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NONCENTRAL LIMIT THEOREM FOR THE CUBIC VARIATION OF A CLASS OF SELF-SIMILAR STOCHASTIC PROCESSES

Используя кратные стохастические интегралы Винера–Ито, мы изучаем вариацию третьего порядка одного класса самоподобных (автомодельных) процессов со стационарными приращениями (процесс Розенблатта с коэффициентом самоподобия $H \in (1/2, 1)$). Исследование мотивировано статистическими целями. Доказывается, что для ренормированной вариации третьего порядка справедлива нецентральная предельная теорема и предел (в смысле $L^2(\Omega)$) снова является процессом Розенблатта.

Ключевые слова и фразы: кратные стохастические интегралы, самоподобные (автомодельные) процессы, процесс Розенблатта, фрактальное броуновское движение, нецентральная предельная теорема, исчисление Мальявена.

1. Introduction. The self-similarity property for a stochastic process means that scaling of time is equivalent to an appropriate scaling of space. That is, a process $(Y_t)_{t \geq 0}$ is self-similar of order $H > 0$ if for all $c > 0$ the processes $(Y_{ct})_{t \geq 0}$ and $(c^H Y_t)_{t \geq 0}$ have the same finite dimensional distributions. The self-similar processes are of interest for various applications, such as economics, internet traffic or hydrology. The fractional Brownian motion is the usual candidate to model phenomena in which the self-similarity property can be observed from the empirical data. Recall that the fractional Brownian motion is a centered Gaussian process with covariance function $R^H(t, s) = (t^{2H} + s^{2H} - |t - s|^{2H})/2$. The parameter $H \in (0, 1)$ characterizes almost all important properties of the process. The fractional Brownian motion can also be defined as the only Gaussian process which is self-similar with stationary increments. In some models the Gaussianity assumption

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could be not plausible and in this case one needs to use a different self-similar process with stationary increments to model the phenomena. Natural candidates are the Hermite processes: these stochastic processes appear as limits in the so-called noncentral limit theorem (see [3], [7], [18], [10]). In contrast with the classical central limit theorem, the noncentral limit theorem deals with sequences of dependent random variable whose renormalized sum converges in some situations to a non-Gaussian distribution. For a complete exposition of limit theorems in probability theory, we refer to [9] or [16]. Except the Gaussian character, these Hermite processes have the same property as the fractional Brownian motion with Hurst parameter $H > 1/2$: self-similarity, stationarity of increments, Hölder continuous path, long-range dependence. While the fractional Brownian motion can be expressed as a Wiener integral with respect to the standard Wiener process, the Hermite process of order $q \geq 2$ is a q iterated integral of a deterministic function with q variables with respect to the Brownian motion. The Rosenblatt process is obtained in the particular case $q = 2$. It will be properly defined in Section 2. These processes have been recently studied by several authors (see [2], [4], [11]–[13], [19], [20]).

The Hurst parameter H characterizes all the important properties of a Hermite process, as seen above. Therefore, estimating H properly is of the utmost importance. Several statistics have been introduced to this end, such as wavelets, k -variations, variograms, maximum likelihood estimators, or spectral methods. Information on these various approaches can be found in [1].

One of the most popular methods to estimate the self-similarity order for stochastic processes is based on the study of their variations. The p -variation of a process $(X_t)_{t \in [0,1]}$ is defined as the limit of the sequence (sometimes the absolute value of the increment is used in the definition)

$$V^{p,N}(X) = \frac{1}{N} \left[\sum_{i=0}^{N-1} \frac{(X_{(i+1)/N} - X_{i/N})^p}{\mathbf{E}(X_{(i+1)/N} - X_{i/N})^p} - 1 \right]. \quad (1)$$

There exists a direct connection between the behavior of the variations and the convergence of an estimator for the self-similarity order based on these variations (see [6], [20]); basically if the renormalized variation satisfies a central limit theorem then the estimator satisfies a central limit theorem and this fact is very useful for statistical aspects.

In a recent paper [20] the quadratic variation of the Rosenblatt process $(Z_t^{(H)})_{t \in [0,1]}$ with self-similarity order $H \in (1/2, 1)$ has been studied. The following facts happen: the normalized sequence $N^{1-H} V^{2,N}(Z^{(H)})$ satisfies a noncentral limit theorem, it converges in L^2 to the Rosenblatt random variable $Z_1^{(H)}$. From this, we can construct an estimator for H whose behavior is still nonnormal. This situation is somehow not good for statistical

applications because one always prefers the estimators which are asymptotically normal. To have normal estimators we need to define some adjusted variations (as in [20]).

In the fractional Brownian motion case the well-known nonnormality of the quadratic variation when $H \in (3/4, 1)$ can be avoided by using «longer filters» (that means replacing the increments $X_{(i+1)/N} - X_{i/N}$ by $X_{(i+1)/N} - 2X_{i/N} + X_{(i-1)/N}$) or higher order variations (choosing a bigger p). In this work we will consider the second choice (the first choice has been treated in [5]): we replace the quadratic variation by the cubic variation for the Rosenblatt processes to see what happens and if it is possible to find a Gaussian distribution as law of the renormalized cubic variation. In the fractional Brownian motion case, this has no sense because the third moment of a centered Gaussian random variable is zero. We use the Wiener chaos expansion for the statistics $V^{3,N}(Z^{(H)})$ and we will decompose it in several terms in the Wiener chaoses 2, 4, and 6. As in other cases [20], [4] the second chaos term is dominant and it has to be renormalized by N^{1-H} to have a nontrivial limit. We note that the rate of convergence N^{1-H} is the same as for quadratic variation, so there is no gain for the speed and moreover the limit is again, modulo a constant, a Rosenblatt random variable with index H (only the constant is changing). This property has been called in [4] the *reproduction property* of the Rosenblatt process because its variation generates again Rosenblatt random variable as limit. We conjecture that the same property holds true for the p -variations.

The organization of our paper is as follows. Section 2 contains the presentation of the basic tools that we will need throughout the paper: multiple Wiener–Itô integrals and their basic properties, the definition of the Rosenblatt process and its characteristics. In Section 3 we estimate the mean square of the cubic variation of the Rosenblatt process and we give its normalization and finally in Section 4 we prove a noncentral limit theorem for the renormalized cubic variation.

2. Preliminaries.

2.1. Multiple stochastic integrals. In this subsection we describe the basic elements of calculus on Wiener chaos. Let $(W_t)_{t \in [0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0,1]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener–Itô integral of f with respect to W . We refer to [14] for a detailed exposition of the construction and the properties of multiple Wiener–Itô integrals.

Let $f \in \mathcal{S}_m$ be an elementary function with m variables that can be written as

$$f = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} 1_{A_{i_1} \times \dots \times A_{i_m}},$$

where the coefficients satisfy $c_{i_1, \dots, i_m} = 0$ if two indices i_k and i_l are equal

and the sets $A_i \in \mathcal{B}([0, 1])$ are disjoint. For such a step function f we define

$$I_m(f) = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} W(A_{i_1}) \cdots W(A_{i_m}),$$

where we put $W([a, b]) = W_b - W_a$. It can be seen that the application I_n from $\mathcal{S}$ to $L^2(\Omega)$ is an isometry on $\mathcal{S}$, i.e.,

$$\mathbf{E}[I_n(f)I_m(g)] = n! \langle f, g \rangle_{L^2([0, 1]^n)} \quad \text{if } m = n \quad (2)$$

and

$$\mathbf{E}[I_n(f)I_m(g)] = 0 \quad \text{if } m \neq n.$$

It also holds that

$$I_n(f) = I_n(\tilde{f}),$$

where $\tilde{f}$ denotes the symmetrization of f defined by $\tilde{f}(x_1, \dots, x_n) = (1/n!) \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

Since the set $\mathcal{S}_n$ is dense in $L^2([0, 1]^n)$ for every $n \geq 1$ the mapping I_n can be extended to an isometry from $L^2([0, 1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension. Note also that I_n can be viewed as an iterated stochastic integral

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_n};$$

here the integrals are of Itô type; this formula is easy to show for elementary f 's, and follows for general $f \in L^2([0, 1]^n)$ by a density argument.

The product for two multiple integrals can be expanded into a sum of multiple integrals (see [14]): if $f \in L^2([0, 1]^n)$ and $g \in L^2([0, 1]^m)$ are symmetric functions, then

$$I_n(f)I_m(g) = \sum_{l=0}^{m \wedge n} l! C_m^l C_n^l I_{m+n-2l}(f \otimes_l g), \quad (3)$$

where the contraction $f \otimes_l g$ belongs to $L^2([0, 1]^{m+n-2l})$ for $l = 0, 1, \dots, m \wedge n$ and it is given by

$$\begin{aligned} & (f \otimes_l g)(s_1, \dots, s_{n-l}, t_1, \dots, t_{m-l}) \\ &= \int_{[0, 1]^l} f(s_1, \dots, s_{n-l}, u_1, \dots, u_l) g(t_1, \dots, t_{m-l}, u_1, \dots, u_l) du_1 \cdots du_l. \end{aligned} \quad (4)$$

When $l = 0$, we use, throughout this paper, the notation $f \otimes g := f \otimes_0 g$.

