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CHARACTERIZATION OF SOLUTIONS TO THE LOG-NORMAL MOMENT PROBLEM

Хорошо известно, что интегрируемая функция на $(0, \infty)$, удовлетворяющая функциональному уравнению $f(qx) = q^{-1/2}xf(x)$, $0 < q < 1$, и такая, что $\int f = 1$, является решением логнормальной проблемы моментов. В настоящей работе вводятся и характеризуются некоторые семейства решений логнормальной проблемы моментов, не удовлетворяющие упомянутому функциональному уравнению.

Ключевые слова и фразы: неопределенная проблема моментов, логнормальное распределение, гауссовское ядро.

1. Introduction. For a real-valued, measurable function f defined on $(0, \infty)$, its n -th moment is defined as $s_n(f) = \int_0^\infty x^n f(x) dx$, $n \in \mathbf{Z}$. If f is a real-valued, measurable function defined on $(0, \infty)$ with moment sequence $(q^{-n^2/2})_{n \in \mathbf{Z}}$, $0 < q < 1$, then we say that f is a solution to the log-normal moment problem.

If f is a complex-valued, measurable function defined on $(0, \infty)$ such that

$$\int_0^\infty x^n f(x) dx = q^{-n^2/2} \quad \forall n \in \mathbf{Z},$$

then $\operatorname{Re} f$ is a solution to the log-normal moment problem.

Throughout this paper we assume that $q \in (0, 1)$ has been fixed, so we can write $q = e^{-\sigma^2}$ for some $\sigma > 0$ fixed, and we will use that $\log_q x = \ln x / \ln q$.

In [7] Stieltjes showed that the log-normal distribution with density on $(0, \infty)$ given by

$$d_\sigma(x) = (2\pi\sigma^2)^{-1/2} x^{-1} \exp(-2^{-1}\sigma^{-2}(\ln x)^2),$$

together with the densities $(a \in [-1, 1])$

$$d_\sigma(x)(1 + a \sin(2\pi\sigma^{-2} \ln x)) \geq 0,$$

have all the moment sequence $(q^{-n^2/2})_{n \in \mathbf{Z}}$. Because the log-normal moment problem has no unique solution it is called M-indeterminate.

We recall that the so-called theta function is given by

$$\theta(x, t) = (4\pi t)^{-1/2} \sum_{n \in \mathbf{Z}} e^{-(x+n)^2/(4t)} \quad \forall (x, t) \in \mathbf{R}_+^2,$$

see [1, p. 59].

When $\beta = 0$ the following result is a consequence of observations made in [2]–[4], and characterizes a class of solutions to the log-normal moment problem.

Theorem 1. Let $\beta \in \mathbf{R}$, $f \in L^1(0, \infty)$. Then f satisfies the functional equation

$$f(qx) = q^{\beta-1/2}xf(x), \quad x > 0, \tag{1}$$

and $\int_0^\infty f(x) dx = q^{-\beta^2/2}$ if and only if there exists a 1-periodic function $\varphi \in L^1(0, 1)$ such that $f(x) = x^\beta d_\sigma(x) \{1 + \varphi(\beta + \log_q x)\}$ and

$$\int_0^1 \varphi(x) \theta(x, 2^{-1}\sigma^{-2}) dx = 0. \tag{2}$$

In any case, $s_n(f) = q^{-(n+\beta)^2/2}$ for all $n \in \mathbf{Z}$.

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When $\beta = 0$ and $\varphi(x) = -a \sin(2\pi x)$ we have the Stieltjes example. Example 2 given in Section 3 shows that φ satisfies (2).

On the other hand, by setting $\beta = 0$ and

$$\varphi_c(x) = -1 + q^{c^2/2} \sigma^2 M_c^{-1} \theta(-x + c, 2^{-1} \sigma^{-2})^{-1},$$

$c \in [0, 1)$, we obtain the well-known solution (see [5, equality (1.9)])

$$f_c(x) = \frac{x^{c-1}}{M_c(q; q)_\infty (-q^{-c+1/2} x; q)_\infty (-q^{c+1/2} x^{-1}; q)_\infty}$$

to the log-normal moment problem, where M_c is the constant that makes $\int_0^\infty f_c(x) dx = 1$, and $(p; q)_\infty = \prod_{n=0}^\infty (1 - pq^n)$. To get more examples see [5].

