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LIMIT THEOREMS FOR SEQUENCES OF BLOCKWISE NEGATIVELY ASSOCIATED RANDOM VARIABLES¹⁾

В статье вводится понятие блочно-отрицательных ассоциированных случайных величин и доказываются усиленный закон больших чисел Марцинкевича и классический усиленный закон больших чисел для последовательности блочно-отрицательных ассоциированных случайных величин.

Ключевые слова и фразы: усиленный закон больших чисел, сходимость почти всюду, отрицательно ассоциированные случайные величины.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathbf{P})$.

A finite family $\{X_1, \dots, X_n\}$ is said to be *negatively associated* (NA) if for any disjoint subset $A, B \subset \{1, \dots, n\}$ and any real coordinatewise nondecreasing functions $f: R^A \rightarrow R$ and $g: R^B \rightarrow R$,

$$\text{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0,$$

assuming of course that the covariance exists. The infinite family $\{X_n, n \geq 1\}$ is said to be NA if every finite subfamily is NA. This concept was introduced by Joag-Dev and Proschan [2].

Let $S_n = \sum_{i=1}^n X_i$. The problem of almost everywhere convergence of $(S_n - \mathbf{E} S_n)/n$ for stationary sequences of NA random variables was studied by Newman [7].

Matula [5] proved a Kolmogorov type inequality for NA random variables and some results on the strong law of large numbers for NA random variables.

Liu et al. [3] extended Hajek–Renyi inequality to NA random variables and also used this inequality to prove Marcinkiewicz strong law of large numbers and integrability of supremum for NA random variables which had not been previously established.

In this paper we will make use to blockwise structure that was used in some of class random variables (for example see [1] for independent random variables and [6] for m -dependent random variables) and introduce a wider class of NA random variables, blockwise negatively associated (blockwise NA) random variables, and then, we will prove the Marcinkiewicz strong law of large numbers and the classical strong law of large numbers for that.

D e f i n i t i o n. Let $\{n_i, i \geq 1\}$ be an arbitrary strictly increasing sequence of natural numbers. We say that a sequence of random variables $\{X_n, n \geq 1\}$ is blockwise NA with respect to $\{n_i, i \geq 1\}$ if the sequence $\{X_n, n_{i-1} \leq n < n_i\}$ be NA for each i .

The family of blockwise NA sequences contains negatively associated (in particular, independent) sequences and some more sequences of random variables which are not much deviated from being negatively associated.

The following example is a blockwise NA, but is not NA.

E x a m p l e. Let $\{n_i, i \geq 1\}$ be an arbitrary strictly increasing sequence of natural numbers and let $\{Y_n, n \geq 1\}$ be i.i.d. $N(0, 1)$. We define $X_{n_{i-1}} = (n_i - n_{i-1} + 1)^{-1/2} (Y_{n_{i-1}} - \sum_{k=n_{i-1}+1}^{n_i-1} Y_k + Y_{n_i})$ for all i , and $X_n = Y_n$ otherwise. We know (see [2]) that a necessary and sufficient condition for Gaussian sequence to be NA is $(\text{Cov}(X_i, X_j) \leq 0, i \neq j)$.

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Therefore, in this example, the sequence of random variables $\{X_n, n \geq 1\}$ is blockwise NA with respect to $\{n_i, i \geq 1\}$, since $\text{Cov}(X_k, X_l) \leq 0$, $n_{i-1} \leq k < l < n_i$, for all i , and therefore, sequence $\{X_n, n_{i-1} \leq n < n_i\}$ is NA for each i .

Note that the sequence of random variables $\{X_n, n \geq 1\}$ is not NA since $\text{Cov}(X_{n_{i-1}}, X_{n_i}) > 0$, for all i .

2. Primary results. From the definition of blockwise NA, we easily obtain the following lemma.

Lemma 1. *Let $\{X_n, n \geq 1\}$ be a sequences of blockwise NA random variables with respect to $\{n_i, i \geq 1\}$ and let $\{f_n, n \geq 1\}$ be a sequence of increasing (or decreasing) continuous functions. Then $\{f_n(X_n), n \geq 1\}$ is also a sequence of blockwise NA random variables with respect to $\{n_i, i \geq 1\}$.*

Denote

$$\begin{aligned} I_k &= \{i: [2^k, 2^{k+1}) \cap [n_i, n_{i+1}) \neq \emptyset\}, \quad \nu_k = \# \{I_k\}, \\ [l_{ki}, r_{ki}) &= [2^k, 2^{k+1}) \cap [n_i, n_{i+1}) \quad \text{if } i \in I_k, \\ \Phi(n) &= \max_{0 \leq j \leq k} \nu_j^{1/2} \quad \text{if } n \in [2^k, 2^{k+1}). \end{aligned}$$

Clearly, $\Phi(n) = O(n^{1/2})$ for any $\{n_i, i \geq 1\}$, and $\Phi(n) = O(1)$ if $\liminf_i n_{i+1}/n_i > 1$.