2.2. The Rosenblatt process. The Rosenblatt process $(Z^{(H)}(t))_{t \in [0,1]}$ appears as a limit in the so-called *noncentral limit theorem* (see [7], [18], [10]). It is not a Gaussian process and can be defined through its representation as double iterated integral with respect to a standard Wiener process (see [19]). More precisely, the Rosenblatt process with self-similarity order $H \in (1/2, 1)$ is defined by

$$Z_t^{(H)} := \int_0^t \int_0^t L_t(y_1, y_2) dW_{y_1} dW_{y_2}, \quad (5)$$

where $(W_t, t \in [0, 1])$ is a Brownian motion,

$$\begin{aligned} L_t^{(H)}(y_1, y_2) &:= L_t(y_1, y_2) \\ &= d(H) 1_{[0,t]}(y_1) 1_{[0,t]}(y_2) \int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du, \end{aligned} \quad (6)$$

with

$$H' := \frac{H+1}{2} \quad \text{and} \quad d(H) := \frac{1}{H+1} \left(\frac{H}{2(2H-1)} \right)^{-1/2}$$

and with K^H the standard kernel appearing in the Wiener integral representation of the fractional Brownian motion (for $t > s$ and $H > 1/2$):

$$K^H(t, s) := c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du, \quad (7)$$

where $c_H = (H(2H-1)/\beta(2-2H, H-1/2))^{1/2}$ and $\beta(\cdot, \cdot)$ is the beta function. The derivative of K^H is

$$\frac{\partial K^H}{\partial t}(t, s) =: \partial_1 K^H(t, s) = c_H \left(\frac{s}{t} \right)^{1/2-H} (t-s)^{H-3/2}. \quad (8)$$

The two-parameter function L_t given by (6) will be called the kernel of the Rosenblatt process. The following key relation is crucial in our calculation and will be used repeatedly in the paper:

$$\int_0^{u \wedge v} \partial_1 K^{H'}(u, y) \partial_1 K^{H'}(v, y) dy = a(H) |u-v|^{2H'-2} \quad (9)$$

with $a(H) = H'(2H'-1)$. Recall several main properties of the Rosenblatt process:

(a) it is H -self-similar in the sense that for any $c > 0$, we have $(Z_{ct}^{(H)}) =^{(d)} (c^H Z_t^{(H)})$, where $=^{(d)}$ means equivalence of all finite dimensional distributions;

(b) it has stationary increments, that is, the joint distribution of $(Z_{t+h}^{(H)} - Z_h^{(H)}, t \in [0, 1])$ is independent of $h > 0$;

(c) $\mathbf{E} |Z_t^{(H)}|^p < \infty$ for any $p > 0$, and $Z^{(H)}$ has the same variance and covariance that a standard fractional Brownian motion with parameter H ;

(d) the Rosenblatt process is Hölder continuous, of order $\delta < H$; this can easily be obtained by the Kolmogorov continuity criterion.

3. Renormalization of the cubic variation.

3.1. Estimation of the mean square. We will study in this paper the cubic variation of the Rosenblatt process obtained by putting $p = 3$ in (1):

$$V^{3,N} = \frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{(Z_{(i+1)/N}^{(H)} - Z_{i/N}^{(H)})^3}{\mathbf{E}(Z_{(i+1)/N}^{(H)} - Z_{i/N}^{(H)})^3} - 1 \right). \quad (10)$$

Note that this expression is immaterial in the case of the fractional Brownian motion because the third moment of a centered Gaussian random variable is zero. By denoting for $i = 1, \dots, N$

$$f_{i,N} = L_{(i+1)/N}^{(H)} - L_{i/N}^{(H)}$$

we obtain $Z_{(i+1)/N}^{(H)} - Z_{i/N}^{(H)} = I_2(f_{i,N})$, where I_2 is a multiple integral of order 2 as defined in Subsection 2.1, and then

$$V^{3,N} = \frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{(I_2(f_{i,N}))^3}{\mathbf{E}(I_2(f_{i,N}))^3} - 1 \right).$$

By using the product formula for multiple Wiener–Itô integrals (3), for any symmetric function $f \in L^2([0, 1]^2)$, we obtain

$$\begin{aligned} (I_2(f))^3 &= I_6((f \tilde{\otimes} f) \otimes f) + 8I_4((f \tilde{\otimes} f) \otimes_1 f) + 4I_4((f \otimes_1 f) \otimes f) \\ &\quad + 12I_2((f \tilde{\otimes} f) \otimes_2 f) + 16I_2((f \otimes_1 f) \otimes_1 f) + 2\langle f, f \rangle_{L^2([0,1]^2)} I_2(f) \\ &\quad + 8\langle (f \otimes_1 f), f \rangle_{L^2([0,1]^2)}. \end{aligned}$$

Here and in the sequel $f \tilde{\otimes} f$ denotes the symmetrization of the function $f \otimes f$ which is not necessarily symmetric even if f is symmetric. Applying this to $f = f_{i,N}$ we obtain

$$(I_2(f_{i,N}))^3 = 8(f_{i,N} \otimes_1 f_{i,N}) \otimes_2 f_{i,N} + I_2(g_{i,N}) + 4I_4(h_{i,N}) + I_6((f_{i,N} \tilde{\otimes} f_{i,N}) \otimes f_{i,N}), \quad (11)$$

where

$$g_{i,N} = 2\|f_{i,N}\|_{L^2}^2 f_{i,N} + 12(f_{i,N} \tilde{\otimes} f_{i,N}) \otimes_2 f_{i,N} + 16(f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N}, \quad (12)$$

$$h_{i,N} = 2(f_{i,N} \tilde{\otimes} f_{i,N}) \otimes_1 f_{i,N} + f_{i,N} \otimes (f_{i,N} \otimes_1 f_{i,N}). \quad (13)$$

Note that $g_{i,N} \in L^2([0, 1]^2)$ and $h_{i,N} \in L^2([0, 1]^4)$. On the other hand, we can simplify a little bit the above expressions, since

$$(f_{i,N} \tilde{\otimes} f_{i,N}) \otimes_2 f_{i,N} = \frac{1}{3} \|f_{i,N}\|_{L^2}^2 f_{i,N} + \frac{2}{3} (f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N}.$$

Hence the kernel of the second chaos term can be written as

$$g_{i,N} = 6\|f_{i,N}\|_{L^2}^2 f_{i,N} + 24(f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N}.$$

We start with the following lemma, where we compute the cubic mean of the increment of the Rosenblatt process. We already observed a significant difference from the Gaussian case: this cubic mean is not zero.

Lemma 1. *Let $(Z_t^{(H)})_{t \in [0,1]}$ be a Rosenblatt process with self-similarity index $H \in (1/2, 1)$. Then, for every $s, t \in [0, 1]$*

$$\mathbf{E} (Z_t^{(H)} - Z_s^{(H)})^3 = C(H) |t - s|^{3H}, \quad (14)$$

where

$$C(H) = 8a(H)^3 d(H)^3 \int_{[0,1]^3} (|u - v| |u - u'| |v - u'|)^{2H'-2} du du' dv. \quad (15)$$

P r o o f. Let us denote

$$f_{s,t}(x, y) = L_t(x, y) - L_s(x, y),$$

where L is the kernel of the Rosenblatt process given by (6) and $x, y \in [0, 1]$. We have, by using relation (9),

$$\begin{aligned} (f_{s,t} \otimes_1 f_{s,t})(x, y) &= \int_0^1 f_{s,t}(x, z) f_{s,t}(y, z) dz \\ &= d(H)^2 a(H) \left(1_{[0,t]}(x, y) \int_x^t \int_y^t \partial_1 K^{H'}(u, x) \partial_1 K^{H'}(v, y) |u - v|^{2H'-2} dv du \right. \\ &\quad - 1_{[0,t]}(x) 1_{[0,s]}(y) \int_x^t \int_y^s \partial_1 K^{H'}(u, x) \partial_1 K^{H'}(v, y) |u - v|^{2H'-2} dv du \\ &\quad - 1_{[0,s]}(x) 1_{[0,t]}(y) \int_x^s \int_y^t \partial_1 K^{H'}(u, x) \partial_1 K^{H'}(v, y) |u - v|^{2H'-2} dv du \\ &\quad \left. + 1_{[0,s]}(x, y) \int_x^s \int_y^s \partial_1 K^{H'}(u, x) \partial_1 K^{H'}(v, y) |u - v|^{2H'-2} dv du \right). \end{aligned}$$

The computation of the cubic mean of a multiple integral in the second chaos (11) implies

$$\mathbf{E} (Z_t^{(H)} - Z_s^{(H)})^3 = 8 \langle f_{s,t} \otimes_1 f_{s,t}, f_{s,t} \rangle_{L^2([0,1]^2)}.$$

We compute, by (9):

$$\begin{aligned} \langle f_{s,t} \otimes_1 f_{s,t}, f_{s,t} \rangle_{L^2([0,1]^2)} &= \int_{[0,1]^2} (f_{s,t} \otimes_1 f_{s,t})(x, y) f_{s,t}(x, y) dx dy \\ &= d(H)^3 a(H)^3 \int_s^t \int_s^t \int_s^t (|u - v| |u - u'| |v - u'|)^{2H'-2} du du' dv. \end{aligned}$$

By the change of variables $\bar{u} = (u - s)/(t - s)$ we will transform the integrals on $[s, t]$ into integrals from 0 to 1. We immediately obtain the relation (14). Lemma 1 is proved.

