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ALMOST SURE LIMIT THEOREMS FOR GAUSSIAN SEQUENCES

Пусть $\{X_n, n \geq 1\}$ — гауссовские случайные величины с нулевым средним, единичной дисперсией и корреляциями $r_{ij} = \mathbf{E} X_i X_j$. Предположим, что существует последовательность $0 \leq \rho_n < 1, n \geq 1$, такая, что $|r_{ij}| \leq \rho_{|j-i|}$ для $i \neq j$ и $\rho_n \ln n (\ln \ln n)^{1+\varepsilon} = O(1)$ при $n \rightarrow \infty$. Для последовательности уровней $\{u_{nk}, 1 \leq k \leq n, n \geq 1\}$, положим $\lambda_n = \min_{1 \leq k \leq n} u_{nk}$, и пусть последовательность $n(1 - \Phi(\lambda_n))$ ограничена. Мы выводим центральные предельные теоремы типа «почти наверное» для $(\ln n)^{-1} \sum_{k=1}^n k^{-1} I(X_1 \leq u_{k1}, \dots, X_k \leq u_{kk})$ и $(\ln n)^{-1} \times \sum_{k=1}^n k^{-1} I(\max_{1 \leq i \leq k} X_i \leq \lambda_k)$.

Ключевые слова и фразы: центральная предельная теорема типа «почти наверное», гауссовская последовательность, логарифмическое среднее.

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1. Introduction. Let $X, X_1, X_2, \dots$ be independent and identically distributed (iid) random variables with $\mathbf{E}X = 0$ and $\mathbf{D}X = 1$. The almost sure central limit theorem (ASCLT) on partial sums was first studied by Brosamler [3], Schatte [12], and Lacey and Phillip [10]. Under some regularity conditions, the ASCLT says that

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I(k^{-1/2} S_k \leq x) \longrightarrow \Phi(x) \quad (1)$$

almost surely, where $I(A)$ denotes the indicator function of A , $S_k = \sum_{i=1}^k X_i$, $k \geq 1$, and $\Phi(x)$ stands for the standard normal distribution function (Brosamler [3] needed the condition $\mathbf{E}|X|^{2+\delta} < \infty$ for some $\delta > 0$, Schatte [12] used $\mathbf{E}|X|^3 < \infty$, and Lacey and Phillip [10] only needed $\mathbf{E}X = 0$ and $\mathbf{E}X^2 = 1$). Ibragimov and Lifshits [8] extended (1) to the functional form:

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} h(k^{-1/2} S_k) \longrightarrow \int_{-\infty}^{\infty} h(x) d\Phi(x)$$

almost surely provided that some regular conditions are fulfilled and the continuous function h satisfies $\int_{-\infty}^{\infty} |h(x)| d\Phi(x) < \infty$. In [Ibragimov and Lifshits, (2000)] the ASCLT was extended for normalized partial sums of a sequence of independent random vectors and for partial sums of a strong mixing sequence. Berkes and Csáki [1] extended the ASCLT to general nonlinear functionals of iid random variables including the maximum case $M_n = \max_{1 \leq k \leq n} X_k$ which was first developed by Fahrner and Stadtmüller [7] and Cheng, Peng, and Qi [4]. If there exist constants $a_n > 0$ and $b_n \in \mathbf{R}$ and a nondegenerate distribution function $G(x)$ such that $\mathbf{P}\{a_n(M_n - b_n) \leq x\} \rightarrow G(x)$ weakly, then the result for the maximum case is that

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I(a_k(M_k - b_k) \leq x) \longrightarrow G(x)$$

almost surely for any continuity point x of G . It is well known that $G(x)$ must be of the same type as the extreme value distribution $G_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma})$, $1 + \gamma x > 0$, $\gamma \in \mathbf{R}$, i.e., $G(x) = G_\gamma(ax + b)$ for some $a > 0$, $b \in \mathbf{R}$. For functional forms of almost sure convergence related to the normalized maximum of iid random variables see [2] and [6]. Csáki and Gonchigdanzan [5] considered the stationary Gaussian case. If $\{X_n\}$ is the standard stationary Gaussian sequence with $\gamma_n \ln n (\ln \ln n)^{1+\varepsilon} = O(1)$ for some $\varepsilon > 0$, where $\gamma_n = \mathbf{E}X_1 X_{n+1}$, then

