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LOCAL AND TRUE MARTINGALES IN DISCRETE TIME¹⁾

Доказывается, что для процесса с дискретным временем и с бесконечным временным горизонтом множество эквивалентных L^p -мартингалов мер плотно в множестве эквивалентных локально-мартингалов мер относительно нормы, определяемой полной вариацией.

Ключевые слова и фразы: эквивалентная мартингаловая мера; локальный мартингал; норма, определяемая полной вариацией.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. All σ -algebras we consider are assumed to be $\mathbf{P}$ -complete sub- σ -algebras of $\mathcal{F}$. Let $(\mathcal{G}_n)_{n \geq 0}$ be a discrete-time filtration and $(S_n)_{n \geq 0}$ an $\mathbf{R}^d$ -valued local martingale with respect to $(\mathcal{G}_n)_{n \geq 0}$.

The following theorem was formulated in [3].

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Theorem 1.1. *There is a probability measure $\mathbf{Q}$ on $\mathcal{F}$ such that $\mathbf{P} \sim \mathbf{Q}$ and S is a $\mathbf{Q}$ -martingale.*

In fact, it is a byproduct of results in [6] on arbitrage theory: if S is a local martingale, then it is known to satisfy the so-called NFLBR property and the main theorem of [6] implies that S admits an equivalent martingale measure. Though the direct proof given in [3] is much shorter, its arguments, as well as those of [6], are still based on the same functional analytic approach (countably normed spaces, infinite dimensional separation, Komlós' theorem, and [1]). In the present paper we give a proof for the stronger result below, using less heavy artillery (namely [1] and [4] which rely on measurable selection techniques).

Theorem 1.2. *For each $\varepsilon > 0$, there is a probability measure $\mathbf{Q}$ on $\mathcal{F}$ such that $\mathbf{P} \sim \mathbf{Q}$, $\|\mathbf{P} - \mathbf{Q}\| \leq \varepsilon$, and S is a $\mathbf{Q}$ -martingale.*

Here $\|\cdot\|$ denotes the total variation norm, i.e.,

$$\|\mathbf{P} - \mathbf{Q}\| = \sup \left\{ \left| \int X d(\mathbf{P} - \mathbf{Q}) \right| : X \text{ random variable, } |X| \leq 1 \right\}.$$

Recall that if λ is a common dominating measure, then the total variation distance can be expressed with the densities:

$$\|\mathbf{P} - \mathbf{Q}\| = \int_{\Omega} \left| \frac{d\mathbf{P}}{d\lambda} - \frac{d\mathbf{Q}}{d\lambda} \right| d\lambda.$$

R e m a r k 1.1. After having read the first version of this paper, Yuri Kabanov has kindly pointed out to us that Theorem 1.2 is a direct consequence of Theorem 1.1 and the main result of [2]. The latter paper uses advanced semimartingale theory, we manage to obtain Theorem 1.2 using only [4] which is the (simpler) discrete-time finite horizon translation of the proof in [2].

Actually, our proof gives much more than Theorem 1.2.

Theorem 1.3. *Let $(Y_n)_{n \geq 0}$ be an adapted $\mathbf{R}$ -valued process. If $(S_n)_{n \geq 0}$ is an $\mathbf{R}^d$ -valued $\mathbf{P}$ -local martingale, then for each $\varepsilon > 0$ there is a probability measure $\mathbf{Q}$ such that $\mathbf{P} \sim \mathbf{Q}$, $\|\mathbf{P} - \mathbf{Q}\| \leq \varepsilon$, $(S_n)_{n \geq 0}$ is a $\mathbf{Q}$ -martingale, and each Y_n is $\mathbf{Q}$ -integrable.*

Theorem 1.3 clearly implies Theorem 1.2. Let $1 \leq p < \infty$. Applying Theorem 1.3 with the choice $Y_n := |S_n|^p$ we get the following theorem.