To calculate $\mathbf{E}(V^{3,N})^2$ we apply the above result and we obtain

$$\begin{aligned}\mathbf{E}(I_2(f_{i,N}))^3 &= 8(f_{i,N} \otimes_1 f_{i,N}) \otimes_2 f_{i,N} \\ &= 8d(H)^3 a(H)^3 \int_{I_i} \int_{I_i} \int_{I_i} dy_1 dy_2 dy_3 (|y_1 - y_2| |y_2 - y_3| |y_3 - y_1|)^{2H'-2} \\ &= 8 \frac{d(H)^3 a(H)^3}{N^{6H'-3}} \int_{[0,1]^3} dy_1 dy_2 dy_3 (|y_1 - y_2| |y_2 - y_3| |y_3 - y_1|)^{2H'-2} \\ &= C(H) N^{-(6H'-3)} = C(H) N^{-3H},\end{aligned}$$

where $I_i = [i/N, (i+1)/N]$, $a(H) = H(H+1)/2$, and $C(H)$ is defined in (15).

We can write the expression of the statistics $V^{3,N}$ as follows:

$$\begin{aligned}V^{3,N} &= \frac{1}{C(H) N^{1-3H}} \sum_{i=0}^{N-1} ((I_2(f_{i,N}))^3 - \mathbf{E}(I_2(f_{i,N}))^3) = \frac{1}{C(H) N^{1-3H}} \\ &\quad \times \sum_{i=0}^{N-1} (I_2(g_{i,N}) + 4I_4(h_{i,N}) + I_6((f_{i,N} \tilde{\otimes} f_{i,N}) \otimes f_{i,N})).\end{aligned}\quad (16)$$

We prove next the following renormalization result.

Proposition 1. *Let $V^{3,N}$ be the cubic variation statistics of the Rosenblatt process. Then*

$$\mathbf{E}(N^{1-H} V^{3,N})^2 \longrightarrow \overline{C}(H), \quad N \rightarrow \infty, \quad (17)$$

where $\tilde{C}(H) := C(H)^2 C_0(H)$ with

$$C_0(H) = 9 + 36C'(H)H(2H-1) + 144[C'(H)H(2H-1)]^2.$$

P r o o f. The isometry property of multiple Wiener–Itô integrals and relation (16) imply

$$\begin{aligned}\mathbf{E}(V^{3,N})^2 &= \frac{1}{N^2 (\mathbf{E}(I_2(f_{i,N}))^3)^2} \\ &\quad \times \sum_{i,j=0}^{N-1} \left[\mathbf{E}(I_2(g_{i,N}) I_2(g_{j,N})) + 16 \mathbf{E}(I_4(h_{i,N}) I_4(h_{j,N})) \right. \\ &\quad \left. + \mathbf{E}(I_6((f_{i,N} \tilde{\otimes} f_{i,N}) \otimes f_{i,N}) I_6((f_{j,N} \tilde{\otimes} f_{j,N}) \otimes f_{j,N})) \right] \\ &= \frac{1}{C(H)^2 N^{2-6H}} \left[\sum_{i,j=0}^{N-1} 2! \langle g_{i,N}, g_{j,N} \rangle_{L^2([0,1]^2)} + \sum_{i,j=0}^{N-1} 4! \cdot 16 \langle \tilde{h}_{i,N}, \tilde{h}_{j,N} \rangle_{L^2([0,1]^4)} \right. \\ &\quad \left. + \sum_{i,j=0}^{N-1} 6! \langle (f_{i,N} \tilde{\otimes} f_{i,N}) \otimes f_{i,N}, (f_{j,N} \tilde{\otimes} f_{j,N}) \otimes f_{j,N} \rangle_{L^2([0,1]^6)} \right] \\ &=: \frac{1}{C(H)^2 N^{2-6H}} [A_N^{(2)} + B_N^{(4)} + D_N^{(6)}].\end{aligned}$$

We use the notation $A_N^{(2)}$ to indicate that this term comes from the estimation of the second chaos summand of $V^{3,N}$, and similarly for the terms $B_N^{(4)}$ and $D_N^{(6)}$. We will try to estimate all the three terms above to see which is the dominant term.

Estimation of the term $A_N^{(2)}$. Taking into account the expression of the second chaos kernel $g_{i,N}$ (12), we get

$$\begin{aligned} A_N^{(2)} &= 2 \sum_{i,j=0}^{N-1} \left[36 \|f_{i,N}\|_{L^2([0,1]^2)}^2 \|f_{j,N}\|_{L^2([0,1]^2)}^2 \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^2)} \right. \\ &\quad + 144 \|f_{i,N}\|_{L^2([0,1]^2)}^2 \langle f_{i,N}, (f_{j,N} \otimes_1 f_{j,N}) \otimes_1 f_{j,N} \rangle_{L^2([0,1]^2)} \\ &\quad \left. + (24)^2 \langle (f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N}, (f_{j,N} \otimes_1 f_{j,N}) \otimes_1 f_{j,N} \rangle_{L^2([0,1]^2)} \right] \\ &=: 2 \left(36 A_{1,N}^{(2)} + 144 A_{2,N}^{(2)} + (24)^2 A_{3,N}^{(2)} \right). \end{aligned}$$

Let us evaluate the term $A_{1,N}^{(2)}$. We have

$$\begin{aligned} 2! \|f_{i,N}\|_{L^2([0,1]^2)}^2 &= \mathbf{E} \left| Z_{(i+1)/N}^{(H)} - Z_{i/N}^{(H)} \right|^2 = N^{-2H}, \\ 2 \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^2)} &= \mathbf{E} \left(Z_{(i+1)/N}^{(H)} - Z_{i/N}^{(H)} \right) \left(Z_{(j+1)/N}^{(H)} - Z_{j/N}^{(H)} \right). \end{aligned}$$

Hence

$$\begin{aligned} A_{1,N}^{(2)} &= \sum_{i,j=0}^{N-1} \left[\|f_{i,N}\|_{L^2([0,1]^2)}^2 \|f_{j,N}\|_{L^2([0,1]^2)}^2 \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^2)} \right] \\ &= \frac{1}{8} N^{-4H} \sum_{i,j=0}^{N-1} \mathbf{E} \left(Z_{(i+1)/N}^{(H)} - Z_{i/N}^{(H)} \right) \left(Z_{(j+1)/N}^{(H)} - Z_{j/N}^{(H)} \right) = \frac{1}{8} N^{-4H} \end{aligned}$$

because $\mathbf{E} \sum_{i,j=0}^{N-1} (Z_{(i+1)/N}^{(H)} - Z_{i/N}^{(H)}) (Z_{(j+1)/N}^{(H)} - Z_{j/N}^{(H)}) = \mathbf{E} (Z_1^{(H)})^2 = 1$, and we have

$$\lim_{N \rightarrow \infty} N^{4H} A_{1,N}^{(2)} = \frac{1}{8}. \quad (18)$$

To evaluate $A_{3,N}^{(2)}$, note that

$$\begin{aligned} &d(H)^{-2} a(H)^{-1} (f_{i,N} \otimes_1 f_{i,N})(x, y) \\ &= 1_{[0, i/N]}^{\otimes 2}(x, y) \int_{I_i} \int_{I_i} \partial_1 K^{H'}(u_1, x) \partial_1 K^{H'}(u_2, y) |u_1 - u_2|^{2H'-2} du_2 du_1 \\ &\quad + 1_{[0, i/N]}(x) 1_{I_i}(y) \int_{I_i} \int_y^{(i+1)/N} \partial_1 K^{H'}(u_1, x) \partial_1 K^{H'}(u_2, y) \\ &\quad \times |u_1 - u_2|^{2H'-2} du_2 du_1 \end{aligned}$$

$$\begin{aligned}
& + 1_{[0,i/N]}(y)1_{I_i}(x) \int_x^{(i+1)/N} \int_{I_i} \partial_1 K^{H'}(u_1, x) \partial_1 K^{H'}(u_2, y) \\
& \quad \times |u_1 - u_2|^{2H'-2} du_2 du_1 \\
& + 1_{I_i}(x)1_{I_i}(y) \int_x^{(i+1)/N} \int_y^{(i+1)/N} \partial_1 K^{H'}(u_1, x) \partial_1 K^{H'}(u_2, y) \\
& \quad \times |u_1 - u_2|^{2H'-2} du_2 du_1.
\end{aligned}$$

