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SINN M.*, KELLER K.*

COVARIANCES OF ZERO CROSSINGS IN GAUSSIAN PROCESSES

Для гауссовского процесса с нулевым средним ковариации пересечений им нулевого уровня можно представить в виде суммы гауссовских мер четырехмерных ортантов. В настоящей статье мы обсуждаем оценивание ковариаций пересечений нуля с помощью одномерных интегралов. Кроме того, мы устанавливаем асимптотику этих ковариаций для больших временных лагов и выводим оценки и аппроксимации. Основываясь на этих результатах, мы анализируем дисперсию эмпирического среднего числа пересечений нуля. Применения наших результатов иллюстрируются на примере процесса авторегрессии (AR), фрактального гауссовского шума и фрактального авторегрессионного интегрального процесса скользящего среднего.

Ключевые слова и фразы: пересечение нуля, бинарные временные ряды, гауссовская мера четырехмерного ортанта, процесс AR(1), фрактальный гауссовский шум, процесс FARIMA(0, d , 0).

1. Introduction. Indicators of zero crossings are widely applied in various fields of engineering and natural science such as the analysis of vibrations, the detection of signals in presence of noise and the modelling of binary time series. A large number of literature has been contributed to the studies of zero crossing analysis. Dating back to the 1940s, telephony engineers found that replacing the original speech signal with rectangular waves having the same zero crossings retained high intelligibility [5]. Since the beginning of digital processing of speech signals, empirical rates of zero crossings have been used for the detection of pitch frequencies and to distinguish voiced and unvoiced intervals [11], [19].

For a discrete-time stationary Gaussian process or a sampled random sinusoid, the zero crossing rate is related to the first-order autocorrelation and to the dominant spectral frequency. Kedem [14] has developed estimators for autocorrelations and spectral frequencies by higher order zero crossings and shows diverse applications. Ho and Sun [12] have proved that the empirical zero crossing rate is asymptotically normally distributed if the

*Universität zu Lübeck, Institut für Mathematik, Wallstr. 40, D-23560 Lübeck, Germany; e-mail: sinn@math.uni-luebeck.de; keller@math.uni-luebeck.de

autocorrelations of the Gaussian process decay faster than $k^{-1/2}$. Coeurjolly [7] has proposed to use zero crossings to estimate the Hurst parameter in fractional Gaussian noise, which generally can be applied to the estimation of monotonic functionals of the first-order autocorrelation. Coeurjolly's estimator has been used to the analysis of hydrological time series [16] and atmospheric turbulence data [20].

Up to now, no closed-form expression is known for the variance of the empirical zero crossing rate. Basically, covariances of zero crossings are sums and products of four-dimensional normal orthant probabilities which can be evaluated only numerically in general. Abrahamson [1] derives an expression involving two-dimensional integrals for the special case of orthoscheme probabilities and gives a representation of any four-dimensional normal orthant probability as the linear combinations of six orthoscheme probabilities. For some simpler correlation structure, Cheng [6] proposes expressions involving the dilogarithm function. Kedem [14], Damsleth and El-Shaarawi [9] introduce approximations for processes with short memory. Most recent approaches apply Monte Carlo sampling for four dimensions and higher (see [8] for an overview).

In this paper, we propose a simple formula for the exact numerical evaluation of zero crossing covariances and derive their asymptotics, bounds and approximations. The results are obtained by analyzing partial derivatives of four-dimensional orthant probabilities with respect to correlation coefficients. In Theorem 3.1, we give a representation of zero crossing covariances and four-dimensional normal orthant probabilities by the sum of four one-dimensional integrals. By a Taylor expansion, we derive Theorem 4.1, which gives asymptotics of zero crossing covariances for large time lags. In particular, when the autocorrelation function of the underlying process decreases to 0 with the same order of magnitude as a function $f(k)$, the zero crossing covariances decrease to 0 with the same order of magnitude as $(f(k))^2$.