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I(M_k \leq u_k) \longrightarrow \exp(-\tau)$$

almost surely provided $n(1 - \Phi(u_n)) \rightarrow \tau$ as $n \rightarrow \infty$. Further, if

$$a_n = (2 \ln n)^{1/2}, \quad b_n = a_n - \frac{1}{2} a_n^{-1} (\ln \ln n + \ln(4\pi)), \quad (2)$$

then

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I(M_k \leq a_k^{-1} x + b_k) \rightarrow \exp(-\exp(-x))$$

almost surely.

The purpose of this paper is to extend Csáki and Gonchigdanzan's results to general standard Gaussian sequences under some regularity conditions. We shall use the following result due to Leadbetter, Lindgren, and Rootzen [11, Theorem 6.1.3].

Theorem A. Let $\{X_n, n \geq 1\}$ be a standard Gaussian sequence with correlations $r_{ij} = \mathbf{E}X_i X_j$ which are dominated by a positive sequence $\{\rho_n, n \geq 1\}$, i.e., $|r_{ij}| \leq \rho_{|j-i|}$ for all

$i, j = 1, 2, \dots$ and $\sup_{n \geq 1} \rho_n < 1$ and $\rho_n \ln n \rightarrow 0$ as $n \rightarrow \infty$. Suppose further that the levels $\{u_{ni}, 1 \leq i \leq n, n \geq 1\}$ are such that

$$\lambda_n = \min_{1 \leq i \leq n} u_{ni} \quad (3)$$

satisfies $\lambda_n \geq c(\ln n)^{1/2}$ for some constant $c > 0$, and $\sum_{i=1}^n (1 - \Phi(u_{ni})) \rightarrow \tau$ as $n \rightarrow \infty$. Then

$$\mathbf{P}(\mathcal{A}_n) \rightarrow \exp(-\tau) \quad (4)$$

as $n \rightarrow \infty$, where

$$\mathcal{A}_n = (X_1 \leq u_{n1}, X_2 \leq u_{n2}, \dots, X_n \leq u_{nn}). \quad (5)$$

Throughout this paper, we assume that the Gaussian sequence $\{X_n, n \geq 1\}$ satisfy

$$\begin{aligned} |r_{ij}| &\leq \rho_{|j-i|}, \quad i \neq j, \quad \rho_n \geq 0, \quad \sup_{n \geq 1} \rho_n = \delta < 1, \\ \rho_n \ln n (\ln \ln n)^{1+\varepsilon} &= O(1), \quad \varepsilon > 0. \end{aligned} \quad (6)$$

2. Main results. This section provides the main results. The proofs are given in Section 3.

Theorem 1. Let (6) hold for the Gaussian sequence $\{X_n, n \geq 1\}$. Let the levels $\{u_{ni}, 1 \leq i \leq n, n \geq 1\}$ be such that $n(1 - \Phi(\lambda_n))$ is bounded, where λ_n is defined in (3). Suppose (4) holds. Then,

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I(\mathcal{A}_k) \rightarrow \exp(-\tau)$$

almost surely, where $\mathcal{A}_k$ is defined by (5).

Theorem 2. Suppose (6) holds for the Gaussian sequence $\{X_n, n \geq 1\}$ and there exists sequence u_n such that $n(1 - \Phi(u_n)) \rightarrow \tau$ as $n \rightarrow \infty$. Then,

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I(M_k \leq u_k) \rightarrow \exp(-\tau)$$

almost surely. Further,

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I(M_k \leq a_k^{-1}x + b_k) \rightarrow \exp(-\exp(-x))$$

almost surely, where a_n and b_n are defined in (2).