Lemma 2. *Let $\{X_n, n \geq 1\}$ be a sequence of blockwise NA with respect to $\{n_i, i \geq 1\}$ with finite second moments and zero mean. Then, for every $\varepsilon > 0$,*

$$\mathbf{P} \left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=2^k}^n X_i \right| > \varepsilon \right\} \leq 8\nu_k \varepsilon^{-2} \sum_{i=2^k}^{2^{k+1}-1} \mathbf{E} X_i^2.$$

P r o o f. We have

$$\max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=2^k}^n X_i \right| \leq \max_{2^k \leq n < 2^{k+1}} \left(0, \sum_{i=2^k}^n X_i \right) + \max_{2^k \leq n < 2^{k+1}} \left(0, - \sum_{i=2^k}^n X_i \right),$$

thus,

$$\begin{aligned} & \mathbf{P} \left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=2^k}^n X_i \right| > \varepsilon \right\} \\ & \leq \mathbf{P} \left\{ \max_{2^k \leq n < 2^{k+1}} \left(0, \sum_{i=2^k}^n X_i \right) > \frac{1}{2} \varepsilon \right\} + \mathbf{P} \left\{ \max_{2^k \leq n < 2^{k+1}} \left(0, - \sum_{i=2^k}^n X_i \right) > \frac{1}{2} \varepsilon \right\} \\ & \leq 4\varepsilon^{-2} \mathbf{E} \left\{ \max_{2^k \leq n < 2^{k+1}} \left(0, \sum_{i=2^k}^n X_i \right) \right\}^2 + 4\varepsilon^{-2} \mathbf{E} \left\{ \max_{2^k \leq n < 2^{k+1}} \left(0, - \sum_{i=2^k}^n X_i \right) \right\}^2 \\ & \leq 4\varepsilon^{-2} \mathbf{E} \left\{ \sum_{j \in I_k} \max_{l_{kj} \leq n < r_{kj}} \left(0, \sum_{i=l_{kj}}^n X_i \right) \right\}^2 + 4\varepsilon^{-2} \mathbf{E} \left\{ \sum_{j \in I_k} \max_{l_{kj} \leq n < r_{kj}} \left(0, - \sum_{i=l_{kj}}^n X_i \right) \right\}^2 \\ & \leq \frac{4\nu_k}{\varepsilon^2} \sum_{j \in I_k} \mathbf{E} \left\{ \max_{l_{kj} \leq n < r_{kj}} \left(0, \sum_{i=l_{kj}}^n X_i \right) \right\}^2 + \frac{4\nu_k}{\varepsilon^2} \sum_{j \in I_k} \mathbf{E} \left\{ \max_{l_{kj} \leq n < r_{kj}} \left(0, - \sum_{i=l_{kj}}^n X_i \right) \right\}^2. \end{aligned}$$

Since $\{X_i, l_{kj} \leq i < r_{kj}\}$, for all $j \in I_k$, are NA random variables, therefore, similar to method of [5, Lemma 4], we have

$$\mathbf{P} \left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=2^k}^n X_i \right| > \varepsilon \right\} \leq 8\nu_k \varepsilon^{-2} \sum_{j \in I_k} \sum_{i=l_{kj}}^{r_{kj}-1} \mathbf{E} X_i^2 = 8\nu_k \varepsilon^{-2} \sum_{i=2^k}^{2^{k+1}-1} \mathbf{E} X_i^2.$$

3. Marcinkiewicz strong law of large numbers. In this section, we only consider such a sequence of random variables $\{Z, X_n, n \geq 1\}$ that for all $t > 0$ and all n ,

$$\frac{1}{n} \sum_{k=1}^n \mathbf{P}\{|X_k| > t\} \leq C \mathbf{P}\{|Z| > t\},$$

where and throughout $C > 0$ is an unimportant constant which may change from line to line. One can easily verify through by parts that for such a sequence it holds that for any $\alpha > 0$ and any $M > 0$,