Theorem 5.1 states sufficient conditions on the autocorrelation structure of the underlying process to obtain lower and upper bounds by setting equal certain correlation coefficients in the orthant probabilities. Approximations of these expressions are given by Theorem 5.3.

In Theorem 6.1 we establish asymptotics of the variance of the empirical zero crossing rate. Furthermore, we discuss how the previous results can be used for a numerical evaluation of the variance. In Section 7, we apply the results to zero crossings in AR(1) processes, fractional Gaussian noise, and FARIMA(0, d , 0) models.

2. Preliminaries. Let $\mathbf{Y} = (Y_k)_{k \in \mathbf{Z}}$ be a stationary and nondegenerate zero-mean Gaussian process on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with autocorrelations $\rho_k = \text{Corr}(Y_0, Y_k)$ for $k \in \mathbf{Z}$. For $k \in \mathbf{Z}$, let

$$C_k := \mathbf{1}_{\{Y_k > 0, Y_{k+1} < 0\}} + \mathbf{1}_{\{Y_k < 0, Y_{k+1} > 0\}}$$

be the *indicator of a zero crossing* at time k . Since $\mathbf{Y}$ is stationary, $\mathbf{P}(C_k = 1)$ is constant in k , and the *empirical zero crossing rate* $\hat{c}_n := n^{-1} \sum_{k=1}^n C_k$ is an unbiased estimator of $\mathbf{P}(C_0 = 1)$, that is, $\mathbf{E} \hat{c}_n = \mathbf{P}(C_0 = 1)$ for all $n \in \mathbf{N}$. Denote the *covariance of zero crossings* by $\gamma_k := \text{Cov}(C_0, C_k)$ for $k \in \mathbf{Z}$. The variance of $\hat{c}_n$ is given by

$$\mathbf{D} \hat{c}_n = \frac{1}{n^2} \left(n \gamma_0 + 2 \sum_{k=1}^{n-1} (n-k) \gamma_k \right). \quad (2.1)$$

This paper investigates the evaluation of $\gamma_0, \gamma_1, \gamma_2, \dots$. Next, we give closed-form expressions for γ_0 and γ_1 based on well known formulas for the evaluation of bi- and trivariate normal orthant probabilities.

2.1. Orthant probabilities. For a nonsingular strictly positive definite and symmetric matrix $\Sigma \in \mathbf{R}^{n \times n}$ with $n \in \mathbf{N}$, let $\varphi(\Sigma, \cdot)$ denote the Lebesgue density of the n -dimensional normal distribution with zero means and the covariance matrix Σ , that is,

$$\varphi(\Sigma, \mathbf{x}) = ((2\pi)^n |\Sigma|)^{-1/2} \exp \left\{ -\frac{1}{2} \mathbf{x}^T \Sigma^{-1} \mathbf{x} \right\}$$

for $\mathbf{x} \in \mathbf{R}^n$, where $|\Sigma|$ denotes the determinant of Σ . The *n -dimensional normal orthant probability* with respect to Σ is given by

$$\Phi(\Sigma) := \int_{[0, \infty)^n} \varphi(\Sigma, \mathbf{x}) \, d\mathbf{x}.$$

If $\mathbf{Z} = (Z_1, \dots, Z_n)$ is a nondegenerate zero-mean Gaussian random vector and $\Sigma = \text{Cov}(\mathbf{Z}) = (\text{Cov}(Z_i, Z_j))_{i,j=1}^n$ is the covariance matrix of $\mathbf{Z}$, then

$$\mathbf{P}(Z_1 > 0, \dots, Z_n > 0) = \Phi(\Sigma).$$

Furthermore, if $a_1, \dots, a_n > 0$ and $\mathbf{A} = \text{diag}(\sqrt{a_1}, \dots, \sqrt{a_n})$ is the $n \times n$ diagonal matrix with entries $\sqrt{a_1}, \dots, \sqrt{a_n}$ on the main diagonal, then $\mathbf{A} \Sigma \mathbf{A}$ is the covariance matrix of $\mathbf{A} \mathbf{Z} = (\sqrt{a_1} Z_1, \dots, \sqrt{a_n} Z_n)$. Consequently, $\Phi(\mathbf{A} \Sigma \mathbf{A}) = \Phi(\Sigma)$. By choosing $a_1 = \dots = a_n = a$ and $a_i = \mathbf{D} Z_i$ for $i = 1, \dots, n$, respectively, we obtain