Corollary 1. Let $\{\eta_k = X_k + m_k, k \geq 1\}$, where $\{X_k, k \geq 1\}$ is the Gaussian sequence satisfying the conditions of Theorem 1 and $\{m_k, k \geq 1\}$ satisfies

$$\beta_n = \max_{1 \leq k \leq n} |m_k| = o(\sqrt{\ln n}),$$

and let m_n^* be chosen such that

$$|m_n^*| \leq \beta_n, \quad a_n \left(\max_{1 \leq i \leq n} m_i - m_n^* \right) \leq C_0,$$

and

$$\frac{1}{n} \sum_{i=1}^n \exp \left(a_n^* (m_i - m_n^*) - \frac{1}{2} (m_i - m_n^*)^2 \right) \rightarrow 1$$

as $n \rightarrow \infty$, where C_0 is a constant and $a_n^* = a_n - \frac{1}{2} a_n^{-1} \ln \ln n$. Then,

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I(M_k^* \leq v_k(x)) \rightarrow \exp(-\exp(-x))$$

almost surely, where $M_n^* = \max_{1 \leq i \leq n} \eta_i$, $v_n = v_n(x) = a_n^{-1}x + b_n + m_n^*$ and a_n and b_n are defined in (2).

3. Proofs of the main results. The main ideas for proving our results are similar to those of [1] and [5]. To prove the main results, we need the following lemmas.

Lemma 1. Let $\xi_1, \xi_2, \dots$ be a sequence of bounded random variables. If

$$\mathbf{D}\left(\sum_{k=1}^n \frac{1}{k} \xi_k\right) \ll \ln^2 n (\ln \ln n)^{-(1+\varepsilon)}$$

for some $\varepsilon > 0$, then

$$\lim_{n \rightarrow \infty} \frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} (\xi_k - \mathbf{E} \xi_k) = 0$$

almost surely.

P r o o f. See Lemma 3.1 in [5].

Lemma 2. Let $\{u_{ni}, 1 \leq i \leq n, n \geq 1\}$ be levels such that $n(1 - \Phi(\lambda_n))$ is bounded, where λ_n is defined in (3). Suppose (6) holds. Then

$$Q_n = \sum_{1 \leq i < j \leq n} |r_{ij}| (1 - r_{ij}^2)^{-1/2} R_{ij}^{(n)} \ll (\ln \ln n)^{-(1+\varepsilon)},$$

where

$$R_{ij}^{(n)} = \exp\left(-\frac{(u_{ni}^2 + u_{nj}^2)/2}{1 + |r_{ij}|}\right) \quad (7)$$

for $1 \leq i < j \leq n$.

P r o o f. For constant K , we have $n(1 - \Phi(\lambda_n)) < K$. Since $\Phi(x)$ is continuous, there exists w_n such that $n(1 - \Phi(w_n)) = K$, which implies

$$w_n < \lambda_n, \quad \exp\left(-\frac{w_n^2}{2}\right) \sim \frac{K\sqrt{2\pi}w_n}{n}, \quad w_n \sim \sqrt{2\ln n} \quad (8)$$

for large enough n . Noting that $\lambda_n = \min_{1 \leq i \leq n} u_{ni} > w_n$ and $\sup_{n \geq 1} \rho_n = \delta < 1$, we have

$$\begin{aligned} Q_n &\ll \sum_{\substack{1 \leq i < j \leq n \\ j-i \leq \theta_n}} |r_{ij}| \exp\left(-\frac{w_n^2}{1 + |r_{ij}|}\right) + \sum_{\substack{1 \leq i < j \leq n \\ j-i > \theta_n}} |r_{ij}| \exp\left(-\frac{w_n^2}{1 + |r_{ij}|}\right) \\ &=: Q_n^{(1)} + Q_n^{(2)}, \end{aligned}$$

where $\theta_n = \exp(\eta w_n^2)$ for $0 < \eta < \eta_0 = (1 - \delta)/(2(1 + \delta))$. Firstly,