R e m a r k 1. If φ satisfies (2), then so does $a\varphi$, $a \in \mathbf{C}$. Moreover, if φ is real-valued and bounded below (above), then there is $a \in \mathbf{R}$ such that $1 + a\varphi \geq 0$. Hence in this case a probability density function f can be obtained by a standard normalizing procedure.

Next, we introduce the fundamental solution of the heat equation on $\mathbf{R}_+^2$, the so-called Gauss kernel $K(x, t) = (4\pi t)^{-1/2} e^{-x^2/(4t)} > 0$.

When $\beta = 0$ the following results provide another class of solutions to the log-normal moment problem but they do not satisfy the functional equation (1). We denote by Φ the Fourier transform operator.

Theorem 2. Let $\beta \in \mathbf{R}$, and $F \in L^2(\mathbf{R})$. The function $g(x) = x^\beta d_\sigma(x) \{1 + F(\beta + \log_q x)\}$ satisfies $s_n(g) = q^{-(n+\beta)^2/2}$ for all $n \in \mathbf{Z}$ if and only if

$$\sum_{n \in \mathbf{Z}} K(\xi + 2n\pi, 2^{-1} \sigma^2) (\Phi F)(\xi + 2n\pi) = 0 \quad \text{for almost all } \xi \in \mathbf{R}. \quad (3)$$

If F is not the zero function, then g does not satisfy the functional equation (1).

If $\gamma \in L^2(\mathbf{R})$ has compact support and satisfies

$$\sum_{n \in \mathbf{Z}} \gamma(\xi + 2n\pi) = 0 \quad \text{for almost all } \xi \in \mathbf{R}, \quad (4)$$

then the function $F \in L^2(\mathbf{R})$ given by $\Phi F = \gamma/K(\cdot, 2^{-1} \sigma^2)$ satisfies (3). For instance, consider $\phi \in C^\infty(\mathbf{R})$ with $\text{supp } \phi \subset [0, 2\pi]$, then $\gamma(\xi) = \phi(\xi) - \phi(\xi - 2\pi)$ fulfills (4), and the corresponding function F is a rapidly decreasing function.

R e m a r k 2. If F satisfies (3), then so does aF , $a \in \mathbf{C}$. Moreover, if F is real-valued and bounded below (above), then there is $a \in \mathbf{R}$ such that $1 + aF \geq 0$. Hence a probability density function can be obtained.

Theorem 3. Let $\beta \in \mathbf{R}$, $\alpha \in \mathbf{R} \setminus 2\pi\mathbf{Z}$, $\lambda \in \mathbf{C}$, and let f be a 1-periodic function in $L^1(0, 1)$. The function $g(x) = x^\beta d_\sigma(x) \{1 + F(\beta + \log_q x)\}$, with $F(x) = (\lambda + f(x))e^{i\alpha x}$, satisfies $s_n(g) = q^{-(n+\beta)^2/2}$ for all $n \in \mathbf{Z}$ if and only if

$$\sqrt{2\pi\sigma^2} K(\alpha, 2^{-1} \sigma^2) \lambda = - \int_{\mathbf{R}} K(x, 2^{-1} \sigma^{-2}) f(x) e^{i\alpha x} dx.$$

If F is not the zero function, then g does not satisfy the functional equation (1).

R e m a r k 3. When $\text{Re } F$ is bounded below (above), with λ as before, there exists $a \in \mathbf{R}$ such that $1 + a \text{Re } F \geq 0$, and a probability density function can be obtained from the corresponding function $\text{Re } g$.

By setting $\beta = 0$, $f(x) = \cos(2\pi x)$ in the previous theorem we obtain the following result.