Sometimes the following expressions are useful:

$$\begin{aligned}
(f_{i,N} \otimes_1 f_{i,N})(x, y) &= d(H)^2 a(H) 1_{[0,(i+1)/N]}^{\otimes 2}(x, y) \\
&\times \int_{I_i} \int_{I_i} \partial_1 K^{H'}(u_1, x) \partial_1 K^{H'}(u_2, y) |u_1 - u_2|^{2H'-2} du_2 du_1 \quad (19)
\end{aligned}$$

and

$$\begin{aligned}
d(H)^{-1} f_{i,N}(x, z) &= 1_{[0,i/N]}^{\otimes 2}(x, z) \int_{I_i} \partial_1 K^{H'}(u_3, x) \partial_1 K^{H'}(u_3, z) du_3 \\
&+ 1_{[0,i/N]}(x)1_{I_i}(z) \int_z^{(i+1)/N} \partial_1 K^{H'}(u_3, x) \partial_1 K^{H'}(u_3, z) du_3 \\
&+ 1_{[0,i/N]}(z)1_{I_i}(x) \int_x^{(i+1)/N} \partial_1 K^{H'}(u_3, x) \partial_1 K^{H'}(u_3, z) du_3 \\
&+ 1_{I_i}(x)1_{I_i}(z) \int_{I_i} \int_{I_i} \partial_1 K^{H'}(u_3, x) \partial_1 K^{H'}(u_3, z) du_3,
\end{aligned}$$

or

$$f_{i,N}(x, z) = d(H) 1_{[0,(i+1)/N]}^{\otimes 2}(x, z) \int_{I_i} \partial_1 K^{H'}(u_3, x) \partial_1 K^{H'}(u_3, z) du_3. \quad (20)$$

Therefore

$$\begin{aligned}
((f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N})(y, z) &= d(H)^3 a(H)^2 \left(1_{[0,(i+1)/N]}^{\otimes 2}(y, z) \right. \\
&\times \left. \int_{I_i} \int_{I_i} \int_{I_i} (|u_1 - u_2| |u_1 - u_3|)^{2H'-2} \partial_1 K^{H'}(u_2, y) \partial_1 K^{H'}(u_3, z) du_3 du_2 du_1 \right).
\end{aligned}$$

The norm has a nicer expression. Using the change of variables $\bar{u} = (u - i/N)N$ (which is now usual and can be used systematically) we have

$$\begin{aligned}
&\langle (f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N}, (f_{j,N} \otimes_1 f_{j,N}) \otimes_1 f_{j,N} \rangle_{L^2([0,1]^2)} \\
&= d(H)^6 a(H)^6 \int_{I_i^3} du_1 du_2 du_3 \int_{I_j^3} dv_1 dv_2 dv_3 \\
&\quad \times |u_1 - u_2|^{2H'-2} |u_1 - u_3|^{2H'-2} |v_1 - v_2|^{2H'-2} |v_1 - v_3|^{2H'-2} \\
&\quad \times |u_2 - v_2|^{2H'-2} |u_3 - v_3|^{2H'-2} \\
&= \frac{d(H)^6 a(H)^6}{N^{12H'-6}} \int_{[0,1]^3} \int_{[0,1]^3} |v_1 - v_2|^{2H'-2} |v_2 - v_3|^{2H'-2} \\
&\quad \times |v_3 - v_4 + i - j|^{2H'-2} |v_4 - v_5|^{2H'-2} |v_5 - v_6|^{2H'-2} \\
&\quad \times |v_6 - v_1 + j - i|^{2H'-2} dv_1 \cdots dv_6.
\end{aligned}$$

The rate of convergence of all terms comes actually from how many product $|u - v|^{2H'-2}$ with $u \in I_i$ and $v \in I_j$ we have. Hence

$$\begin{aligned}
A_{3,N}^{(2)} &= \frac{d(H)^6 a(H)^6}{N^{12H'-6}} \sum_{i,j=1}^N \int_{[0,1]^3} \int_{[0,1]^3} |v_1 - v_2|^{2H'-2} |v_2 - v_3|^{2H'-2} \\
&\quad \times |v_3 - v_4 + i - j|^{2H'-2} |v_4 - v_5|^{2H'-2} |v_5 - v_6|^{2H'-2} \\
&\quad \times |v_6 - v_1 + j - i|^{2H'-2} dv_1 \cdots dv_6 \\
&= \frac{2d(H)^6 a(H)^6}{N^{12H'-6}} \sum_{i>j=1}^N \int_{[0,1]^3} \int_{[0,1]^3} |v_1 - v_2|^{2H'-2} |v_2 - v_3|^{2H'-2} \\
&\quad \times |v_3 - v_4 + i - j|^{2H'-2} |v_4 - v_5|^{2H'-2} |v_5 - v_6|^{2H'-2} \\
&\quad \times |v_1 - v_6 + i - j|^{2H'-2} dv_1 \cdots dv_6 \\
&= \frac{2d(H)^6 a(H)^6}{N^{12H'-6}} \sum_{k=0}^{N-1} (N-k) \int_{[0,1]^3} \int_{[0,1]^3} |v_1 - v_2|^{2H'-2} |v_2 - v_3|^{2H'-2} \\
&\quad \times |v_3 - v_4 + k|^{2H'-2} |v_4 - v_5|^{2H'-2} |v_5 - v_6|^{2H'-2} \\
&\quad \times |v_1 - v_6 + k|^{2H'-2} dv_1 \cdots dv_6.
\end{aligned}$$

We put

$$\begin{aligned}
\overline{A}_{3,N}^{(2)} &:= \frac{1}{N^{12H'-6}} \sum_{k=0}^{N-1} (N-k) |v_3 - v_4 + k|^{2H'-2} |v_1 - v_6 + k|^{2H'-2} \\
&= \frac{1}{N^{8H'-4}} \frac{1}{N} \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right) \left| \frac{v_3 - v_4}{N} + \frac{k}{N} \right|^{2H'-2} \left| \frac{v_1 - v_6}{N} + \frac{k}{N} \right|^{2H'-2}
\end{aligned}$$

and we conclude easily by a Riemann sum convergence that

$$N^{4H} \overline{A}_{3,N}^{(2)} = N^{8H'-4} \overline{A}_{3,N}^{(2)} \xrightarrow{N \rightarrow \infty} \int_0^1 (1-x) x^{4H'-4} dx = \frac{1}{2H-1} - \frac{1}{2H},$$

because the terms $(v_3 - v_4)/N$ are negligible with respect to k/N for large enough k . This implies that

$$N^{4H} A_{3,N}^{(2)} \xrightarrow{N \rightarrow \infty} \frac{d(H)^6 a(H)^6}{H(2H-1)} (C'(H))^2 = \frac{H^2(2H-1)^2}{8} (C'(H))^2, \quad (21)$$

where $C'(H) = \int_{[0,1]^3} |v_1 - v_2|^{2H'-2} |v_2 - v_3|^{2H'-2} dv_1 dv_2 dv_3$. Now, we estimate the term $A_{2,N}^{(2)}$. We have

$$\begin{aligned}
&\langle f_{i,N}, (f_{j,N} \otimes_1 f_{j,N}) \otimes_1 f_{j,N} \rangle_{L^2([0,1]^2)} \\
&= d(H)^4 a(H)^4 \int_{I_i} \int_{I_j^3} |u_1 - u_2|^{2H'-2} |u_2 - u_3|^{2H'-2} |u_3 - u_4|^{2H'-2} \\
&\quad \times |u_4 - u_1|^{2H'-2} du_1 \cdots du_4
\end{aligned}$$

$$= \frac{d(H)^4 a(H)^4}{N^{8H'-4}} \int_{[0,1]^4} |v_1 - v_2 + i - j|^{2H'-2} |v_2 - v_3|^{2H'-2} |v_3 - v_4|^{2H'-2} \\ \times |v_4 - v_1 + j - i|^{2H'-2} dv_1 \cdots dv_4.$$

Then

$$\begin{aligned} A_{2,N}^{(2)} &= \|f_{i,N}\|_{L^2([0,1]^2)}^2 \frac{d(H)^4 a(H)^4}{N^{8H'-4}} \int_{[0,1]^4} |v_2 - v_3|^{2H'-2} |v_3 - v_4|^{2H'-2} \\ &\quad \times \left(\sum_{i,j=1}^N |v_1 - v_2 + i - j|^{2H'-2} |v_4 - v_1 + j - i|^{2H'-2} \right) dv_1 \cdots dv_4 \\ &= \frac{N^{-2H}}{2} \frac{d(H)^4 a(H)^4}{N^{8H'-4}} \int_{[0,1]^4} |v_2 - v_3|^{2H'-2} |v_3 - v_4|^{2H'-2} \\ &\quad \times \left(2 \sum_{k=0}^{N-1} (N-k) |v_1 - v_2 + k|^{2H'-2} |v_4 - v_1 + k|^{2H'-2} \right) dv_1 \cdots dv_4 \\ &= N^{-2H} \frac{d(H)^4 a(H)^4}{N^{4H'-2}} \int_{[0,1]^4} |v_2 - v_3|^{2H'-2} |v_3 - v_4|^{2H'-2} \\ &\quad \times \left(\frac{1}{N} \sum_{k=0}^{N-1} \left(1 - \frac{k}{N} \right) \left| \frac{v_1 - v_2}{N} + \frac{k}{N} \right|^{2H'-2} \left| \frac{v_4 - v_1}{N} + \frac{k}{N} \right|^{2H'-2} \right) dv_1 \cdots dv_4 \\ &= \frac{d(H)^4 a(H)^4}{N^{4H}} \int_{[0,1]^4} |v_2 - v_3|^{2H'-2} |v_3 - v_4|^{2H'-2} \\ &\quad \times \left(\frac{1}{N} \sum_{k=0}^{N-1} \left(1 - \frac{k}{N} \right) \left| \frac{v_1 - v_2}{N} + \frac{k}{N} \right|^{2H'-2} \left| \frac{v_4 - v_1}{N} + \frac{k}{N} \right|^{2H'-2} \right) dv_1 \cdots dv_4. \end{aligned}$$