Theorem 1.4. *For each $1 \leq p < \infty$ and each $\varepsilon > 0$, there is a probability measure $\mathbf{Q}$ on $\mathcal{F}$ such that $\mathbf{P} \sim \mathbf{Q}$, $\|\mathbf{P} - \mathbf{Q}\| \leq \varepsilon$, and S is a $\mathbf{Q}$ -martingale belonging to $L^p(\mathbf{Q})$.*

Now let $(S_k)_{k \geq 0}$ be an arbitrary process on $(\Omega, \mathcal{F}, \mathbf{P})$ adapted to $(\mathcal{G}_k)_{k \geq 0}$. Let $\mathcal{M}^p(\mathcal{M}_{\text{loc}})$ denote the set of $\mathbf{Q} \sim \mathbf{P}$ such that S is a $\mathbf{Q}$ -martingale in $L^p(\mathbf{Q})$ (respectively, $\mathbf{Q}$ -local martingale) with respect to the given filtration. The following result is an obvious consequence of Theorem 1.4.

Corollary 1.1. $\mathcal{M}^p$ is dense in $\mathcal{M}_{\text{loc}}$ with respect to $\|\cdot\|$.

In the usual setting of mathematical finance S represents the price process of an asset. The sets $\mathcal{M}^p$, $\mathcal{M}_{\text{loc}}$ correspond to pricing functionals for derivative securities and play a prominent role in the arbitrage theory of financial markets.

2. Proof. We recall that, for arbitrary nonnegative random variable Y and σ -algebra $\mathcal{G}$, one can define $\mathbf{E}(Y | \mathcal{G})$ (finite or infinite). If both the positive and negative parts X_+ , X_- of a random variable X satisfy $\mathbf{E}(X_+ | \mathcal{G}) < \infty$, $\mathbf{E}(X_- | \mathcal{G}) < \infty$, then $\mathbf{E}(X | \mathcal{G}) := \mathbf{E}(X_+ | \mathcal{G}) - \mathbf{E}(X_- | \mathcal{G})$ can be defined. Multidimensional extension of this notion is straightforward.

D e f i n i t i o n 2.1. An $\mathbf{R}^d$ -valued process $(S_n)_{n \geq 0}$ adapted to $(\mathcal{G}_n)_{n \geq 0}$ is called a generalized martingale if, for all $n \geq 0$, $\mathbf{E}(S_{n+1} | \mathcal{G}_n) = S_n$ in the above generalized sense.

The following result goes back to Meyer, see [7, Chap. VII, Theorem 1].

Proposition 2.1. *On a discrete-time filtration $(\mathcal{G}_k)_{k \geq 0}$ every local martingale is a generalized martingale.*

We note that if $\mathcal{G}_0$ is trivial, then the reverse implication also holds. Our arguments will be based on the following lemma.

Lemma 2.1. *Let $X: \Omega \rightarrow \mathbf{R}^h$, $Y: \Omega \rightarrow \mathbf{R}_+$ be random variables and $\mathcal{G} \subset \mathcal{F}$ a σ -algebra. Assume that $\mathbf{E}(X|\mathcal{G})$ exists and $\mathbf{E}(X|\mathcal{G}) = 0$. Then for any $\varepsilon > 0$ there is a bounded random variable $Z > 0$ such that*

- (i) $\mathbf{E}|Z - 1| < \varepsilon$,
- (ii) $\mathbf{E}(ZX|\mathcal{G})$ exists and $\mathbf{E}(ZX|\mathcal{G}) = 0$,
- (iii) ZY is bounded.

P r o o f. Define $\mathcal{H} := \mathcal{G} \vee \sigma(X) \vee \sigma(Y)$. We recall one direction of Theorem 2.4 in [1].

Theorem 2.1. *Let $\mathcal{G} \subset \mathcal{H}$ be (complete) sub- σ -algebras of $\mathcal{F}$, and let $W: \Omega \rightarrow \mathbf{R}^h$ be an $\mathcal{H}$ -measurable random variable satisfying*

$$\langle W, R \rangle \geq 0 \text{ a.s.} \quad \Rightarrow \quad \langle W, R \rangle = 0 \text{ a.s.} \quad (1)$$

for each $\mathcal{G}$ -measurable h -dimensional random variable R ; here $\langle \cdot, \cdot \rangle$ denotes scalar product. Then there exists an $\mathcal{H}$ -measurable scalar random variable L such that $0 < L \leq 1$ a.s., $\mathbf{E}(|W|L) < \infty$ and $\mathbf{E}(WL|\mathcal{G}) = 0$ a.s.