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E} |\bar{X}_k|^\alpha \leq C \mathbf{E} |\bar{Z}|^\alpha,$$

where

$$\begin{aligned} \bar{X}_k &= X_k I(|X_k| \leq M) + MI(X_k > M) - MI(X_k < -M), \quad k \geq 1, \\ \bar{Z} &= Z I(|Z| \leq M) + MI(Z > M) - MI(Z < -M). \end{aligned}$$

Noting that if $\{X_k, k \geq 1\}$ be a sequences of blockwise NA random variables with respect to $\{n_i, i \geq 1\}$, then, from Lemma 1, $\{\bar{X}_k, k \geq 1\}$ is also a sequences of blockwise NA random variables with respect to $\{n_i, i \geq 1\}$.

Theorem 1. *If $\{X_n, n \geq 1\}$ is blockwise NA with respect to $\{n_i, i \geq 1\}$ and $\mathbf{E} |Z|^r < \infty$, $1 \leq r < 2$, then*

$$\frac{1}{n^{1/r} \Phi(n)} \sum_{k=1}^n (X_k - \mathbf{E} X_k) \rightarrow 0 \quad \text{a.e.} \quad (3.1)$$

P r o o f. Set

$$\begin{aligned} \bar{X}_n &= X_n I(|X_n| \leq n^{1/r}) + n^{1/r} I(X_n > n^{1/r}) - n^{1/r} I(X_n < n^{1/r}), \quad n \geq 1, \\ \bar{Z}_n &= Z I(|Z| \leq n^{1/r}) + n^{1/r} I(Z > n^{1/r}) - n^{1/r} I(Z < n^{1/r}), \quad n \geq 1. \end{aligned}$$

First, we prove that as $k \rightarrow \infty$,

$$\frac{1}{2^{k/r} \Phi(2^k)} \max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=2^k}^n (\bar{X}_i - \mathbf{E} \bar{X}_i) \right| \rightarrow 0 \quad \text{a.e.} \quad (3.2)$$

From Lemma 1, we know that $\{\bar{X}_n, n \geq 1\}$ is a sequence of blockwise NA random variables with respect to $\{n_i, i \geq 1\}$, therefore, from Lemma 2, we can obtain

$$\begin{aligned} & \sum_{k=1}^{\infty} \mathbf{P} \left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=2^k}^n (\bar{X}_i - \mathbf{E} \bar{X}_i) \right| > 2^{k/r} \Phi(2^k) \varepsilon \right\} \\ & \leq C \sum_{k=1}^{\infty} \frac{\nu_k}{2^{2k/r} \Phi^2(2^k)} \sum_{i=2^k}^{2^{k+1}-1} \mathbf{E} (\bar{X}_i - \mathbf{E} \bar{X}_i)^2 \leq C \sum_{k=1}^{\infty} \frac{1}{2^{2k/r}} \sum_{i=2^k}^{2^{k+1}-1} \mathbf{E} \bar{X}_i^2 \\ & \leq C \sum_{k=0}^{\infty} \frac{2^k \mathbf{E} \bar{Z}_{2^k}^2}{2^{2k/r}} \leq C \sum_{k=0}^{\infty} 2^k \int_0^1 t \mathbf{P}\{|Z| > 2^{k/r} t\} dt \leq C \int_0^1 t \mathbf{E} |Z/t|^r dt < \infty, \end{aligned}$$

therefore, by the Borel–Cantelli lemma,

$$\mathbf{P} \left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=2^k}^n (\bar{X}_i - \mathbf{E} \bar{X}_i) \right| > 2^{k/r} \Phi(2^k) \varepsilon, \quad \text{i.o.} \right\} = 0,$$

which yields (3.2).

For any n there exists a $k = k(n)$ such that $n \in [2^k, 2^{k+1})$, and we have

$$\begin{aligned} \frac{1}{n^{1/r}\Phi(n)} \left| \sum_{i=1}^n Y_i \right| &\leq \frac{1}{2^{k/r}\Phi(2^k)} \left| \sum_{i=1}^{2^k-1} Y_i \right| + \frac{1}{2^{k/r}\Phi(2^k)} \max_{2^k \leq j < 2^{k+1}} \left| \sum_{i=2^k}^j Y_i \right| \\ &\leq \frac{1}{2^{k/r}} \sum_{j=0}^{k-1} 2^{j/r} \frac{1}{2^{j/r}\Phi(2^j)} \left| \sum_{i=2^j}^{2^{j+1}-1} Y_i \right| + \frac{1}{2^{k/r}\Phi(2^k)} \max_{2^k \leq j < 2^{k+1}} \left| \sum_{i=2^k}^j Y_i \right|, \end{aligned}$$

where $Y_i = \bar{X}_i - \mathbf{E} \bar{X}_i$, $i \geq 1$. It follows that

$$\frac{1}{n^{1/r}\Phi(n)} \sum_{i=1}^n (\bar{X}_i - \mathbf{E} \bar{X}_i) \rightarrow 0 \quad \text{a.e.} \quad (3.3)$$