We obtain that $N^{4H} A_{2,N}^{(2)}$ converges as $N \rightarrow \infty$ to

$$\begin{aligned} &\left(\int_0^1 (1-x) x^{4H'-4} dx \right) \left(d(H)^4 a(H)^4 \right. \\ &\quad \times \left. \int_{[0,1]^3} |v_2 - v_3|^{2H'-2} |v_3 - v_4|^{2H'-2} dv_2 dv_3 dv_4 \right) \\ &= \left(\frac{1}{2H-1} - \frac{1}{2H} \right) d(H)^4 a(H)^4 \\ &\quad \times \int_{[0,1]^3} |v_2 - v_3|^{2H'-2} |v_3 - v_4|^{2H'-2} dv_2 dv_3 dv_4. \end{aligned}$$

Thus

$$N^{4H} A_{2,N}^{(2)} \xrightarrow{N \rightarrow \infty} C'(H) \frac{H(2H-1)}{8}. \quad (22)$$

From (18), (21), and (22), we obtain that

$$N^{4H} A_N^{(2)} = 2 \left(36 N^{4H} A_{1,N}^{(2)} + 144 N^{4H} A_{2,N}^{(2)} + (24)^2 N^{4H} A_{3,N}^{(2)} \right) \quad (23)$$

converges to $(9 + 36C'(H)H(2H - 1) + 144[C'(H)H(2H - 1)]^2) =: C_0(H)$ as $N \rightarrow \infty$. Consequently

$$\frac{N^{4H} A_N^{(2)}}{C_0(H)} \xrightarrow{N \rightarrow \infty} 1. \quad (24)$$

Estimation of the term $D_N^{(6)}$. Now, we study the convergence of $D_N^{(6)}$. By the symmetry property of every $f_{i,N}$, $i = 0, \dots, N-1$, on $[0, 1]^2$, there exist positive combinatorial constants c_1 , c_2 , and c_3 such that

$$\begin{aligned} D_N^{(6)} &= \sum_{i,j=0}^{N-1} 6! \langle (f_{i,N} \tilde{\otimes} f_{j,N}) \tilde{\otimes} f_{j,N}, (f_{j,N} \tilde{\otimes} f_{i,N}) \tilde{\otimes} f_{i,N} \rangle_{L^2([0,1]^6)} \\ &= c_1 \sum_{i,j=0}^{N-1} \left(\langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^2)} \right)^3 + c_2 \sum_{i,j=0}^{N-1} \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^2)} \\ &\quad \times \int_{[0,1]^4} f_{i,N}(x_1, x_2) f_{j,N}(x_2, x_3) f_{i,N}(x_3, x_4) f_{j,N}(x_4, x_1) dx_1 \cdots dx_4 \\ &\quad + c_3 \sum_{i,j=0}^{N-1} \int_{[0,1]^6} f_{i,N}(x_1, x_2) f_{j,N}(x_2, x_3) f_{i,N}(x_3, x_4) f_{j,N}(x_4, x_5) \\ &\quad \times f_{i,N}(x_5, x_6) f_{j,N}(x_6, x_1) dx_1 \cdots dx_6 \\ &=: c_1 D_{1,N}^{(6)} + c_2 D_{2,N}^{(6)} + c_3 D_{3,N}^{(6)}. \end{aligned}$$

By using the same argument as above, we obtain

$$\begin{aligned} D_{1,N}^{(6)} &= \sum_{i,j=0}^{N-1} \left(\langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^2)} \right)^3 \\ &= \frac{d(H)^6 a(H)^6}{N^{6H}} \sum_{i,j=0}^{N-1} \left(\int_{[0,1]^2} |x_1 - x_2 + i - j|^{2H-2} dx_1 dx_2 \right)^3 \\ &= \frac{d(H)^6 a(H)^6}{N^{6H}} 2 \sum_{k=0}^{N-1} (N - k) \left(\int_{[0,1]^2} |x_1 - x_2 + k|^{2H-2} dx_1 dx_2 \right)^3 \\ &= \frac{2d(H)^6 a(H)^6}{N^4} \sum_{k=0}^{N-1} \frac{1}{N} \left(1 - \frac{k}{N} \right) \left(\int_{[0,1]^2} \left| \frac{x_1 - x_2}{N} + \frac{k}{N} \right|^{2H-2} dx_1 dx_2 \right)^3, \end{aligned}$$

and clearly, since $H < 1$, we have

$$\lim_{N \rightarrow \infty} N^{4H} D_{1,N}^{(6)} = 0. \quad (25)$$

In the same way, we obtain

$$\begin{aligned} D_{2,N}^{(6)} &= \sum_{i,j=0}^{N-1} \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^2)} \\ &\quad \times \int_{[0,1]^4} f_{i,N}(x_1, x_2) f_{j,N}(x_2, x_3) f_{i,N}(x_3, x_4) f_{j,N}(x_4, x_1) dx_1 \cdots dx_4 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j=0}^{N-1} N^{-4} d(H)^4 a(H)^4 \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^2)} \int_{[0,1]^4} \left| \frac{x_1 - x_2}{N} + \frac{i - j}{N} \right|^{2H'-2} \\
&\quad \times \left| \frac{x_3 - x_2}{N} + \frac{i - j}{N} \right|^{2H'-2} \left| \frac{x_3 - x_4}{N} + \frac{i - j}{N} \right|^{2H'-2} \\
&\quad \times \left| \frac{x_1 - x_4}{N} + \frac{i - j}{N} \right|^{2H'-2} dx_1 \cdots dx_4 \\
&= \frac{2d(H)^6 a(H)^6}{N^6} \sum_{k=0}^{N-1} (N - k) \int_{[0,1]^2} \left| \frac{x_5 - x_6}{N} + \frac{k}{N} \right|^{2H-2} dx_5 dx_6 \\
&\quad \times \int_{[0,1]^4} \left| \frac{x_1 - x_2}{N} + \frac{k}{N} \right|^{2H'-2} \left| \frac{x_3 - x_2}{N} + \frac{k}{N} \right|^{2H'-2} \left| \frac{x_3 - x_4}{N} + \frac{k}{N} \right|^{2H'-2} \\
&\quad \times \left| \frac{x_1 - x_4}{N} + \frac{k}{N} \right|^{2H'-2} dx_1 \cdots dx_4 \\
&= \frac{2d(H)^6 a(H)^6}{N^4} \sum_{k=0}^{N-1} \frac{1}{N} \left(1 - \frac{k}{N} \right) \int_{[0,1]^2} \left| \frac{x_5 - x_6}{N} + \frac{k}{N} \right|^{2H-2} dx_5 dx_6 \\
&\quad \times \int_{[0,1]^4} \left| \frac{x_1 - x_2}{N} + \frac{k}{N} \right|^{2H'-2} \left| \frac{x_3 - x_2}{N} + \frac{k}{N} \right|^{2H'-2} \left| \frac{x_3 - x_4}{N} + \frac{k}{N} \right|^{2H'-2} \\
&\quad \times \left| \frac{x_1 - x_4}{N} + \frac{k}{N} \right|^{2H'-2} dx_1 \cdots dx_4.
\end{aligned}$$

This implies that

$$\lim_{N \rightarrow \infty} N^{4H} D_{2,N}^{(6)} = 0. \quad (26)$$

In the same manner as in previous results, we have

$$\begin{aligned}
D_{3,N}^{(6)} &= \sum_{i,j=0}^{N-1} \int_{[0,1]^6} f_{i,N}(x_1, x_2) f_{j,N}(x_2, x_3) f_{i,N}(x_3, x_4) f_{j,N}(x_4, x_5) \\
&\quad \times f_{i,N}(x_5, x_6) f_{j,N}(x_6, x_1) dx_1 \cdots dx_6 \\
&= \frac{d(H)^6 a(H)^6}{N^6} \sum_{i,j=0}^{N-1} \int_{[0,1]^6} \left| \frac{x_1 - x_2}{N} + \frac{i - j}{N} \right|^{2H'-2} \left| \frac{x_3 - x_2}{N} + \frac{i - j}{N} \right|^{2H'-2} \\
&\quad \times \left| \frac{x_3 - x_4}{N} + \frac{i - j}{N} \right|^{2H'-2} \left| \frac{x_5 - x_4}{N} + \frac{i - j}{N} \right|^{2H'-2} \\
&\quad \times \left| \frac{x_5 - x_6}{N} + \frac{i - j}{N} \right|^{2H'-2} \left| \frac{x_1 - x_6}{N} + \frac{i - j}{N} \right|^{2H'-2} dx_1 \cdots dx_6 \\
&= \frac{2d(H)^6 a(H)^6}{N^4} \sum_{k=0}^{N-1} \frac{1}{N} \left(1 - \frac{k}{N} \right) \int_{[0,1]^4} \left| \frac{x_1 - x_2}{N} + \frac{k}{N} \right|^{2H'-2} \\
&\quad \times \left| \frac{x_3 - x_2}{N} + \frac{k}{N} \right|^{2H'-2} \left| \frac{x_3 - x_4}{N} + \frac{k}{N} \right|^{2H'-2}
\end{aligned}$$

$$\begin{aligned} & \times \left| \frac{x_5 - x_4}{N} + \frac{k}{N} \right|^{2H'-2} \left| \frac{x_5 - x_6}{N} + \frac{k}{N} \right|^{2H'-2} \\ & \times \left| \frac{x_1 - x_6}{N} + \frac{k}{N} \right|^{2H'-2} dx_1 \cdots dx_6. \end{aligned}$$