We claim that (1) holds for $W := X$. Indeed, if $\langle X, R \rangle \geq 0$ a.s., then

$$\mathbf{E}\langle X, R \rangle = \mathbf{E}\mathbf{E}(\langle X, R \rangle|\mathcal{G}) = \mathbf{E}\langle \mathbf{E}(X|\mathcal{G}), R \rangle = 0,$$

showing (1). Define now a new probability $\mathbf{P}_1 \sim \mathbf{P}$ by

$$\frac{d\mathbf{P}_1}{d\mathbf{P}} := \frac{(1+Y)^{-1}}{\mathbf{E}(1+Y)^{-1}}.$$

Note that (1) continues to hold under $\mathbf{P}_1$ hence applying Theorem 2.1 (under $\mathbf{P}_1$) we may set

$$\frac{d\mathbf{P}_2}{d\mathbf{P}_1} := \frac{L}{\mathbf{E}L}$$

and get that X is $\mathbf{P}_2$ -integrable with $\mathbf{E}_2(X|\mathcal{G}) = 0$. Here and in what follows we use $\mathbf{E}_2$ for the expectation with respect to $\mathbf{P}_2$. Now we restate Proposition 2.3 of [4].

Lemma 2.2. *Let σ -algebras $\mathcal{G}$ and $\mathcal{H}$ satisfy $\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}$. Let $W: \Omega \rightarrow \mathbf{R}^d$, $F: \Omega \rightarrow \mathbf{R}_+ \setminus \{0\}$ be $\mathcal{H}$ -measurable random variables with*

$$\mathbf{E}|W| < \infty, \quad \mathbf{E}(W|\mathcal{G}) = 0, \quad \mathbf{E}(F|\mathcal{G}) = 1, \quad \mathbf{E}(WF|\mathcal{G}) = 0.$$

Then there is a sequence of strictly positive $\mathcal{H}$ -measurable bounded random variables F_n such that

$$\mathbf{E}(F_n|\mathcal{G}) = 1, \quad \mathbf{E}(WF_n|\mathcal{G}) = 0 \quad (2)$$

and we also have

$$F_n \rightarrow F \quad \text{a.s.} \quad (3)$$

Use the above result under $\mathbf{P}_2$ with the choice $W := X$ and

$$F := \frac{\zeta}{\mathbf{E}(\zeta|\mathcal{G})}, \quad \text{where } \zeta = \frac{d\mathbf{P}}{d\mathbf{P}_2}.$$

Note that $1/\zeta = \frac{d\mathbf{P}_2}{d\mathbf{P}}$ is bounded and so is Y/ζ by the choice of $\mathbf{P}_1$. Define

$$Z_n := \frac{F_n}{\zeta} \frac{\mathbf{E}_2(\zeta \wedge n|\mathcal{G})}{\mathbf{E}_2(\zeta \wedge n)}.$$

Since F_n , $1/\zeta$, $\zeta \wedge n$, and Y/ζ are all bounded, so are Z_n and $Z_n Y$. By (2),

$$\mathbf{E}_2(XF_n|\mathcal{G}) = 0,$$

consequently,

$$\mathbf{E}(XZ_n | \mathcal{G}) = 0. \quad (4)$$

By construction,

$$\mathbf{E} Z_n = \mathbf{E}_2 \left(\mathbf{E}_2(F_n | \mathcal{G}) \frac{\mathbf{E}_2(\zeta \wedge n | \mathcal{G})}{\mathbf{E}_2(\zeta \wedge n)} \right) = 1,$$

and

$$Z_n \rightarrow \frac{F}{\zeta} \frac{\mathbf{E}_2(\zeta | \mathcal{G})}{\mathbf{E}_2 \zeta} = 1 \quad \text{a.s.}$$

by (3) and by dominated convergence theorem, thus Scheffé's theorem shows that choosing n large enough Z_n will satisfy all the statements (i), (ii), (iii) above. Lemma 2.1 is proved.