$$\Phi(\Sigma) = \Phi(a \Sigma) \quad \text{and} \quad \Phi(\text{Corr}(\mathbf{Z})) = \Phi(\text{Cov}(\mathbf{Z})). \quad (2.2)$$

The following closed-form expressions for two- and three-dimensional normal orthant probabilities are well known (see, e.g., [2]).

Lemma 2.1. *Let (Z_1, Z_2, Z_3) be a zero-mean nondegenerate Gaussian random vector and $\rho_{ij} = \text{Corr}(Z_i, Z_j)$ for $i, j \in \{1, 2, 3\}$. Then*

$$\begin{aligned} \mathbf{P}(Z_1 > 0, Z_2 > 0) &= \frac{1}{4} + \frac{1}{2\pi} \arcsin \rho_{12}, \\ \mathbf{P}(Z_1 > 0, Z_2 > 0, Z_3 > 0) &= \frac{1}{8} + \frac{1}{4\pi} \arcsin \rho_{12} + \frac{1}{4\pi} \arcsin \rho_{13} \\ &\quad + \frac{1}{4\pi} \arcsin \rho_{23}. \end{aligned}$$

Lemma 2.1 allows us to derive a closed-form expression for the probability of a change, namely,

$$\begin{aligned}\mathbf{P}(C_0 = 1) &= 1 - \mathbf{P}(Y_0 > 0, Y_1 > 0) - \mathbf{P}(Y_0 < 0, Y_1 < 0) \\ &= \frac{1}{2} - \frac{1}{\pi} \arcsin \rho_1.\end{aligned}\quad (2.3)$$

Furthermore,

$$\begin{aligned}\gamma_0 &= \mathbf{P}(C_0 = 1)(1 - \mathbf{P}(C_0 = 1)) \\ &= (1 - 2\mathbf{P}(Y_0 > 0, Y_1 > 0)) \cdot 2\mathbf{P}(Y_0 > 0, Y_1 > 0) \\ &= \frac{1}{4} - \frac{1}{\pi^2} (\arcsin \rho_1)^2\end{aligned}\quad (2.4)$$

and

$$\begin{aligned}\gamma_1 &= \mathbf{P}(C_0 = 1, C_1 = 1) - (\mathbf{P}(C_0 = 1))^2 \\ &= 2\mathbf{P}(Y_0 > 0, -Y_1 > 0, Y_2 > 0) - (1 - 2\mathbf{P}(Y_0 > 0, Y_1 > 0))^2 \\ &= \frac{1}{2\pi} \arcsin \rho_2 - \frac{1}{\pi^2} (\arcsin \rho_1)^2.\end{aligned}\quad (2.5)$$

If $k > 1$, then γ_k can be expressed as the sum and product of bi- and quadrivariate normal orthant probabilities,

$$\begin{aligned}\gamma_k &= \text{Cov}(1 - C_0, 1 - C_k) \\ &= 2\mathbf{P}(Y_0 > 0, Y_1 > 0, Y_k > 0, Y_{k+1} > 0) \\ &\quad + 2\mathbf{P}(Y_0 > 0, Y_1 > 0, -Y_k > 0, Y_{k+1} > 0) \\ &\quad - 4\mathbf{P}(Y_0 > 0, Y_1 > 0) \mathbf{P}(Y_k > 0, Y_{k+1} > 0).\end{aligned}\quad (2.6)$$

Note that, in general, no closed-form expression is available for normal orthant probabilities of dimension $n \geq 4$.