Corollary 1. *The function $d_\sigma(x)\{1+(\lambda+\cos(2\pi\log_q x))\cos(\alpha\log_q x)\}$ is a solution to the log-normal moment problem that does not satisfy the functional equation (1), provided that $\alpha \in \mathbf{R} \setminus 2\pi\mathbf{Z}$ and*

$$\lambda = -e^{\sigma^{-2}\alpha^2/2} \int_{\mathbf{R}} K(x, 2^{-1}\sigma^{-2}) \cos(2\pi x) \cos(\alpha x) dx.$$

Theorem 4. *Let $\beta \in \mathbf{R}$. There exists no function $h \in L^2(\mathbf{R})$ with $g(x) = x^\beta d_\sigma(x)h(\beta + \log_q x)$ satisfying $s_n(g) = q^{-(n+\beta)^2/2}$ for all $n \in \mathbf{Z}$.*

This note is organized as follows. Preliminaries are given in the following section, and the last section contains the proofs of the theorems.

2. Notation and preliminaries. We follow the notation in [6, Chap. 9]. For $f \in L^1(\mathbf{R})$, we define its Fourier transform as

$$(\Phi f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \frac{dx}{\sqrt{2\pi}}.$$

The Fourier transform operator Φ is a bijective map from $L^2(\mathbf{R})$ onto $L^2(\mathbf{R})$.

If $f, g \in L^2(\mathbf{R})$, then the Parseval formula holds:

$$\int_{-\infty}^{\infty} f(x) \overline{g(x)} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} (\Phi f)(\xi) \overline{(\Phi g)(\xi)} \frac{d\xi}{\sqrt{2\pi}}.$$

The translation operators τ_x are defined by

$$(\tau_x f)(y) = f(y - x),$$

and for $f \in L^1(\mathbf{R})$ we have $\Phi(\tau_x f)(\xi) = e^{ix\xi}(\Phi f)(\xi)$.

For $(x, t) \in \mathbf{R}_+^2$, let (see [1, p. 33, 59])

$$K(x, t) = (2\pi)^{-1/2} \Phi^{-1}(e^{-t\xi^2})(x) = (4\pi t)^{-1/2} e^{-x^2/(4t)}, \quad (5)$$

$$\theta(x, t) = \sum_{n \in \mathbf{Z}} K(x + n, t) = \sum_{n \in \mathbf{Z}} e^{-4\pi^2 n^2 t + 2\pi n i x}. \quad (6)$$

The positive functions K, θ satisfy the heat equation on $\mathbf{R}_+^2$. Clearly $\theta(\cdot, t)$ is a 1-periodic continuous function for all $t > 0$. Moreover,

$$\int_0^1 \theta(x, t) dx = \int_{\mathbf{R}} K(x, t) dx = 1 \quad \forall t > 0. \quad (7)$$

When $\beta = 1/2$ the following observation can be found in [3], [4].

Proposition 1. *Let $0 < q < 1$, $\beta \in \mathbf{R}$ fixed. If $f \in L^1(0, \infty)$ satisfies the functional equation (1), then $s_n(f) = q^{-n^2/2-\beta n} \int_0^\infty f(x) dx$, $n \in \mathbf{Z}$.*

P r o o f. By induction we can see that f satisfies the functional equation

$$f(q^n x) = q^{n^2/2+(\beta-1)n} x^n f(x) \quad \forall n \in \mathbf{Z},$$

thus

$$\int_0^\infty x^n f(x) dx = q^{-n^2/2-\beta n} \int_0^\infty q^n f(q^n x) dx = q^{-n^2/2-\beta n} \int_0^\infty f(x) dx.$$

E x a m p l e 1. For $\beta \in \mathbf{R}$, the positive function $x^\beta d_\sigma(x)$ is integrable on $(0, \infty)$ and satisfies the functional equation (1).

