sqrt{2}u \frac{x^{p+1}}{p+1} \Gamma\left(\frac{1-\alpha}{2}\right) {}_2F_2\left(\frac{1}{2} + \frac{p}{2}, \frac{1}{2} - \frac{\alpha}{2}; \frac{3}{2} + \frac{p}{2}, \frac{3}{2}; \left(\frac{i\sqrt{2}ux}{2}\right)^2\right). \end{aligned} \quad (5.4)$$

P r o o f. The computation of the integral can be found in [2].

Proposition 5.2. *The characteristic function of the TID distribution with parameter $(\alpha, C_+, C_-, \lambda_+, \lambda_-, p_+, p_-)$, mean m and with spectral measure (5.3) is*

$$\phi(u) = \exp(ium + C_+ B(iu; \alpha, r_+, p_+) + C_- B(-iu; \alpha, r_-, p_-)), \quad (5.5)$$

where

$$\begin{aligned}
B(iu; \alpha, r, p) &= 2^{-\alpha/2-1} \frac{1}{p} \Gamma\left(-\frac{\alpha}{2}\right) \\
&\quad \times \left({}_2F_2\left(\frac{p}{2}, -\frac{\alpha}{2}; 1 + \frac{p}{2}, \frac{1}{2}; \left(\frac{i\sqrt{2}ur}{2}\right)^2\right) - 1 \right) \\
&\quad + 2^{-\alpha/2-1} \frac{i\sqrt{2}ur}{p+1} \Gamma\left(\frac{1-\alpha}{2}\right) \\
&\quad \times \left({}_2F_2\left(\frac{1}{2} + \frac{p}{2}, \frac{1}{2} - \frac{\alpha}{2}; \frac{3}{2} + \frac{p}{2}, \frac{3}{2}; \left(\frac{i\sqrt{2}ur}{2}\right)^2\right) - 1 \right).
\end{aligned}$$

P r o o f. Since the support of the measure R is bounded, by Proposition 2.3 the distribution has exponential moments of any order and in particular finite mean. By Theorem 2.3, we can consider the representation (2.23) and by Lemma 5.1 the characteristic exponent can be computed.

5.3. Example 3: MTS distribution. A parametric example of TID distributions has been already considered in the literature, the MTS distribution, see [9], [7]. The Lévy measure of a MTS distribution is defined as

$$\begin{aligned}
M(dx) &= \left(C_+ \frac{\lambda_+^{(\alpha+1)/2} K_{(\alpha+1)/2}(\lambda_+ x)}{x^{(\alpha+1)/2}} I_{\{x>0\}} \right. \\
&\quad \left. + C_- \frac{\lambda_-^{(\alpha+1)/2} K_{(\alpha+1)/2}(\lambda_- |x|)}{x^{(\alpha+1)/2}} I_{\{x<0\}} \right) dx,
\end{aligned}$$

where $\lambda_+, \lambda_-, C_+, C_- > 0$, $\alpha \in (0, 2)$, and $\alpha \neq 1$. The tempering function q and the measure σ are of the form

$$\begin{aligned}
q(r, \pm 1) &= \frac{2^{(1-\alpha)/2}}{\Gamma((\alpha+1)/2)} (\lambda_{\pm} r)^{(\alpha+1)/2} K_{(\alpha+1)/2}(\lambda_{\pm} r), \\
\sigma(\pm 1) &= 2^{(\alpha-1)/2} C_{\pm} \Gamma\left(\frac{\alpha+1}{2}\right).
\end{aligned} \tag{5.6}$$

Lemma 5.2. *Let $z > 0$ and let $K_{\nu}(x)$ be the modified Bessel function of second kind. Then the equality*

$$2z^{\nu/2} K_{\nu}(2\sqrt{z}) = \int_0^{\infty} e^{-zt-1/t} t^{-\nu-1} dt \tag{5.7}$$

is satisfied.

P r o o f. The equality follows by [4, 8.432(7)].

Lemma 5.3. *Let μ be a MTS distribution, then*

$$Q(ds | \pm 1) = \frac{2^{(1-\alpha)/2}}{\Gamma((\alpha+1)/2)} e^{-\lambda_{\pm}^2/2s^2} s^{-\alpha-2} \lambda_{\pm}^{\alpha+1} ds, \tag{5.8}$$

and

$$R(dx) = \left(C_+ \lambda_+^{\alpha+1} e^{-\lambda_+^2 x^2/2} I_{\{x>0\}} + C_- \lambda_-^{\alpha+1} e^{-\lambda_-^2 x^2/2} I_{\{x<0\}} \right) dx. \quad (5.9)$$

P r o o f. The equalities are proved in [2].

Lemma 5.4. *Let $\alpha \in (0, 2)$ and $\alpha \neq 1$. Then*

$$\begin{aligned} & \int_0^\infty \left(\Gamma\left(-\frac{\alpha}{2}\right) M\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{i\sqrt{2}ux}{2}\right)^2\right) \right. \\ & \quad \left. + i\sqrt{2}ux \Gamma\left(\frac{1-\alpha}{2}\right) M\left(\frac{1}{2} - \frac{\alpha}{2}, \frac{3}{2}; \left(\frac{i\sqrt{2}ux}{2}\right)^2\right) \right) e^{-x^2 \lambda^2/2} dx \\ &= \frac{1}{\lambda} \sqrt{\frac{\pi}{2}} \Gamma\left(-\frac{\alpha}{2}\right) \left(1 + \frac{u^2}{\lambda^2}\right)^{\alpha/2} \\ & \quad + \frac{i\sqrt{2}u}{\lambda^2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) {}_2F_1\left(1, \frac{1}{2} - \frac{\alpha}{2}; \frac{3}{2}; -\frac{u^2}{\lambda^2}\right). \end{aligned} \quad (5.10)$$

P r o o f. The computation of the integral can be found in [2].

Proposition 5.3. *The characteristic function of the MTS distribution with parameter $(\alpha, C_+, C_-, \lambda_+, \lambda_-)$, mean m and with spectral measure (5.9) is*

$$\phi(u) = \exp(ium + C_+ H(iu; \alpha, \lambda_+) + C_- H(-iu; \alpha, \lambda_-)), \quad (5.11)$$

where

$$\begin{aligned} H(iu; \alpha, \lambda) &= \frac{\lambda^\alpha \sqrt{\pi}}{2^{\alpha/2+3/2}} \Gamma\left(-\frac{\alpha}{2}\right) \left(\left(1 + \frac{u^2}{\lambda^2}\right)^{\alpha/2} - 1 \right) \\ & \quad + \frac{i\lambda^{\alpha-1}u}{2^{\alpha/2+1/2}} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) \left({}_2F_1\left(1, \frac{1}{2} - \frac{\alpha}{2}; \frac{3}{2}; -\frac{u^2}{\lambda^2}\right) - 1 \right). \end{aligned} \quad (5.12)$$

P r o o f. By definition of the measure R , by Proposition 2.3, the distribution has finite mean. By Theorem 2.3, we can consider the representation (2.23) and by Lemma 5.4 the characteristic exponent can be computed.

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