2.2. Context of the investigations. We consider the problem of evaluating γ_k for $k > 1$ in a more general context. For this, let $\mathcal{R}$ denote the set of $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in [-1, 1]^6$ such that the matrix

$$\Sigma(\mathbf{r}) := \begin{pmatrix} 1 & r_1 & r_2 & r_3 \\ r_1 & 1 & r_4 & r_5 \\ r_2 & r_4 & 1 & r_6 \\ r_3 & r_5 & r_6 & 1 \end{pmatrix}$$

is strictly positive definite, that is, $\mathbf{x}^T \Sigma(\mathbf{r}) \mathbf{x} > 0$ for $\mathbf{x} \in \mathbf{R}^4 \setminus \{\mathbf{0}\}$. Note that $\Sigma(\mathcal{R})$ is the set of 4×4 correlation matrices of nondegenerate Gaussian random vectors, and $\mathbf{r} \in \mathcal{R}$ implies that all components of $\mathbf{r}$ lie within $(-1, 1)$.

For $h \in [-1, 1]$ consider the diagonal matrix $\mathbf{I}_h := \text{diag}(1, h, h, h, h, 1)$. If $\mathbf{r}, \mathbf{s} \in \mathcal{R}$, then

$$\mathbf{x}^T \Sigma(h\mathbf{r} + (1-h)\mathbf{s}) \mathbf{x} = h \mathbf{x}^T \Sigma(\mathbf{r}) \mathbf{x} + (1-h) \mathbf{x}^T \Sigma(\mathbf{s}) \mathbf{x} > 0$$

for all $\mathbf{x} \in \mathbf{R}^4 \setminus \{\mathbf{0}\}$ and $h \in [0, 1]$, in other words, $\mathcal{R}$ is convex. Furthermore, $\mathbf{r} \in \mathcal{R}$ implies $\mathbf{I}_{-1} \mathbf{r} \in \mathcal{R}$. This can be seen as follows: If $\mathbf{r} \in \mathcal{R}$, then there exists a zero-mean nondegenerate Gaussian random vector $\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4)$ such that $\text{Cov}(\mathbf{Z}) = \Sigma(\mathbf{r})$. Since $\mathbf{Z}' = (Z'_1, Z'_2, Z'_3, Z'_4) := (Z_1, Z_2, -Z_3, -Z_4)$ is also nondegenerate Gaussian, the matrix $\text{Cov}(\mathbf{Z}') = \Sigma(\mathbf{I}_{-1} \mathbf{r})$ is strictly positive definite, too, and hence $\mathbf{I}_{-1} \mathbf{r} \in \mathcal{R}$.

Now, since $\mathbf{I}_1 \mathbf{r} = \mathbf{r}$ and

$$\mathbf{I}_h \mathbf{r} = \frac{1+h}{2} \mathbf{I}_1 \mathbf{r} + \frac{1-h}{2} \mathbf{I}_{-1} \mathbf{r}$$

for all $\mathbf{r} \in \mathcal{R}$ and $h \in [-1, 1]$, the convexity of $\mathcal{R}$ implies that $\mathbf{I}_h \mathbf{r} \in \mathcal{R}$ for all $\mathbf{r} \in \mathcal{R}$ and $h \in [-1, 1]$, a fact we will repeatedly use in the rest of the paper.

For $\mathbf{r} \in \mathcal{R}$, write $\Phi(\mathbf{r}) = \Phi(\Sigma(\mathbf{r}))$, and define

$$\Psi(\mathbf{r}) := 2\Phi(\mathbf{r}) + 2\Phi(\mathbf{I}_{-1} \mathbf{r}) - 4\Phi(\mathbf{I}_0 \mathbf{r}). \quad (2.7)$$