Hence

$$\lim_{N \rightarrow \infty} N^{4H} D_{3,N}^{(6)} = 0. \quad (27)$$

Thus, from (25)–(27), it follows that

$$\lim_{N \rightarrow \infty} N^{4H} D_N^{(6)} = 0. \quad (28)$$

Estimation of the term $B_N^{(4)}$. The same argument as above yields that there exist constants c'_1 and c'_2 such that

$$\begin{aligned} & \langle \tilde{h}_{i,N}, \tilde{h}_{j,N} \rangle_{L^2([0,1]^4)} \\ &= c'_1 \sum_{i,j=0}^{N-1} \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^2)} \\ & \quad \times \int_{[0,1]^4} f_{i,N}(x_1, x_2) f_{j,N}(x_2, x_3) f_{i,N}(x_3, x_4) f_{j,N}(x_4, x_1) dx_1 \cdots dx_4 \\ & \quad + c'_2 \sum_{i,j=0}^{N-1} \int_{[0,1]^6} f_{i,N}(x_1, x_2) f_{j,N}(x_2, x_3) f_{i,N}(x_3, x_4) f_{j,N}(x_4, x_5) \\ & \quad \times f_{i,N}(x_5, x_6) f_{j,N}(x_6, x_1) dx_1 \cdots dx_6 \\ &= c'_1 D_{2,N}^{(6)} + c'_2 D_{3,N}^{(6)}. \end{aligned}$$

The same terms as in the estimation of the sixth chaos kernel appear. Thus, from the convergences (26) and (27),

$$\begin{aligned} \lim_{N \rightarrow \infty} N^{4H} B_N^{(4)} &= \lim_{N \rightarrow \infty} N^{4H} \sum_{i,j=0}^{N-1} \langle \tilde{h}_{i,N}, \tilde{h}_{j,N} \rangle_{L^2([0,1]^4)} \\ &= \lim_{N \rightarrow \infty} \left(c'_1 N^{4H} D_{2,N}^{(6)} + c'_2 N^{4H} D_{3,N}^{(6)} \right) = 0. \end{aligned} \quad (29)$$

As a consequence of the convergences (24), (28), and (29), we have proved that for every $H > 1/2$ and with the notation $\overline{C}(H) = C(H)^2/C_0(H)$,

$$\frac{C(H)^2}{C_0(H)} N^{2-2H} \mathbf{E} (V^{3,N})^2 = \mathbf{E} \left(\sqrt{\overline{C}(H)} N^{1-H} V^{3,N} \right)^2 \xrightarrow{N \rightarrow \infty} 1. \quad (30)$$

Proposition 1 is proved.

3.2. Nonconvergence to a Gaussian limit. We prove that the normalized variation does not converge in distribution to the normal law. Of course, this is somehow superfluous taking into account that in the next section we show a noncentral limit theorem for this statistics, but we found the calculations instructive to see why it does not converges to a Gaussian limit. Recall that by [15, Theorem] a sequence $F_N = I_q(f_N)$ in the q Wiener chaos with $\mathbf{E} F_N^2 \rightarrow 1$, $N \rightarrow \infty$, converges to the normal law $N(0, 1)$ if and only if $\|DF_N\|_{L^2([0,1])}^2$ converges to q in $L^2(\Omega)$ as $N \rightarrow \infty$. Here D denotes the Malliavin derivative; if $f \in L^2([0, T]^n)$ is a symmetric function, we will use the following rule to differentiate in the Malliavin sense:

$$D_t I_n(f) = n I_{n-1}(f(\cdot, t)), \quad t \in [0, 1].$$

We put

$$T_N = \frac{\sqrt{C(H)} N^{1-H}}{C(H) N^{1-3H}} \sum_{i=0}^N I_2(g_{i,N}) = \frac{N^{2H}}{\sqrt{C_0(H)}} \sum_{i=0}^N I_2(g_{i,N}).$$

We derive T_N in the Malliavin sense and we obtain $D_t T_N = (2N^{2H}/\sqrt{C_0(H)}) \sum_{i=0}^N I_1(g_{i,N}(\cdot, t))$ and thus

$$\begin{aligned} \|DT_N\|_{L^2([0,1]^2)}^2 &= \frac{4N^{4H}}{C_0(H)} \int_0^1 \left(\sum_{i=0}^N I_1(g_{i,N}(\cdot, t)) \right)^2 dt \\ &= \frac{4N^{4H}}{C_0(H)} \int_0^1 \left(\sum_{i,j=0}^N I_1(g_{i,N}(\cdot, t)) I_1(g_{j,N}(\cdot, t)) \right) dt \\ &= \frac{4N^{4H}}{C_0(H)} \left(\sum_{i,j=0}^N \int_0^1 I_0(g_{i,N} \otimes_1 g_{j,N}) dt + \sum_{i,j=0}^N \int_0^1 I_2(g_{i,N} \otimes_0 g_{j,N}) dt \right) \\ &=: \frac{4N^{4H}}{C_0(H)} (J_{1,N} + J_{2,N}), \end{aligned}$$

where

$$J_{1,N} := \sum_{i,j=0}^N \int_0^1 I_0(g_{i,N} \otimes_1 g_{j,N}) dt = \sum_{i,j=0}^N \langle g_{i,N}, g_{j,N} \rangle_{L^2([0,1]^2)} = \frac{1}{2} A_N^{(2)}.$$

From (24) we obtain

$$\frac{4N^{4H}}{C_0(H)} J_{1,N} \xrightarrow{N \rightarrow \infty} 2 \quad (31)$$

in $L^2(\Omega)$ because the term $A_N^{(2)}$ is deterministic. To prove that $\|DT_N\|_{L^2([0,1]^2)}^2$ does not converge in $L^2(\Omega)$ to 2, it is sufficient to show that

$$\lim_{N \rightarrow \infty} \mathbf{E} \left(\frac{4N^{4H}}{C_0(H)} J_{2,N} \right)^2 > 0.$$

where $J_{2,N} = \sum_{i,j=0}^N \int_0^1 I_2(g_{i,N}(\cdot, t)g_{j,N}(\cdot, t)) dt$.

We calculate the mean square of this term:

$$\begin{aligned}
\mathbf{E}(J_{2,N})^2 &= 2 \int_{[0,1]^2} \left(\sum_{i,j=0}^N \int_0^1 I_2(g_{i,N}(r, t)g_{j,N}(s, t)) dt \right)^2 dr ds \\
&= 2 \sum_{i,j,k,l=0}^N \int_{[0,1]^4} g_{i,N}(r, t)g_{j,N}(s, t)g_{k,N}(r, u)g_{l,N}(s, u) dr ds dt du \\
&\geq \sum_{i,j,k,l=0}^N \int_{[0,1]^4} f_{i,N}(r, t)f_{j,N}(s, t)f_{k,N}(r, u)f_{l,N}(s, u) dr ds dt du \\
&= 2d(H)^4 a(H)^4 N^{-8H} \\
&\quad \times \sum_{i,j,k,l=0}^N \int_{I_i} \int_{I_j} \int_{I_k} \int_{I_l} (|r-t||s-t||r-u||s-u|)^{2H'-2} dr ds dt du \\
&= 2d(H)^4 a(H)^4 N^{-8H} \sum_{i,j,k,l=0}^N \int_{[0,1]^4} dr ds dt du \\
&\quad \times \frac{1}{N^4} \left(\left| \frac{r-t+i-j}{N} \right| \left| \frac{t-s+j-k}{N} \right| \left| \frac{s-u+k-l}{N} \right| \right. \\
&\quad \left. \times \left| \frac{u-r+l-i}{N} \right| \right)^{2H'-2}.
\end{aligned}$$

By using Riemann sums approximations, we obtain

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbf{E}(N^{4H} J_{2,N})^2 &\geq 2d(H)^4 a(H)^4 \int_{[0,1]^4} dx_1 dx_2 dx_3 dx_4 \\
&\quad \times (|x_1 - x_2| |x_2 - x_3| |x_3 - x_4| |x_4 - x_1|)^{2H'-2} > 0.
\end{aligned}$$