$$\begin{aligned} Q_n^{(1)} &\leq \delta \sum_{\substack{1 \leq i < j \leq n \\ j-i \leq \theta_n}} \exp\left(-\frac{w_n^2}{1 + \delta}\right) \leq \delta \theta_n n \exp\left(-\frac{w_n^2}{1 + \delta}\right) \\ &\ll n \left(\frac{w_n}{n}\right)^{-2(\eta-1/(1+\delta))} \ll n^{2(\eta-\eta_0)} (\ln n)^{1/(1+\delta)-\eta} \ll (\ln \ln n)^{-(1+\varepsilon)} \end{aligned}$$

for $\eta < \eta_0$. Now consider $Q_n^{(2)}$. By (8), for large n , there exists a constant $c > 0$ such that $w_n > c\sqrt{\ln n}$ and hence $\theta_n = \exp(\eta w_n^2) \geq \exp(c^2 \eta \ln n) = n^\alpha$, $\alpha = c^2 \eta$ for large n . Denoting $\delta_x = \sup_{j \geq x} \rho_j$ and $p = [n^\alpha]$, we have

$$\begin{aligned} Q_n^{(2)} &\leq \delta_p n^2 \exp\left(-\frac{w_n^2}{1 + \delta_p}\right) \ll \left(n(1 - \Phi(w_n))\right)^2 \delta_p w_n^2 \exp(\delta_p w_n^2) \\ &\ll \delta_p (\ln p) \exp(\delta_p \ln p) \ll (\ln \ln p)^{-(1+\varepsilon)} \ll (\ln \ln n)^{-(1+\varepsilon)} \end{aligned}$$

for $\delta_p \ln p (\ln \ln p)^{1+\varepsilon} \leq \sup_{j \geq p} \rho_j \ln j (\ln \ln j)^{1+\varepsilon} = O(1)$ for large n . The result of the lemma follows.

Lemma 3. Under the conditions of Lemma 2, for $1 \leq k < n$, we have

$$\mathbf{E} |I(\mathcal{A}_n) - I(\mathcal{B}_{kn})| \leq \frac{k}{n} + (\ln \ln n)^{-(1+\varepsilon)}$$

for large enough n , where

$$\mathcal{B}_{kn} = (X_{k+1} \leq u_{n(k+1)}, X_{k+2} \leq u_{n(k+2)}, \dots, X_n \leq u_{nn}) \quad (9)$$

and $\mathcal{A}_n$ is defined in (5).

P r o o f. Note that

$$\begin{aligned} \mathbf{E} |I(\mathcal{A}_n) - I(\mathcal{B}_{kn})| &= \mathbf{P}(\mathcal{B}_{kn}) - \mathbf{P}(\mathcal{A}_n) \\ &\leq \left| \mathbf{P}(\mathcal{B}_{kn}) - \prod_{j=k+1}^n \Phi(u_{nj}) \right| + \left| \mathbf{P}(\mathcal{A}_n) - \prod_{j=1}^n \Phi(u_{nj}) \right| \\ &\quad + \left(\prod_{j=k+1}^n \Phi(u_{nj}) - \prod_{j=1}^n \Phi(u_{nj}) \right) \\ &=: A_n + B_n + C_n. \end{aligned}$$

By the Normal Comparison Lemma and Lemma 2, we know that A_n and B_n are both dominated by

$$C \sum_{1 \leq i < j \leq n} |r_{ij}| (1 - r_{ij}^2)^{-1/2} R_{ij}^{(n)} \ll (\ln \ln n)^{-(1+\varepsilon)},$$

where $R_{ij}^{(n)}$ is given by (7). It is easy to check that

$$0 \leq C_n \leq 1 - \prod_{j=1}^k \Phi(u_{nj}) \leq \sum_{j=1}^k (1 - \Phi(u_{nj})) \leq k(1 - \Phi(\lambda_n)) = \frac{k}{n} [n(1 - \Phi(\lambda_n))] \ll \frac{k}{n}.$$

The result of the lemma follows.