Proposition 2. *Let $\beta \in \mathbf{R}$, $h \in L^1(\mathbf{R}, q^{(x-n)^2/2} dx)$ for all $n \in \mathbf{Z}$. Then the function $g(x) = x^\beta d_\sigma(x)h(\beta + \log_q x)$ satisfies $s_n(g) = q^{-(n+\beta)^2/2}$ for all $n \in \mathbf{Z}$ if and only if*

$$\int_{\mathbf{R}} K(x, 2^{-1}\sigma^{-2})h(x+n) dx = 1 \quad \forall n \in \mathbf{Z}. \quad (8)$$

P r o o f. Since $q = e^{-\sigma^2}$, we can write $d_\sigma(x) = (2\pi\sigma^2)^{-1/2}x^{-1}q^{(\log_q x)^2/2}$. Notice that $x^\lambda d_\sigma(x) = q^{\lambda-\lambda^2/2}d_\sigma(xq^\lambda)$ for all $x > 0$, $\lambda \in \mathbf{R}$, so by making the change of variable $y = n + \beta + \log_q x$, we obtain

$$\begin{aligned} s_n(g) &= q^{-(\beta+n)^2/2} \int_{-\infty}^{\infty} (2\pi\sigma^{-2})^{-1/2} q^{y^2/2} h(y-n) dy \\ &= q^{-(\beta+n)^2/2} \int_{-\infty}^{\infty} K(y, 2^{-1}\sigma^{-2}) h(y-n) dy, \end{aligned} \quad (9)$$

and the result follows.

R e m a r k 4. In q -calculus a function f defined on $(0, \infty)$ is q -periodic if $f(qx) = f(x)$ for all $x > 0$. If f is a q -periodic function defined on $(0, \infty)$, then $\psi(x) = f(q^x)$ is a 1-periodic function (in the usual sense) on $\mathbf{R}$. So, we have that f is a q -periodic function if and only if there exists a 1-periodic function ψ such that $f(x) = \psi(\log_q x)$.

R e m a r k 5. If $f, g: (0, \infty) \rightarrow \mathbf{R}$ satisfy the functional equation (1) and g is a positive function, then $(f/g)(x) = (f/g)(qx)$, therefore f/g is a q -periodic function. By the previous remark there exists a 1-periodic function ψ such that $(f/g)(x) = \psi(\log_q x)$. In particular, we can write $f(x) = x^\beta d_\sigma(x) \psi(\log_q x)$, where ψ is a 1-periodic function.

3. Proofs of Theorems 1–4.

P r o o f o f T h e o r e m 1. Remark 5 implies that there exists a 1-periodic function ψ such that $f(x) = x^\beta d_\sigma(x) \psi(\log_q x)$, $x > 0$. By (9), with $h = \psi(\cdot - \beta)$, we obtain

$$\begin{aligned} q^{-\beta^2/2} &= s_0(f) = q^{-\beta^2/2} \int_{\mathbf{R}} K(x, 2^{-1}\sigma^{-2}) \psi(x - \beta) dx \\ &= q^{-\beta^2/2} \int_0^1 \psi(x - \beta) \theta(x, 2^{-1}\sigma^{-2}) dx. \end{aligned}$$

By (7) we have that $\varphi(x) = -1 + \psi(x - \beta)$ is a 1-periodic function satisfying (2).

On the other hand, if φ is a 1-periodic function, then $f(x) = x^\beta d_\sigma(x) \{1 + \varphi(\beta + \log_q x)\}$ satisfies (1). The previous computations, (2), and (7) show that $\int_0^\infty f(x) dx = q^{-\beta^2/2}$. Theorem 1 is proved.

E x a m p l e 2. If $\varphi(x) = \sum_{n \in \mathbf{Z}} c_n e^{2\pi n i x} \in L^2(\mathbf{T})$, then φ satisfies (2) if and only if

$$\sum_{n \in \mathbf{Z}} c_n e^{-2\sigma^{-2}\pi^2 n^2} = 0$$

(see (6)). When $c_{-1} = -ai/2$, $c_0 = 0$, $c_1 = ai/2$, $c_n = 0$ for all $|n| > 1$, $a \in [-1, 1]$, we have the Stieltjes example with $\varphi(x) = -a \sin(2\pi x)$.