Note that if $\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4)$ is a zero-mean nondegenerate Gaussian random vector with covariance matrix $\text{Cov}(\mathbf{Z}) = \Sigma(\mathbf{r})$, then

$$\begin{aligned} \Psi(\mathbf{r}) &= 2\mathbf{P}(Z_1 > 0, Z_2 > 0, Z_3 > 0, Z_4 > 0) \\ &\quad + 2\mathbf{P}(Z_1 > 0, Z_2 > 0, -Z_3 > 0, -Z_4 > 0) \\ &\quad - 4\mathbf{P}(Z_1 > 0, Z_2 > 0) \mathbf{P}(Z_3 > 0, Z_4 > 0). \end{aligned}$$

Thus, according to (2.6),

$$\gamma_k = \Psi(\rho_1, \rho_k, \rho_{k+1}, \rho_{k-1}, \rho_k, \rho_1) \quad (2.8)$$

for $k > 1$. The evaluation of Φ and Ψ is the main concern of this paper. In Sections 3 and 4, we consider the general problem to evaluate $\Phi(\mathbf{r})$ and $\Psi(\mathbf{r})$ for arbitrary $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathcal{R}$. In Section 5, we focus on the special case, where $r_1 = r_6$ and $r_2 = r_5$.

3. Numerical evaluation. The following lemma establishes basic equations and closed-form expressions for Φ and Ψ in some special cases.

Lemma 3.1. *For every $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathcal{R}$,*

$$\begin{aligned} \Psi(\mathbf{r}) &= \Psi(\mathbf{I}_{-1} \mathbf{r}) = \Psi(-r_1, -r_2, r_3, r_4, -r_5, -r_6) \\ &= \Psi(-r_1, r_2, -r_3, -r_4, r_5, -r_6). \end{aligned}$$

If $r_2 = r_3 = r_4 = r_5 = 0$, then $\Psi(\mathbf{r}) = 0$ and

$$\Phi(\mathbf{r}) = \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin r_1 \right) \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin r_6 \right). \quad (3.1)$$

P r o o f. The first equation follows by the definition of Ψ and $\mathbf{I}_0 \mathbf{I}_{-1} = \mathbf{I}_0$. Now, let $\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4)$ be zero-mean Gaussian with $\text{Cov}(\mathbf{Z}) = \Sigma(\mathbf{r})$. Define $\mathbf{Z}' = (Z'_1, Z'_2, Z'_3, Z'_4) := (Z_1, -Z_2, -Z_3, Z_4)$ and $\mathbf{r}' := (-r_1, -r_2, r_3, r_4, -r_5, -r_6)$. Since $\text{Cov}(\mathbf{Z}') = \Sigma(\mathbf{r}')$, the second equation follows because

$$\begin{aligned} \Psi(\mathbf{r}) &= \text{Cov}(\mathbf{1}_{\{Z_1>0, Z_2>0\}} + \mathbf{1}_{\{Z_1<0, Z_2<0\}}, \mathbf{1}_{\{Z_3>0, Z_4>0\}} + \mathbf{1}_{\{Z_3<0, Z_4<0\}}) \\ &= \text{Cov}(\mathbf{1}_{\{Z_1>0, -Z_2<0\}} + \mathbf{1}_{\{Z_1<0, -Z_2>0\}}, \mathbf{1}_{\{Z_3>0, -Z_4<0\}} + \mathbf{1}_{\{Z_3<0, -Z_4>0\}}) \\ &= \Psi(\mathbf{r}'). \end{aligned}$$

Applying $\Psi(\mathbf{r}) = \Psi(\mathbf{I}_{-1} \mathbf{r})$ to $\mathbf{r} = (-r_1, -r_2, r_3, r_4, -r_5, -r_6)$ yields the third equation.

Now, assume $r_2 = r_3 = r_4 = r_5 = 0$. Since $\mathbf{r} = \mathbf{I}_{-1} \mathbf{r} = \mathbf{I}_0 \mathbf{r}$, we obtain $\Psi(\mathbf{r}) = 0$. Furthermore, if $\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4)$ is zero-mean nondegenerate Gaussian with $\text{Cov}(\mathbf{Z}) = \Sigma(\mathbf{r})$, then (Z_1, Z_2) and (Z_3, Z_4) are independent. Thus, (3.1) follows from Lemma 2.1. Lemma 3.1 is proved.