Next,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{P}\{X_n \neq \bar{X}_n\} &= \sum_{n=1}^{\infty} \mathbf{P}\{|X_n| > n^{1/r}\} \leq \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \mathbf{P}\{|X_n| > 2^{k/r}\} \\ &\leq C \sum_{k=0}^{\infty} 2^k \mathbf{P}\{|Z| > 2^{k/r}\} \leq C \mathbf{E}|Z|^r < \infty, \end{aligned}$$

which implies $\mathbf{P}\{X_n \neq \bar{X}_n, \text{ i.o.}\} = 0$, and hence,

$$\frac{1}{n^{1/r}} \sum_{k=1}^n (X_k - \bar{X}_k) \rightarrow 0 \quad \text{a.e.} \quad (3.4)$$

Finally, we prove that

$$\frac{1}{n^{1/r}} \sum_{k=1}^n (\mathbf{E} X_k - \mathbf{E} \bar{X}_k) \rightarrow 0. \quad (3.5)$$

Clearly,

$$\frac{1}{n^{1/r}} \left| \sum_{k=1}^n (\mathbf{E} X_k - \mathbf{E} \bar{X}_k) \right| \leq \frac{1}{n^{1/r}} \sum_{k=1}^n \mathbf{E} |X_k| I(|X_k| > k^{1/r}).$$

In the case $r = 1$,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \mathbf{E} |X_k| I(|X_k| > k) &\leq \frac{1}{n} \sum_{k=1}^N \mathbf{E} |X_k| I(|X_k| > k) + \frac{1}{n} \sum_{k=1}^n \mathbf{E} |X_k| I(|X_k| > N) \\ &\leq \frac{O(1)}{n} + C \mathbf{E} |Z| I(|Z| > N). \end{aligned}$$

Letting $n \rightarrow \infty$ and then $N \rightarrow \infty$ we get (3.5).

In the case $1 < r < 2$, since

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{E} |X_n| I(|X_n| > n^{1/r}) / n^{1/r} \\ &\leq \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \mathbf{E} |X_n| I(|X_n| > 2^{k/r}) / 2^{k/r} \leq C \sum_{k=0}^{\infty} 2^k \mathbf{E} |Z| I(|Z| > 2^{k/r}) / 2^{k/r} \\ &\leq C \sum_{k=0}^{\infty} 2^k \int_1^{\infty} \mathbf{P}\{|Z| > 2^{k/r} t\} dt \leq C \int_1^{\infty} \mathbf{E} |Z/t|^r dt < \infty, \end{aligned}$$

via the Kroneker lemma, we also get (3.5). Combining (3.3)–(3.5) we obtain (3.1). The theorem is proved.

4. Classical strong law of large numbers.

Theorem 2. Let $\{X_n, n \geq 1\}$ be a sequence of blockwise NA random variables with respect to $\{n_i, i \geq 1\}$ and let $\{b_n, n \geq 1\}$ be a positive strictly increasing sequence, tending to infinity as n tends to infinity such that

$$0 < a = \inf_{i \geq 1} b_{n_i} b_{n_{i+1}}^{-1} \leq \sup_{i \geq 1} b_{n_i} b_{n_{i+1}}^{-1} = c < 1.$$

If

$$\sum_{i=1}^{\infty} \sum_{j=n_{i-1}}^{n_i-1} \frac{\mathbf{D}X_j}{b_{n_i}^2} < \infty, \quad (4.1)$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n (X_k - \mathbf{E} X_k) = 0 \quad \text{a.e.} \quad (4.2)$$