R e m a r k 2.1. If $h = 1$, then there is an elementary proof of Lemma 2.1. Unfortunately, this does not provide a proof of Theorems 1.2 and 1.4 in the one-dimensional case. The reason is that our arguments (see the proof of Theorem 1.3 below) require Lemma 2.1 for $h := d + 1$ -dimensional random variables. We also note that the rest of this section relies on elementary measure-theoretic techniques only. In particular, we use the following observation, which is a useful corollary of the monotone convergence theorem.

Proposition 2.2. *Let X_n be a sequence of integrable random variables such that*

$$\sum_n \mathbf{E} |X_n - X_{n+1}| < \infty.$$

Then X_n is convergent both in L^1 and a.s.

Lemma 2.1 extends easily to finite sequences of generalized martingale differences. Under this we mean a sequence $\{(X_k, \mathcal{G}_k): 0 \leq k \leq n\}$ such that

- (i) X_k is a $\mathcal{G}_k$ -measurable $\mathbf{R}^h$ -valued random variable for $k = 0, \dots, n$,
- (ii) $\mathbf{E}(X_k | \mathcal{G}_{k-1})$ exists and is a.s. zero, for $k = 1, \dots, n$.

Lemma 2.3. *Let $\{(X_k, \mathcal{G}_k): 0 \leq k \leq n\}$ be a finite sequence of generalized martingale differences, and let Y be a $\mathcal{G}_n$ -measurable nonnegative random variable, and $\varepsilon > 0$. Then there is a $\mathcal{G}_n$ -measurable positive random variable D such that*

- (i) $\mathbf{E} D = 1$, $\mathbf{E} |D - 1| < \varepsilon$,
- (ii) D and YD are bounded,
- (iii) $\mathbf{E}(X_k D | \mathcal{G}_{k-1}) = 0$ for $k = 1, \dots, n$.

P r o o f. Let $\mathcal{G}_{-1} = \{\emptyset, \Omega\}$. Fix $\eta > 0$. First we apply Lemma 2.1 with

$$X = X_n, \quad Y = Y, \quad \mathcal{G} = \mathcal{G}_{n-1}, \quad \text{and} \quad \varepsilon = \eta. \quad (5)$$

This gives a bounded Z_n such that $Z_n Y$ is also bounded and $\mathbf{E} |Z_n - 1| < \eta$.

If $Z_n, \dots, Z_{k+1}$ ($0 \leq k < n$) are defined, then we apply Lemma 2.1 with

$$X = \begin{cases} X_k & \text{if } k > 0, \\ 0 & \text{if } k = 0, \end{cases} \quad Y = \frac{1}{\mathbf{E}(Z_{k+1} | \mathcal{G}_k)}, \quad \mathcal{G} = \mathcal{G}_{k-1}, \quad \varepsilon = \eta$$

to obtain a bounded Z_k such that $Z_k / \mathbf{E}(Z_{k+1} | \mathcal{G}_k)$ is also bounded and $\mathbf{E} |Z_k - 1| < \eta$.

With this choice the following variables are all bounded:

$$\begin{array}{ccccccc} Z_n, & Z_{n-1}, & \dots, & Z_1, & Z_0, \\ Y Z_n, & \frac{Z_{n-1}}{\mathbf{E}(Z_n | \mathcal{G}_{n-1})}, & \dots, & \frac{Z_1}{\mathbf{E}(Z_2 | \mathcal{G}_1)}, & \frac{Z_0}{\mathbf{E}(Z_1 | \mathcal{G}_0)}. \end{array}$$

Put

$$D = \frac{Z_n}{\mathbf{E}(Z_n | \mathcal{G}_{n-1})} \cdots \frac{Z_1}{\mathbf{E}(Z_1 | \mathcal{G}_0)} \frac{Z_0}{\mathbf{E} Z_0}.$$

Then D , DY are bounded, and for $1 \leq k \leq n$,

$$\begin{aligned} \mathbf{E}(X_k D | \mathcal{G}_{k-1}) &= \mathbf{E}(\mathbf{E}(D | \mathcal{G}_k) X_k | \mathcal{G}_{k-1}) \\ &= \frac{Z_{k-1}}{\mathbf{E}(Z_k | \mathcal{G}_{k-1})} \cdots \frac{Z_0}{\mathbf{E}(Z_1 | \mathcal{G}_0)} \frac{1}{\mathbf{E} Z_0} \mathbf{E}(Z_k X_k | \mathcal{G}_{k-1}) = 0. \end{aligned}$$