P r o o f o f T h e o r e m 2. Since $q^{(x-n)^2/2}$ is a rapidly decreasing function, we have $q^{(x-n)^2/2} F \in L^1(\mathbf{R})$ for all $n \in \mathbf{Z}$. The Parseval formula and (5) imply that

$$\begin{aligned} a_n &:= \int_{\mathbf{R}} K(x, 2^{-1}\sigma^{-2}) F(x+n) dx \\ &= \int_{\mathbf{R}} \Phi^{-1}(e^{-2^{-1}\sigma^{-2}\xi^2})(x) (\tau_{-n} F)(x) \frac{dx}{\sqrt{2\pi}} \\ &= \int_{\mathbf{R}} e^{-2^{-1}\sigma^{-2}\xi^2} (\Phi F)(\xi) e^{-in\xi} \frac{d\xi}{\sqrt{2\pi}} \\ &= \int_0^{2\pi} \left\{ \sum_{m \in \mathbf{Z}} e^{-2^{-1}\sigma^{-2}(\xi+2\pi m)^2} (\Phi F)(\xi+2\pi m) \right\} e^{-in\xi} \frac{d\xi}{\sqrt{2\pi}}. \end{aligned}$$

Therefore a_n is a multiple of the n -th Fourier coefficient of the function $\sum_{m \in \mathbf{Z}} K(\xi + 2\pi m, 2^{-1}\sigma^2) (\Phi F)(\xi + 2\pi m) \in L^1(\mathbf{T})$. By using Proposition 2, with $h = 1 + F$, and (7) we get that $a_n = 0$ for all $n \in \mathbf{Z}$ and the result follows.

If g satisfies the functional equation (1), Remark 5 implies that F is a 1-periodic function in $L^2(\mathbf{R})$, therefore $F = 0$ a.e. in $\mathbf{R}$. Theorem 2 is proved.

P r o o f o f T h e o r e m 3. Clearly $q^{(x-n)^2/2}F \in L^1(\mathbf{R})$ for all $n \in \mathbf{Z}$, since

$$\int_{\mathbf{R}} K(x-n, 2^{-1}\sigma^{-2})|f(x)| dx = \int_0^1 \theta(x, 2^{-1}\sigma^{-2})|f(x)| dx < \infty \quad \forall n \in \mathbf{Z}.$$

Proposition 2 and (7) imply that

$$\lambda \int_{\mathbf{R}} K(x, 2^{-1}\sigma^{-2})e^{i\alpha x} dx = \int_{\mathbf{R}} K(x, 2^{-1}\sigma^{-2})f(x)e^{i\alpha x} dx.$$

By (5), noting that $\Phi^{-1}(e^{-t\xi^2})(x) = \Phi(e^{-t\xi^2})(x)$ and using the inversion formula we get that

$$\begin{aligned} \int_{\mathbf{R}} K(x, 2^{-1}\sigma^{-2})e^{i\alpha x} dx &= \int_{\mathbf{R}} \Phi(e^{-2^{-1}\sigma^{-2}\xi^2})(x)e^{i\alpha x} \frac{dx}{\sqrt{2\pi}} \\ &= \Phi^{-1}\Phi(e^{-2^{-1}\sigma^{-2}\xi^2})|_{\xi=\alpha} = e^{-2^{-1}\sigma^{-2}\alpha^2}. \end{aligned}$$

If F is not the zero function, then it is not a 1-periodic function, and Remark 5 implies that g does not satisfy the functional equation (1). Theorem 3 is proved.

P r o o f o f T h e o r e m 4. Let $h \in L^2(\mathbf{R})$, then $q^{(x-n)^2/2}h \in L^1(\mathbf{R})$ for all $n \in \mathbf{Z}$. By the Parseval formula we have that $a_n = \int_{\mathbf{R}} K(x, 2^{-1}\sigma^{-2})h(x+n) dx$ is a multiple of the n -th Fourier coefficient of the function $\sum_{m \in \mathbf{Z}} K(\xi + 2\pi m, 2^{-1}\sigma^2)(\Phi h)(\xi + 2\pi m) \in L^1(\mathbf{T})$. Proposition 2 would imply that $a_n = 1$ for all $n \in \mathbf{Z}$, but this contradicts the Riemann–Lebesgue lemma. Theorem 4 is proved.

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