4. The noncentral limit theorem for the cubic variation of the Rosenblatt process. Denote by L_t the kernel of the Rosenblatt process:

$$L_t(x, y) = d(H) 1_{[0,t]^{\otimes 2}}(x, y) \int_{x \vee y}^t \partial_1 K^{H'}(s, x) \partial_1 K^{H'}(s, y) ds$$

and recall the notation

$$f_{i,N}(x, y) = L_{(i+1)/N}^{(H)}(x, y) - L_{i/N}^{(H)}(x, y).$$

We proved in the previous section that the dominant term of the statistics $V^{3,N}$ which gives its normalization is

$$C(H)^{-1} N^{3H-1} \sum_{i=0}^{N-1} I_2(g_{i,N}),$$

where

$$\begin{aligned} g_{i,N} &= 6\|f_{i,N}\|_{L^2([0,1]^2)}^2 f_{i,N} + 24(f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N} \\ &= 3N^{-2H} f_{i,N} + 24(f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N} =: 3g_{i,N}^{(1)} + 24g_{i,N}^{(2)}. \end{aligned}$$

More precisely, it follows from the proof of Proposition 1 that

$$\begin{aligned} \mathbf{E} \left[N^{1-H} \left(N^{3H-1} \sum_{i=0}^{N-1} I_2(g_{i,N}^{(1)}) \right) \right]^2 &= N^{4H} A_{1,N}^{(2)} \xrightarrow{N \rightarrow \infty} \frac{1}{8}, \\ \mathbf{E} \left[N^{1-H} \left(N^{3H-1} \sum_{i=0}^{N-1} I_2(g_{i,N}^{(2)}) \right) \right]^2 &= N^{4H} A_{3,N}^{(2)} \xrightarrow{N \rightarrow \infty} \frac{H^2(2H-1)^2}{8} (C'(H))^2. \end{aligned}$$

Consequently, the limit of the sequence $V^{3,N}$ is the same as the limit of the sequence

$$C(H)^{-1} N^{1-H} N^{3H-1} \left(3 \sum_{i=0}^{N-1} I_2(g_{i,N}^{(1)}) + 24 \sum_{i=0}^{N-1} I_2(g_{i,N}^{(2)}) \right).$$

We prove here our main result.

Theorem 1. *The renormalized cubic variation statistics based on the Rosenblatt process $N^{1-H}V^{3,N}$ with $V^{3,N}$ given by (10) converges in $L^2(\Omega)$ as $N \rightarrow \infty$ to the Rosenblatt random variable $D(H)Z_1^H$, where $D(H) = C(H)^{-1}(3 + 24d(H)^2a(H)^2C'(H))$.*

P r o o f. To find the limit of $N^{1-H}V^{3,N}$ we need to study the convergence of $N^{1-H}N^{3H-1} \sum_{i=0}^{N-1} I_2(g_{i,N}^{(1)})$ and of $N^{1-H}N^{3H-1} \sum_{i=0}^{N-1} I_2(g_{i,N}^{(2)})$.

Is easy to treat the first part. In fact we have

$$N^{1-H}N^{3H-1} \sum_{i=0}^{N-1} I_2(g_{i,N}^{(1)}) = N^{2H} \sum_{i=0}^{N-1} N^{-2H} I_2(f_{i,N}) = \sum_{i=0}^{N-1} I_2(f_{i,N}) = Z_1^H, \quad (32)$$

where Z_1^H is a Rosenblatt random variable with self-similarity order H .

We find then the limit of the second part of the dominant term. We have

$$N^{1-H}N^{3H-1} \sum_{i=0}^{N-1} I_2(g_{i,N}^{(2)}) = N^{2H} \sum_{i=0}^{N-1} I_2((f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N}).$$

Introduce the notation

$$\begin{aligned} l^{H'}(x, y, z, t) &:= \partial_1 K^{H'}(x, y) \partial_1 K^{H'}(z, t), \\ l_0^{H'}(x, y, z) &:= \partial_1 K^{H'}(x, y) \partial_1 K^{H'}(x, z) = l^{H'}(x, y, x, z), \\ l_1^{H'}(x, y, z) &:= \partial_1 K^{H'}(x, z) \partial_1 K^{H'}(y, z) = l^{H'}(x, z, y, z). \end{aligned}$$

Using relations (19) and (20) we get

$$\begin{aligned} ((f_{i,N} \otimes_1 f_{i,N}) \otimes_1 f_{i,N})(y_1, y_2) &= d(H)^3 a(H)^2 1_{[0,i/N]}^{\otimes 2}(y_1, y_2) \\ &\times \int_{I_i^3} du_1 du_2 du_3 l^{H'}(u_1, y_1, u_3, y_2) [|u_1 - u_2| |u_2 - u_3|]^{2H'-2} \\ &=: b_{i,N}^{(1)}(y_1, y_2) + b_{i,N}^{(2)}(y_1, y_2) \end{aligned}$$

with

$$\begin{aligned} b_{i,N}^{(1)}(y_1, y_2) &= d(H)^3 a(H)^2 1_{[0,i/N]}^{\otimes 2}(y_1, y_2) \\ &\times \int_{I_i^3} du_1 du_2 du_3 l^{H'}(u_1, y_1, u_3, y_2) [|u_1 - u_2| |u_2 - u_3|]^{2H'-2} \end{aligned}$$

and

$$\begin{aligned} b_{i,N}^{(2)}(y_1, y_2) &= d(H)^3 a(H)^2 \\ &\times \left[1_{I_i}(y_1) 1_{[0,i/N]}(y_2) \int_{y_1}^{(i+1)/N} du_1 \int_{I_i^2} du_2 du_3 l^{H'}(u_1, y_1, u_3, y_2) \right. \\ &\quad \times [|u_1 - u_2| |u_2 - u_3|]^{2H'-2} \\ &+ 1_{[0,i/N]}(y_1) 1_{I_i}(y_2) \int_{I_i^2} du_1 du_2 \int_{y_2}^{(i+1)/N} du_3 l^{H'}(u_1, y_1, u_3, y_2) \\ &\quad \times [|u_1 - u_2| |u_2 - u_3|]^{2H'-2} \\ &+ 1_{I_i^2}(y_1, y_2) \int_{y_1}^{(i+1)/N} du_1 \int_{I_i} du_2 \int_{y_2}^{(i+1)/N} du_3 l^{H'}(u_1, y_1, u_3, y_2) \\ &\quad \left. \times [|u_1 - u_2| |u_2 - u_3|]^{2H'-2} \right]. \end{aligned}$$

We show that $I_2(N^{2H} \sum_{i=0}^{N-1} b_{i,N}^{(2)})$ converges to zero in $L^2(\Omega)$ and it makes no contribution to the limit. Indeed,

$$\begin{aligned} \mathbf{E} \left(I_2 \left(N^{2H} \sum_{i=0}^{N-1} b_{i,N}^{(2)} \right) \right)^2 &= 2N^{4H} \int_0^1 \int_0^1 dy_1 dy_2 \left(\sum_{i=0}^{N-1} b_{i,N}^2(y_1, y_2) \right)^2 \\ &\leq 2d(H)^6 a(H)^4 N^{4H} \\ &\quad \times \left(\sum_{i=0}^{N-1} \int_0^1 \int_0^1 dy_1 dy_2 \int_{I_i^6} du_1 du'_1 du_2 du'_2 du_3 du'_3 1_{[y_1, (i+1)/N]^2}(u_1, u'_1) \right. \\ &\quad \times 1_{[y_2, (i+1)/N]^2}(u_3, u'_3) l_1^{H'}(u_1, u'_1, y_1) l_1^{H'}(u_3, u'_3, y_2) \\ &\quad \times [|u_1 - u_2| |u_2 - u_3| |u'_1 - u'_2| |u'_2 - u'_3|]^{2H'-2} \Big) \\ &\leq 2d(H)^6 a(H)^4 N^{4H} \left(\sum_{i=0}^{N-1} \int_{I_i^6} du_1 du'_1 du_2 du'_2 du_3 du'_3 \right. \end{aligned}$$

$$\begin{aligned}
& \times [|u_1 - u_2||u_2 - u_3||u'_1 - u'_2||u'_2 - u'_3|]^{2H'-2} \\
& \times \int_0^{u_1 \wedge u'_1} dy_1 l_1^{H'}(u_1, u'_1, y_1) \int_0^{u_3 \wedge u'_3} dy_2 l_1^{H'}(u_3, u'_3, y_2) \Big) \\
& \leq 2d(H)^6 a(H)^4 N^{4H} \left(\sum_{i=0}^{N-1} \int_{I_i^6} du_1 du'_1 du_2 du'_2 du_3 du'_3 \right. \\
& \quad \times [|u_1 - u_2||u_2 - u_3||u'_1 - u'_2||u'_2 - u'_3||u_1 - u'_1||u_1 - u'_3|]^{2H'-2} \Big) \\
& \leq 2d(H)^6 a(H)^4 N^{4H} N N^{-6} N^{12-12H'} \left(\int_{[0,1]^6} dv_1 dv'_1 dv_2 dv'_2 dv_3 dv'_3 \right. \\
& \quad \times [|v_1 - v_2||v_2 - v_3||v'_1 - v'_2||v'_2 - v'_3||v_1 - v'_1||v_1 - v'_3|]^{2H'-2} \Big) \\
& \leq 2cd(H)^6 a(H)^4 N^{1-2H}.
\end{aligned}$$