We can carry out the above construction for all $\eta > 0$ and in this way we get a sequence of random variables with the obvious notation

$$D_\eta = \frac{Z_n(\eta) \cdots Z_0(\eta)}{\mathbf{E}(Z_n(\eta) | \mathcal{G}_{n-1}) \cdots \mathbf{E} Z_0(\eta)}.$$

Now let $\eta_m > 0$ be such that $\sum_m \eta_m < \infty$. Let us apply Proposition 2.2 to $X_n := Z_n$ noting that $\mathbf{E}|Z_{n+1} - Z_n| \leq \eta_{n+1} + \eta_n$. We get that $\lim_{m \rightarrow \infty} Z_k(\eta_m) = 1$ and $\lim_{m \rightarrow \infty} \mathbf{E}(Z_k(\eta_m) | \mathcal{G}_{k-1}) = 1$ both in L_1 and a.s. for $0 \leq k \leq n$. But this implies that

$$D_{\eta_m} \rightarrow 1 \quad \text{a.s. as } m \rightarrow \infty.$$

Since $\mathbf{E} D_{\eta_m} = 1$, Scheffé's theorem gives that for large m , $D = D_{\eta_m}$ satisfies $\mathbf{E}|D - 1| \leq \varepsilon$ also. Lemma 2.3 is proved.

We can finally prove Theorem 1.3.

P r o o f. Let $X_0 = S_0$, $X_n = S_n - S_{n-1}$ for $n > 0$. Take $\varepsilon_n > 0$ such that $\sum_{n=0}^{\infty} \varepsilon_n < \min(\varepsilon, 1)/3$. We may and will suppose that $Y_n \geq |X_n|$ for each n .

Using Lemma 2.3 (recall Proposition 2.1) we define a sequence (D_n) in a recursive way. Besides (D_n) a sequence of positive numbers (c_n) is to be defined. These sequences will satisfy the following properties for each n :

- (i) $0 < D_n(1 + Y_n) \leq c_n$,
- (ii) $\mathbf{E}|D_n - 1| \leq \varepsilon_n / (1 + \prod_{k \leq n} c_k)$,
- (iii) for $k \leq n$, $\mathbf{E}(D_0 \cdots D_n X_k | \mathcal{G}_{k-1}) = 0$,
- (iv) D_n is $\mathcal{G}_n$ -measurable.

This will prove the statement by the following reasoning. Put $H_n = \prod_{k \leq n} D_k$. Then

$$\mathbf{E}|H_{n+1} - H_n| = \mathbf{E}(H_n |D_{n+1} - 1|) \leq \prod_{k \leq n} c_k \mathbf{E}|D_{n+1} - 1| \leq \varepsilon_{n+1}.$$

Since ε_n is summable, we can use Proposition 2.2 to obtain that H_n is convergent in L_1 and a.s., the limit is denoted by H . It is also clear that

$$\mathbf{E}|H - 1| \leq \sum_{n=0}^{\infty} \varepsilon_n < \frac{\min(\varepsilon, 1)}{3}.$$

This implies that $d\mathbf{Q} = (H/\mathbf{E} H) d\mathbf{P}$ defines a probability measure $\mathbf{Q} \ll \mathbf{P}$ on $\mathcal{F}$. Elementary calculation gives that

$$\|\mathbf{P} - \mathbf{Q}\| = \mathbf{E} \left| \frac{H}{\mathbf{E} H} - 1 \right| \leq \varepsilon.$$

To prove that $\mathbf{Q}$ is actually equivalent we need that $H > 0$ a.s. This easily follows from

$$\sum_{n=0}^{\infty} |D_n - 1| < \infty \quad \text{a.s.}$$

which is true, since by monotone convergence

$$\mathbf{E} \left(\sum_{n=0}^{\infty} |D_n - 1| \right) = \sum_{n=0}^{\infty} \mathbf{E}|D_n - 1| \leq \sum_{n=0}^{\infty} \varepsilon_n < \infty.$$