Note that bounds for $\Psi(\mathbf{r})$ can be obtained by the Berman inequality, namely,

$$|\Psi(\mathbf{r})| \leq \frac{2}{\pi} \sum_{k=2}^5 \frac{|r_k|}{\sqrt{1-r_k}}$$

(see [17]). In the remaining part of this section, we show how to compute $\Psi(\mathbf{r})$ and $\Phi(\mathbf{r})$ for any $\mathbf{r} \in \mathcal{R}$ by the numerical evaluation of four one-dimensional integrals. According to a formula first given by David [10], this also allows us to evaluate normal orthant probabilities of dimension $n = 5$. Further, we derive explicit formulas for the partial derivatives of Φ and Ψ with respect to r_2, r_3, r_4 , and r_5 .

3.1. Partial derivatives. For $\mathbf{r} \in \mathcal{R}$ and $i, j \in \{1, 2, 3, 4\}$, let $\sigma'_{ij}(\mathbf{r})$ denote the (i, j) -th component of the inverse $(\Sigma(\mathbf{r}))^{-1}$ of $\Sigma(\mathbf{r})$. It is well known that the inverse and any principal submatrix of a symmetric strictly positive definite matrix is symmetric and strictly positive definite (see [13]). Now, for fixed $k \in \{1, 2, \dots, 6\}$, let $\{i, j\}$ with $i \neq j$ be the unique subset of $\{1, 2, 3, 4\}$ such that r_k does not lie in the i -th row and j -th column of $\Sigma(\mathbf{r})$. Using the so-called reduction formula for normal orthant probabilities (see [18], [4]), we obtain the first partial derivative of Φ with respect to r_k ,

$$\frac{\partial \Phi}{\partial r_k}(\mathbf{r}) = \frac{1}{2\pi\sqrt{1-r_k^2}} \Phi \left(\begin{pmatrix} \sigma'_{ii}(\mathbf{r}) & \sigma'_{ij}(\mathbf{r}) \\ \sigma'_{ij}(\mathbf{r}) & \sigma'_{jj}(\mathbf{r}) \end{pmatrix}^{-1} \right)$$

for $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathcal{R}$. Note that the argument of Φ is a principal submatrix of $(\Sigma(\mathbf{r}))^{-1}$ and thus strictly positive definite. By the first

equation in (2.2),

$$\frac{\partial \Phi}{\partial r_k}(\mathbf{r}) = \frac{1}{2\pi\sqrt{1-r_k^2}} \Phi \left(\begin{pmatrix} \sigma'_{ii}(\mathbf{r}) & -\sigma'_{ij}(\mathbf{r}) \\ -\sigma'_{ij}(\mathbf{r}) & \sigma'_{jj}(\mathbf{r}) \end{pmatrix} \right).$$

Now, let $\sigma_{ij}(\mathbf{r}) = -|\Sigma(\mathbf{r})| \sigma'_{ij}(\mathbf{r})$ if $i \neq j$, and $\sigma_{ij}(\mathbf{r}) = |\Sigma(\mathbf{r})| \sigma'_{ij}(\mathbf{r})$ if $i = j$ for $i, j \in \{1, 2, 3, 4\}$. By the second equation in (2.2) and the formula for two-dimensional normal orthant probabilities in Lemma 2.1,

$$\frac{\partial \Phi}{\partial r_2}(\mathbf{r}) = \frac{1}{2\pi\sqrt{1-r_2^2}} \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{\sigma_{24}(\mathbf{r})}{\sqrt{\sigma_{22}(\mathbf{r})\sigma_{44}(\mathbf{r})}} \right), \quad (3.2)$$