P r o o f. We set $n_0 = 1$, $b_0 = 0$ and $T_k = b_{n_k}^{-1} \sum_{j=n_{k-1}}^{n_k-1} Y_j$, where $Y_j = X_j - \mathbf{E} X_j$. For any positive integer n , there exists a positive integer m such that $n_{m-1} \leq n < n_m$. Note that $m \rightarrow \infty$ as $n \rightarrow \infty$. Without loss of generality, we assume that $n > n_0$ and, therefore, $n_{m-1} \geq 1$ and $b_n \geq b_{n_{m-1}} > 0$. We can show that

$$\frac{1}{b_n} \sum_{j=1}^n Y_j = \frac{b_{n_{m-1}}}{b_n} \sum_{j=1}^{m-1} \frac{b_{n_j}}{b_{n_{m-1}}} T_j + \frac{1}{b_n} \sum_{j=n_{m-1}+1}^n Y_j.$$

Since $b_{n_{m-1}} \geq ab_{n_m}$, we conclude that

$$\left| \frac{1}{b_n} \sum_{j=1}^n Y_j \right| \leq \left| \sum_{j=1}^{m-1} \frac{b_{n_j}}{b_{n_{m-1}}} T_j \right| + \frac{1}{ab_{n_m}} \max_{n_{m-1} \leq l < n_m} \left| \sum_{j=n_{m-1}}^l Y_j \right|. \quad (4.3)$$

In order to prove (4.2) it suffices to demonstrate that each of the two terms on the right-hand side of (4.3) converges to zero almost everywhere as $n \rightarrow \infty$. The first term does so due to the Toeplitz lemma (see [4]) provided that

$$\lim_{j \rightarrow \infty} T_j = 0 \quad \text{a.e.}, \quad \sup_{m \geq 2} \sum_{j=1}^{m-1} \frac{b_{n_j}}{b_{n_{m-1}}} < \infty, \quad \lim_{n \rightarrow \infty} \frac{b_{n_j}}{b_{n_{m-1}}} = 0 \quad \text{for every } j.$$

The third condition is satisfied since by the hypothesis the sequence $\{b_n, n \geq 1\}$ monotonically increases without bounds. The second condition holds since

$$\frac{b_{n_j}}{b_{n_{m-1}}} = \prod_{i=j}^{m-2} \frac{b_{n_i}}{b_{n_{i+1}}} \leq c^{m-j-1},$$

$$\sum_{j=1}^{m-1} \frac{b_{n_j}}{b_{n_{m-1}}} \leq \sum_{j=1}^{m-1} c^{m-j-1} = \frac{1-c^m}{1-c} < \frac{1}{1-c},$$

since by the hypothesis $b_{n_j} \leq cb_{n_{j+1}}$, $c \in (0, 1)$. Thus, the first term on the right-hand side of (4.3) converges to zero almost everywhere as $m \rightarrow \infty$ if the sequence $\{T_k, k \geq 1\}$ also does so.

By the hypothesis, let ε be an arbitrary positive number. We know that the sequence of $\{Y_j, n_{k-1} \leq j \leq n_k - 1\}$ is NA random variables for all k . Therefore, by using of the Markov inequality, we obtain

$$\varepsilon^2 \sum_{k=2}^{\infty} \mathbf{P}\{|T_k| > \varepsilon\} \leq \sum_{k=2}^{\infty} \mathbf{E}|T_k|^2 \leq \sum_{k=2}^{\infty} b_{n_k}^{-2} \mathbf{D}(S_{n_k-1} - S_{n_{k-1}-1}) \leq \sum_{k=1}^{\infty} \sum_{i=n_{k-1}}^{n_k-1} \frac{\mathbf{D}X_i}{b_{n_k}^2} < \infty.$$

The finiteness of the last series on the right-hand side is guaranteed by condition (4.1). In view of the Borel–Cantelli lemma, the sequence $\{T_k, k \geq 1\}$ converges to zero a.e.

Let us turn to the second term on the right-hand side of (4.3). Since $\{Y_j, n_{m-1} \leq j \leq n_m - 1\}$ is NA random variables with zero mean, we apply the Kolmogorov-type inequality, valid for partial sums of NA random variables (see [5]). Hence, from (4.1), we have

$$\varepsilon^2 \sum_{m=2}^{\infty} \mathbf{P} \left\{ \frac{1}{b_{n_m}} \max_{n_{m-1} \leq l < n_m} \left| \sum_{j=n_{m-1}}^l Y_j \right| > \varepsilon \right\} \leq 8 \sum_{m=2}^{\infty} b_{n_m}^{-2} \left\{ \sum_{i=n_{m-1}}^{n_m-1} \mathbf{D}X_i \right\} < \infty.$$

By virtue of the Borel–Cantelli lemma, the sequence

$$\left\{ \frac{1}{b_{n_m}} \max_{n_{m-1} \leq l < n_m} \left| \sum_{j=n_{m-1}}^l Y_j \right| \right\}_{m \geq 1}$$

converges to zero almost everywhere. Thus, the theorem is proved.