Lemma 4. Under the conditions of Lemma 2, for $1 \leq k < n$, we have

$$|\text{Cov}(I(\mathcal{A}_k), I(\mathcal{B}_{kn}))| \ll (\ln \ln n)^{-(1+\varepsilon)}$$

for large n , where $\mathcal{A}_k$ and $\mathcal{B}_{kn}$ are defined in (5) and (9), respectively.

P r o o f. By the Normal Comparison Lemma, we have

$$\begin{aligned} |\text{Cov}(I(\mathcal{A}_k), I(\mathcal{B}_{kn}))| &= |\mathbf{P}(\mathcal{A}_k \cap \mathcal{B}_{kn}) - \mathbf{P}(\mathcal{A}_k)\mathbf{P}(\mathcal{B}_{kn})| \\ &\leq C \sum_{\substack{1 \leq i \leq k, \\ k+1 \leq j \leq n, \\ j-i \leq \theta_n}} |r_{ij}| (1 - r_{ij}^2)^{-1/2} R_{ij}^{(n)} + C \sum_{\substack{1 \leq i \leq k, \\ k+1 \leq j \leq n, \\ j-i > \theta_n}} |r_{ij}| (1 - r_{ij}^2)^{-1/2} R_{ij}^{(n)} \\ &=: D_n + E_n, \end{aligned}$$

where $R_{ij}^{(n)}$ is defined by (7) and θ_n as in the proof of Lemma 2. Note by (8) that

$$\begin{aligned} D_n &\ll \sum_{\substack{1 \leq i \leq k, \\ k+1 \leq j \leq n, \\ j-i \leq \theta_n}} \exp\left(-\frac{(w_k^2 + w_n^2)/2}{1 + \delta}\right) \leq k\theta_n \exp\left(-\frac{(w_k^2 + w_n^2)/2}{1 + \delta}\right) \\ &= k \left(\exp\left(-\frac{w_k^2}{2}\right) \right)^{1/(1+\delta)} \left(\exp\left(-\frac{w_n^2}{2}\right) \right)^{1/(1+\delta)-2\eta} \\ &\ll k \left(\frac{w_k}{k} \right)^{1/(1+\delta)} \left(\frac{w_n}{n} \right)^{1/(1+\delta)-2\eta} \\ &\ll k^{1-1/(1+\delta)} (\ln k)^{1/(2(1+\delta))} n^{2\eta-1/(1+\delta)} (\ln n)^{1/(2(1+\delta))-\eta} \\ &\leq n^{-2(\eta_0-\eta)} (\ln n)^{1/(1+\delta)-\eta} \ll (\ln \ln n)^{-(1+\varepsilon)} \end{aligned}$$

and

$$\begin{aligned} E_n &\ll \delta_{[n^\alpha]} k \exp\left(-\frac{w_k^2}{2}\right) \exp\left(\frac{1}{2} w_k^2 \delta_{[n^\alpha]}\right) n \exp\left(-\frac{w_n^2}{2}\right) \exp\left(\frac{1}{2} w_n^2 \delta_{[n^\alpha]}\right) \\ &\ll \delta_{[n^\alpha]} \ln n \exp\left(\frac{1}{2} w_k^2 \delta_{[n^\alpha]}\right) \exp\left(\frac{1}{2} w_n^2 \delta_{[n^\alpha]}\right) \ll (\ln \ln n)^{-(1+\varepsilon)}, \end{aligned}$$

where $\delta_{[n^\alpha]}$ is as in the proof of Lemma 2. The result of the lemma follows by combining the above bounds.