Combining it with the fact that $H > 1/2$, we conclude that

$$I_2 \left(N^{2H} \sum_{i=0}^{N-1} b_{i,N}^{(2)} \right) \xrightarrow{N \rightarrow \infty} 0 \quad \text{in } L^2(\Omega), \quad (33)$$

and then we need to find the limit of

$$\begin{aligned}
N^{2H} \sum_{i=0}^{N-1} b_{i,N}^{(1)} &= d(H)^3 a(H)^2 N^{2H} \sum_{i=0}^{N-1} 1_{[0,i/N]^2}(y_1, y_2) \\
&\quad \times \int_{I_i^3} du_1 du_2 du_3 l^{H'}(u_1, y_1, u_3, y_2) [|u_1 - u_2||u_2 - u_3|]^{2H'-2} \\
&= d(H)^3 a(H)^2 \sum_{i=0}^{N-1} 1_{[0,i/N]^2}(y_1, y_2) N^{-1} \\
&\quad \times \int_{[0,1]^3} dv_1 dv_2 dv_3 l^{H'} \left(\frac{v_1 + i}{N}, y_1, \frac{v_3 + i}{N}, y_2 \right) \\
&\quad \times [|v_1 - v_2||v_2 - v_3|]^{2H'-2}.
\end{aligned}$$

The last sequence has the same pointwise limit (for every y_1, y_2) as

$$\begin{aligned}
& d(H)^3 a(H)^2 \int_{[0,1]^3} dv_1 dv_2 dv_3 [|v_1 - v_2||v_2 - v_3|]^{2H'-2} \\
& \quad \times \sum_{i=0}^{N-1} 1_{[0,i/N]^2}(y_1, y_2) N^{-1} l^{H'} \left(\frac{i}{N}, y_1, \frac{i}{N}, y_2 \right). \quad (34)
\end{aligned}$$

This last term is a Riemann sum that converges to

$$\begin{aligned}
& d(H)^3 a(H)^2 \int_{[0,1]^3} dv_1 dv_2 dv_3 [|v_1 - v_2||v_2 - v_3|]^{2H'-2} \\
& \quad \times \int_{y_1 \vee y_2}^1 dx l^{H'}(x, y_1, x, y_2)
\end{aligned}$$

$$\begin{aligned}
&= d(H)^3 a(H)^2 C'(H) \int_{y_1 \vee y_2}^1 dx \partial_1 K^{H'}(x, y_1) \partial_1 K^{H'}(x, y_2) \\
&= d(H)^2 a(H)^2 C'(H) L_1^{(H)}(y_1, y_2),
\end{aligned}$$

where $L_1^{(H)}$ is the standard kernel of the Rosenblatt process, see (6).

We need a Cauchy sequence argument as in [20] to conclude the proof. That is, we will show that the sequence $N^{2H} \sum_{i=0}^{N-1} b_{i,N}^{(1)}$ (or equivalently $N^{2H} \sum_{i=0}^{N-1} g_{i,N}^{(2)}$) is Cauchy in the Hilbert space $L^2([0, 1]^2)$. This will imply that the sequence of random variables $I_2(N^{2H} \sum_{i=0}^{N-1} g_{i,N}^{(2)})$ is Cauchy, so convergent, in the space $L^2(\Omega)$ and it is easy to deduce that its limit coincides with the multiple integral of the pointwise limit of the kernel. We compute, for $M, N \geq 1$

$$\begin{aligned}
&\left\| N^{2H} \sum_{i=0}^{N-1} b_{i,N}^{(1)} - M^{2H} \sum_{i=0}^{M-1} b_{i,M}^{(1)} \right\|_{L^2([0,1]^2)}^2 \\
&= d(H)^6 a(H)^4 \left[N^{4H} \sum_{i,j=0}^{N-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du'_1 du'_2 du'_3 \right. \\
&\quad \times [|u_1 - u_2| |u_2 - u_3| |u'_1 - u'_2| |u'_2 - u'_3|]^{2H'-2} \\
&\quad \times \int_0^{u_1 \wedge u'_1} dy_1 \int_0^{u_3 \wedge u'_3} dy_2 l^{H'}(u_1, y_1, u_3, y_2) l^{H'}(u'_1, y_1, u'_3, y_2) \\
&\quad + M^{4H} \sum_{i,j=0}^{M-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du'_1 du'_2 du'_3 \\
&\quad \times [|u_1 - u_2| |u_2 - u_3| |u'_1 - u'_2| |u'_2 - u'_3|]^{2H'-2} \\
&\quad \times \int_0^{u_1 \wedge u'_1} dy_1 \int_0^{u_3 \wedge u'_3} dy_2 l^{H'}(u_1, y_1, u_3, y_2) l^{H'}(u'_1, y_1, u'_3, y_2) \\
&\quad - 2N^{2H} M^{2H} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du'_1 du'_2 du'_3 \\
&\quad \times [|u_1 - u_2| |u_2 - u_3| |u'_1 - u'_2| |u'_2 - u'_3|]^{2H'-2} \\
&\quad \times \int_0^{u_1 \wedge u'_1} dy_1 \int_0^{u_3 \wedge u'_3} dy_2 l^{H'}(u_1, y_1, u_3, y_2) l^{H'}(u'_1, y_1, u'_3, y_2) \Big] \\
&= d(H)^6 a(H)^4 \left[N^{4H} \sum_{i,j=0}^{N-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du'_1 du'_2 du'_3 \right. \\
&\quad \times [|u_1 - u_2| |u_2 - u_3| |u'_1 - u'_2| |u'_2 - u'_3| |u_1 - u'_1| |u_3 - u'_3|]^{2H'-2} \\
&\quad + M^{4H} \sum_{i,j=0}^{M-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du'_1 du'_2 du'_3 \\
&\quad \times [|u_1 - u_2| |u_2 - u_3| |u'_1 - u'_2| |u'_2 - u'_3| |u_1 - u'_1| |u_3 - u'_3|]^{2H'-2}
\end{aligned}$$

$$\begin{aligned}
& -2N^{2H}M^{2H} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \int_{I_i^3} \int_{I_j^3} du_1 du_2 du_3 du'_1 du'_2 du'_3 \\
& \times [|u_1 - u_2| |u_2 - u_3| |u'_1 - u'_2| |u'_2 - u'_3| |u_1 - u'_1| |u_3 - u'_3|]^{2H'-2} \Big]
\end{aligned}$$

and this is equal to

$$\begin{aligned}
& d(H)^6 a(H)^4 \left[N^{-2H} \sum_{i,j=0}^{N-1} \int_{[0,1]^6} du_1 du_2 du_3 du'_1 du'_2 du'_3 \right. \\
& \times [|v_1 - v_2| |v_2 - v_3| |v'_1 - v'_2| |v'_2 - v'_3| |v_1 - v'_1 + i - j| |v_3 - v'_3 + i - j|]^{2H'-2} \\
& + M^{-2H} \sum_{i,j=0}^{M-1} \int_{[0,1]^6} du_1 du_2 du_3 du'_1 du'_2 du'_3 \cdot \\
& \times [|v_1 - v_2| |v_2 - v_3| |v'_1 - v'_2| |v'_2 - v'_3| |v_1 - v'_1 + i - j| |v_3 - v'_3 + i - j|]^{2H'-2} \\
& - 2N^{-1}M^{-1} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \int_{[0,1]^6} du_1 du_2 du_3 du'_1 du'_2 du'_3 \\
& \times \left[|v_1 - v_2| |v_2 - v_3| |v'_1 - v'_2| |v'_2 - v'_3| \left| \frac{v_1}{N} - \frac{v'_1}{M} + \frac{i}{N} - \frac{j}{M} \right| \right. \\
& \left. \times \left| \frac{v_3}{N} - \frac{v'_3}{M} + \frac{i}{N} - \frac{j}{M} \right| \right]^{2H'-2} \Big].
\end{aligned}$$

In the same way as above this last term as $N \rightarrow \infty$ and $M \rightarrow \infty$ converges to

$$\begin{aligned}
& d(H)^6 a(H)^4 \left[2 \int_0^1 (1-x) x^{2H-2} dx + 2 \int_0^1 (1-y) y^{2H-2} dy \right. \\
& \left. + 2 \int_0^1 \int_0^1 |x-y|^{2H-2} dx dy \right] \\
& = \frac{1}{H(2H-1)} + \frac{1}{H(2H-1)} - \frac{2}{H(2H-1)} = 0.
\end{aligned}$$

We obtained that $\{N^{2H} \sum_{i=0}^{N-1} b_{i,N}^{(1)}, N \geq 0\}$ is a Cauchy sequence. Theorem 1 is proved.

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