In a similar way we can prove that S is a $\mathbf{Q}$ -martingale and each Y_k is $\mathbf{Q}$ -integrable. To show this we need that for each k , $|X_k| \leq Y_k$ is in $L^1(\mathbf{Q})$ and $\mathbf{E}_{\mathbf{Q}}(X_k | \mathcal{G}_{k-1}) = 0$. In other words, we need that

$$\mathbf{E}(HY_k) < \infty \quad \text{and} \quad \mathbf{E}(HX_k | \mathcal{G}_{k-1}) = 0.$$

Both follows if we show that $H_n Y_k$, $n \geq 0$, is a Cauchy sequence in $L^1(\mathbf{P})$, since it implies that $H_n X_k \rightarrow H X_k$ not only a.s. but also in $L^1(\mathbf{P})$ and we can use property (iii).

Now for $n \geq k$

$$\begin{aligned} \mathbf{E} |H_{n+1} Y_k - H_n Y_k| &= \mathbf{E} (|D_{n+1} - 1| D_n \cdots D_{k+1} (D_k Y_k) D_{k-1} \cdots D_0) \\ &\leq \mathbf{E} (|D_{n+1} - 1|) \prod_{m \leq n} c_m \leq \varepsilon_{n+1}. \end{aligned}$$

So the theorem is proved provided that we can carry out the recursive definition of (D_n) and (c_n) with the properties listed above.

In the recursive definition of the sequence (D_n) we actually use the following equivalent form of property (iii):

$$\mathbf{E} (\mathbf{E} (H_n | \mathcal{G}_k) S_k | \mathcal{G}_{k-1}) = \mathbf{E} (H_n | \mathcal{G}_{k-1}) S_{k-1}, \quad n \geq 0, \quad k > 0, \quad (6)$$

where $H_n = D_0 \cdots D_n$. It is obviously true for $k > n$ by the $\mathcal{G}_n$ -measurability of H_n , while it follows from property (iii) for $k \leq n$. It is also clear that (6) implies property (iii) by the tower rule.

First we define D_0, c_0 . Let $Z_n = 1/(1+Y_0/n)$, and let n be so large that $D_0 = Z_n/\mathbf{E} Z_n$ satisfies $\mathbf{E} |D_0 - 1| \leq \varepsilon_0$. Clearly D_0 and $D_0 Y_0$ are bounded and we can define c_0 as the sum of their essential suprema.

Assume that D_k, c_k , $0 \leq k < n$, are already defined, with the properties listed above. We can apply Lemma 2.3 to $Y = Y_n$, ε chosen according to (ii), and the differences of the generalized martingale $X_k := S'_k - S'_{k-1}$, $1 \leq k \leq n$, $X_0 := S'_0$, where

$$S'_k = \mathbf{E} (H_{n-1} | \mathcal{G}_k) \begin{pmatrix} S_k \\ 1 \end{pmatrix}, \quad k = 0, \dots, n.$$

The fact that S'_k , $k = 0, \dots, n$, is a generalized martingale is obvious for the last coordinate, while for the rest of the coordinates it follows from property (iii) of H_{n-1} , cf. (6). Note that S'_k has one extra coordinate $\mathbf{E} (H_{n-1} | \mathcal{G}_k)$.

In this way we get D_n ; c_n is defined as in (i). We have to show property (iii). By the $\mathcal{G}_{n-1}$ -measurability of H_{n-1} we get

$$\mathbf{E} (H_n | \mathcal{G}_{n-1}) = \mathbf{E} (D_n | \mathcal{G}_{n-1}) \mathbf{E} (H_{n-1} | \mathcal{G}_{n-1}).$$

The last coordinate of $D_n S'$ satisfies, by the choice of D_n ,

$$\mathbf{E} (D_n (\mathbf{E} (H_{n-1} | \mathcal{G}_k) - \mathbf{E} (H_{n-1} | \mathcal{G}_{k-1})) | \mathcal{G}_{k-1}) = 0, \quad 1 < k \leq n.$$