P r o o f o f T h e o r e m 1. Let $\varsigma_k = I(\mathcal{A}_k) - \mathbf{P}(\mathcal{A}_k)$, where $\mathcal{A}_k$ is defined in (5). Then,

$$\mathbf{D}\left(\sum_{k=1}^n \frac{1}{k} I(\mathcal{A}_k)\right) = \mathbf{E}\left(\sum_{k=1}^n \frac{1}{k} \varsigma_k\right)^2 \leq \sum_{k=1}^n \frac{1}{k^2} + 2 \sum_{1 \leq k < l \leq n} \frac{|\mathbf{E}(\varsigma_k \varsigma_l)|}{kl}. \quad (10)$$

The first term on the right-hand side of (10) is convergent. For the second term, consider $\mathbf{E}|\varsigma_k \varsigma_l|$. By Lemmas 3 and 4, for $1 \leq k < l$,

$$\begin{aligned} \mathbf{E}|\varsigma_k \varsigma_l| &= \left| \text{Cov}(I(\mathcal{A}_k), I(\mathcal{A}_l)) \right| \leq \mathbf{E}|I(\mathcal{A}_k) - I(\mathcal{B}_{kl})| + \left| \text{Cov}(I(\mathcal{A}_k), I(\mathcal{B}_{kl})) \right| \\ &\ll \frac{k}{l} + (\ln \ln l)^{-(1+\varepsilon)} \end{aligned}$$

for sufficiently large l , where $\mathcal{B}_{kl}$ is defined in (9). By the estimation method of [5], we have

$$\mathbf{D}\left(\sum_{k=1}^n \frac{1}{k} I(\mathcal{A}_k)\right) \leq \sum_{k=1}^n \frac{1}{k^2} + 2 \sum_{1 \leq k < l \leq n} \frac{|\mathbf{E}(\varsigma_k \varsigma_l)|}{kl} \ll \ln^2 n (\ln \ln n)^{-(1+\varepsilon)}.$$

The result of the theorem follows by Lemma 1.

P r o o f o f T h e o r e m 2. A special case of Theorem 1.

P r o o f o f C o r o l l a r y 1. Define $u_{ni}(x) = u_n(x) + m_n^* - m_i$ and $u_n(x) = a_n^{-1}x + b_n$. We have

$$\mathbf{P}(M_n^* \leq v_n(x)) = \mathbf{P}(\mathcal{A}_n) \longrightarrow \exp(-\exp(-x))$$

weakly as $n \rightarrow \infty$ (cf. Theorem 6.2.1 in [11]), where $\mathcal{A}_n$ is defined in (5). Hence, we only need to prove

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I(\mathcal{A}_k) \longrightarrow \exp(-\exp(-x))$$

almost surely. By Theorem 1, it is sufficient to check that $n(1 - \Phi(\lambda_n))$ is bounded with $\lambda_n = \lambda_n(x) = \min_{1 \leq i \leq n} u_{ni}(x) = u_n(x) + m_n^* - \max_{1 \leq i \leq n} m_i$. Noting the restrictions on β_n and $|m_n^*|$, we have

$$\begin{aligned} &u_n\left(\max_{1 \leq i \leq n} m_i - m_n^*\right) - \frac{1}{2}\left(\max_{1 \leq i \leq n} m_i - m_n^*\right)^2 \\ &= o(1) + \sqrt{2 \ln n} \left(\max_{1 \leq i \leq n} m_i - m_n^*\right) \left(1 - \frac{\ln \ln n + \ln(4\pi)}{4 \ln n} - \frac{\max_{1 \leq i \leq n} m_i - m_n^*}{2\sqrt{2 \ln n}}\right) \\ &\leq o(1) + C_0 \end{aligned}$$

for large n . Hence,

$$\begin{aligned} n(1 - \Phi(\lambda_n)) &\sim n\lambda_n^{-1} \exp\left(-\frac{1}{2}\left(u_n + m_n^* - \max_{1 \leq i \leq n} m_i\right)^2\right) \\ &\sim n(1 - \Phi(u_n)) \exp\left(-u_n\left(m_n^* - \max_{1 \leq i \leq n} m_i\right) - \frac{1}{2}\left(m_n^* - \max_{1 \leq i \leq n} m_i\right)^2\right) \\ &\ll n(1 - \Phi(u_n)) \exp(C_0) \end{aligned}$$

for large n . The result of the corollary follows.

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