By using Lemma 2 and the same way as used in Theorem 2, we can prove the following theorem.

Theorem 3. *Let $\{X_n, n \geq 1\}$ be a sequence of blockwise NA random variables with respect to $\{n_i, i \geq 1\}$ and let $\{b_n, n \geq 1\}$ be a positive strictly increasing sequence, tending to infinity as n tends to infinity such that*

$$0 < a = \inf_{k \geq 1} b_{2^k} b_{2^{k+1}}^{-1} \leq \sup_{k \geq 1} b_{2^k} b_{2^{k+1}}^{-1} = c < 1.$$

If

$$\sum_{i=1}^{\infty} \sum_{j=2^{k-1}}^{2^k-1} \frac{\mathbf{D}X_j}{b_{2^k}^2} < \infty, \quad (4.4)$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{b_n \Phi(n)} \sum_{k=1}^n (X_k - \mathbf{E} X_k) = 0 \quad a.e. \quad (4.5)$$

Corollary 1. *Let $\{X_n, n \geq 1\}$ be a sequence of blockwise NA random variables with respect to $\{n_k, k = 1, 2, \dots\}$ such that $n_{k+1}/n_k > 1$ and*

$$\mathbf{D}X_j = O(1), \quad (4.6)$$

for all $j \geq 1$. Then for all $\delta > 1$,

$$\frac{\sum_{j=1}^n (X_j - \mathbf{E} X_j)}{(n \ln n (\ln \ln n)^{\delta})^{1/2}} \rightarrow 0 \quad a.e. \quad \text{as } n \rightarrow \infty.$$

P r o o f. Under condition (4.6), there exists a constant B such that

$$\sum_{i=2^{k-1}}^{2^k-1} \mathbf{D}X_i \leq B(2^k - 2^{k-1}) \leq 2^k B.$$

The sequence $b_n = (n \ln n (\ln \ln n)^{\delta})^{1/2}$, for all $\delta > 1$, satisfy the hypotheses of Theorem 3, which this proves the Corollary 1.

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СПИСОК ЛИТЕРАТУРЫ

1. Гапошкин В. Ф. Об усиленном законе больших чисел для блочно-независимых и блочно-ортогональных случайных величин. — Теория вероятн. и ее примен., 1994, т. 39, в. 4, с. 804–812.
2. Joag-Dev K., Proschan F. Negative association random variable, with application. — Ann. Statist., 1983, v. 11, № 1, p. 286–295.
3. Liu J., Gan S., Chen P. The Hajek–Rényi inequality for the NA random variables and its application. — Statist. Probab. Lett., 1999, v. 43, № 1, p. 99–105.
4. Лозе М. Теория вероятностей. М.: Мир, 1967, 144 с.
5. Matula P. A note on the almost sure convergence of sums of negatively dependent variables. — Statist. Probab. Lett., 1992, v. 15, № 3, p. 209–213.
6. Móricz F. Strong limit theorems for blockwise m -dependent and blockwise quasiorthogonal sequences of random variables. — Proc. Amer. Math. Soc., 1987, v. 101, p. 709–715.
7. Newman C.M. Asymptotic independence and limit theorems for positively and negatively dependent random variables. — Inequalities in Statistics Probability: Proceedings of the Symposium on Inequalities in Statistics and Probability, Y. L. Tong ed., Institute of Mathematics Statistics, Hayward, CA, 1984, p. 127–140.

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LOCAL AND TRUE MARTINGALES IN DISCRETE TIME¹⁾

Доказывается, что для процесса с дискретным временем и с бесконечным временным горизонтом множество эквивалентных L^p -мартингалов мер плотно в множестве эквивалентных локально-мартингалов мер относительно нормы, определяемой полной вариацией.

Ключевые слова и фразы: эквивалентная мартингаловая мера; локальный мартингал; норма, определяемая полной вариацией.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. All σ -algebras we consider are assumed to be $\mathbf{P}$ -complete sub- σ -algebras of $\mathcal{F}$. Let $(\mathcal{G}_n)_{n \geq 0}$ be a discrete-time filtration and $(S_n)_{n \geq 0}$ an $\mathbf{R}^d$ -valued local martingale with respect to $(\mathcal{G}_n)_{n \geq 0}$.

The following theorem was formulated in [3].

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