This implies

$$\mathbf{E} (\mathbf{E} (D_n | \mathcal{G}_k) \mathbf{E} (H_{n-1} | \mathcal{G}_k) | \mathcal{G}_{k-1}) = \mathbf{E} (D_n | \mathcal{G}_{k-1}) \mathbf{E} (H_{n-1} | \mathcal{G}_{k-1}),$$

for $1 \leq k \leq n$. Applying it for $k = n-1$ gives that

$$\begin{aligned} \mathbf{E} (H_n | \mathcal{G}_{n-2}) &= \mathbf{E} (\mathbf{E} (H_n | \mathcal{G}_{n-1}) | \mathcal{G}_{n-2}) \\ &= \mathbf{E} (\mathbf{E} (D_n | \mathcal{G}_{n-1}) \mathbf{E} (H_{n-1} | \mathcal{G}_{n-1}) | \mathcal{G}_{n-2}) \\ &= \mathbf{E} (D_n | \mathcal{G}_{n-2}) \mathbf{E} (H_{n-1} | \mathcal{G}_{n-2}). \end{aligned}$$

Repeating this we get that

$$\mathbf{E} (H_n | \mathcal{G}_k) = \mathbf{E} (D_n | \mathcal{G}_k) \mathbf{E} (H_{n-1} | \mathcal{G}_k), \quad 0 \leq k \leq n.$$

We can apply this and $\mathbf{E} (D_n (S'_k - S'_{k-1}) | \mathcal{G}_{k-1}) = 0$ to calculate

$$\begin{aligned} \mathbf{E} (H_n S_k | \mathcal{G}_{k-1}) &= \mathbf{E} (\mathbf{E} (H_n | \mathcal{G}_k) S_k | \mathcal{G}_{k-1}) \\ &= \mathbf{E} (\mathbf{E} (D_n | \mathcal{G}_k) \mathbf{E} (H_{n-1} | \mathcal{G}_k) S_k | \mathcal{G}_{k-1}) \\ &= \mathbf{E} (D_n | \mathcal{G}_{k-1}) \mathbf{E} (H_{n-1} | \mathcal{G}_{k-1}) S_{k-1} = \mathbf{E} (H_n | \mathcal{G}_{k-1}) S_{k-1}, \end{aligned}$$

which is just the equivalent form (6) of property (iii). So the recursion can be continued and Theorem 1.3 is proved.

3. Examples. The continuous-time counterpart of Theorem 1.1 is not true in general. For completeness we give here two simple counterexamples.

E x a m p l e 3.1. Let X be the three-dimensional squared Bessel process starting from some positive value. Then X satisfies the following equation:

$$dX_t = 2\sqrt{X_t} dW_t + 3 dt, \quad X_0 > 0,$$

with a Wiener process W . It is well known (see, e.g., [5, Chap. XI]) that the solution of the above equation is strong, so for deterministic X_0 , the process (X_t) is adapted to the filtration of W and $(Y_t = 1/\sqrt{X_t}, \mathcal{G}_t)$ is a local martingale, where $\mathcal{G}_t = \sigma(W_s, s \leq t)$. This is a well-known example of a local martingale not being a martingale.

Assume that there is $\mathbf{Q} \ll \mathbf{P}$ such that, under $\mathbf{Q}$, Y is a martingale. Put

$$D = \frac{d\mathbf{Q}|_{\mathcal{G}_\infty}}{d\mathbf{P}|_{\mathcal{G}_\infty}}, \quad D_t = \mathbf{E}(D | \mathcal{G}_t).$$

It is easy to see that the martingale property under $\mathbf{Q}$ would mean

$$\mathbf{E}_{\mathbf{Q}}(Y_t | \mathcal{G}_s) = \frac{\mathbf{E}(D_t Y_t | \mathcal{G}_s)}{D_s} = Y_s,$$

in other words, $(DY, \mathcal{G})$ is a martingale under $\mathbf{P}$. Since $\mathcal{G}$ is a Brownian filtration, $D_t = D_0 + \int_0^t H_s dW_s$ with some predictable H . This gives that

$$d(DY)_t = D_t dY_t + Y_t dD_t + d\langle Y, D \rangle_t.$$

Here the first two terms are local martingales, so if DY were a local martingale also, then $\langle Y, D \rangle$ would be identically zero. However

$$d\langle Y, D \rangle_t = -H_t Y_t^2 dt,$$

since $dY_t = -Y_t^2 dW_t$. As $Y_t > 0$ a.s. for all t , $\langle Y, D \rangle = 0$ would imply $H = 0$ and $D = D_0$. Since D is $\mathcal{G}_0$ -measurable and $\mathcal{G}_0$ is trivial, $D = 1$ and $\mathbf{Q}|_{\mathcal{G}_\infty} = \mathbf{P}|_{\mathcal{G}_\infty}$ is a contradiction.

E x a m p l e 3.2. This example is due to Christophe Stricker. Let

$$X_t := \exp\left(W_t - \frac{t}{2}\right), \quad t \geq 0,$$

where W is Brownian motion and $(\mathcal{F}_t)_{t \geq 0}$ is its natural filtration. Define the (a.s. finite) stopping time

$$\tau := \inf\left\{t: X_t = \frac{1}{2}\right\},$$

and set

$$S_t := X_{\tau \wedge \text{tg } t}, \quad 0 \leq t < \frac{\pi}{2}, \quad S_t = \frac{1}{2}, \quad t \geq \frac{\pi}{2}.$$

We can unify the notation, by using the convention $\text{tg } t = \infty$, for $t \geq \pi/2$ in this example, since τ is finite a.s. Then $S_t = X_{\tau \wedge \text{tg } t}$ and we can define $\mathcal{G}_t := \mathcal{F}_{\text{tg } t}$. We find that (S_t) is a $(\mathcal{G}_t)$ -adapted continuous process. It cannot be a martingale under any measure $\mathbf{P}' \sim \mathbf{P}$, since $S_{\pi/2} = 1/2$ and S_t is not constant for $t < \pi/2$.

It is, however, a $(\mathcal{G}_t)_{t \geq 0}$ -local martingale. Take

$$\sigma_n := \inf\{t: X_{t \wedge \tau} = n\},$$

then $\tau_n := (\arctg \sigma_n)I_{\{\sigma_n < \infty\}} + \infty I_{\{\sigma_n = \infty\}}$, $n \geq 0$, is a sequence of $(\mathcal{G}_t)_{t \geq 0}$ -stopping times tending a.s. to infinity.

Since X_t is an $(\mathcal{F}_t)_{t \geq 0}$ -martingale, the stopped process $X_{t \wedge \tau \wedge \sigma_n}$ is a (bounded) $(\mathcal{F}_t)_{t \geq 0}$ -martingale. Then $X_{\text{tg } t \wedge \tau \wedge \text{tg } \tau_n} = X_{\text{tg}(t \wedge \tau_n) \wedge \tau} = S_{t \wedge \tau_n}$ is a $(\mathcal{G}_t)_{t \geq 0}$ -martingale, showing that S is indeed a local martingale.

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ALMOST SURE LIMIT THEOREMS FOR GAUSSIAN SEQUENCES

Пусть $\{X_n, n \geq 1\}$ — гауссовские случайные величины с нулевым средним, единичной дисперсией и корреляциями $r_{ij} = \mathbf{E} X_i X_j$. Предположим, что существует последовательность $0 \leq \rho_n < 1, n \geq 1$, такая, что $|r_{ij}| \leq \rho_{|j-i|}$ для $i \neq j$ и $\rho_n \ln n (\ln \ln n)^{1+\varepsilon} = O(1)$ при $n \rightarrow \infty$. Для последовательности уровней $\{u_{nk}, 1 \leq k \leq n, n \geq 1\}$, положим $\lambda_n = \min_{1 \leq k \leq n} u_{nk}$, и пусть последовательность $n(1 - \Phi(\lambda_n))$ ограничена. Мы выводим центральные предельные теоремы типа «почти наверное» для $(\ln n)^{-1} \sum_{k=1}^n k^{-1} I(X_1 \leq u_{k1}, \dots, X_k \leq u_{kk})$ и $(\ln n)^{-1} \times \sum_{k=1}^n k^{-1} I(\max_{1 \leq i \leq k} X_i \leq \lambda_k)$.

Ключевые слова и фразы: центральная предельная теорема типа «почти наверное», гауссовская последовательность, логарифмическое среднее.

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