$$\frac{\partial \Phi}{\partial r_3}(\mathbf{r}) = \frac{1}{2\pi\sqrt{1-r_3^2}} \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{\sigma_{23}(\mathbf{r})}{(\sqrt{\sigma_{22}(\mathbf{r})\sigma_{33}(\mathbf{r})})} \right), \quad (3.3)$$

$$\frac{\partial \Phi}{\partial r_4}(\mathbf{r}) = \frac{1}{2\pi\sqrt{1-r_4^2}} \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{\sigma_{14}(\mathbf{r})}{\sqrt{\sigma_{11}(\mathbf{r})\sigma_{44}(\mathbf{r})}} \right), \quad (3.4)$$

$$\frac{\partial \Phi}{\partial r_5}(\mathbf{r}) = \frac{1}{2\pi\sqrt{1-r_5^2}} \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{\sigma_{13}(\mathbf{r})}{\sqrt{\sigma_{11}(\mathbf{r})\sigma_{33}(\mathbf{r})}} \right) \quad (3.5)$$

for $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathcal{R}$. Note that $\sigma_{ij}(\mathbf{r})$ is equal to the determinant of the matrix obtained by deleting the i -th row and the j -th column of $\Sigma(\mathbf{r})$, multiplied with $(-1)^{i+j+1}$ if $i \neq j$ (see [13]). We obtain

$$\sigma_{11}(\mathbf{r}) = 1 - r_4^2 - r_5^2 - r_6^2 + 2r_4r_5r_6, \quad (3.6)$$

$$\sigma_{22}(\mathbf{r}) = 1 - r_2^2 - r_3^2 - r_6^2 + 2r_2r_3r_6, \quad (3.7)$$

$$\sigma_{33}(\mathbf{r}) = 1 - r_1^2 - r_3^2 - r_5^2 + 2r_1r_3r_5, \quad (3.8)$$

$$\sigma_{44}(\mathbf{r}) = 1 - r_1^2 - r_2^2 - r_4^2 + 2r_1r_2r_4, \quad (3.9)$$

$$\sigma_{13}(\mathbf{r}) = r_2 - r_1r_4 + r_3r_4r_5 - r_2r_5^2 - r_3r_6 + r_1r_5r_6, \quad (3.10)$$

$$\sigma_{14}(\mathbf{r}) = r_3 - r_1r_5 + r_2r_4r_5 - r_3r_4^2 - r_2r_6 + r_1r_4r_6, \quad (3.11)$$

$$\sigma_{23}(\mathbf{r}) = r_4 - r_1r_2 + r_2r_3r_5 - r_4r_3^2 - r_5r_6 + r_1r_3r_6, \quad (3.12)$$

$$\sigma_{24}(\mathbf{r}) = r_5 - r_1r_3 + r_2r_3r_4 - r_5r_2^2 - r_4r_6 + r_1r_2r_6. \quad (3.13)$$

The following corollary is an immediate consequence of (3.2)–(3.5) and (3.6)–(3.13).

Corollary 3.1. *For every $\mathbf{r} \in \mathcal{R}$, the partial derivatives of Φ of any order exist and are continuous at $\mathbf{r}$.*

The following lemma gives the partial derivatives of Ψ with respect to r_i for $i = 2, 3, 4, 5$. For $\mathbf{r} \in \mathcal{R}$, let

$$\psi_i(\mathbf{r}) := \frac{\partial \Psi}{\partial r_i}(\mathbf{r}).$$

Lemma 3.2. *For every $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathcal{R}$,*

$$\psi_2(\mathbf{r}) = \frac{1}{\pi^2 \sqrt{1 - r_2^2}} \arcsin \frac{\sigma_{24}(\mathbf{r})}{\sqrt{\sigma_{22}(\mathbf{r})\sigma_{44}(\mathbf{r})}}, \quad (3.14)$$

$$\psi_3(\mathbf{r}) = \frac{1}{\pi^2 \sqrt{1 - r_3^2}} \arcsin \frac{\sigma_{23}(\mathbf{r})}{\sqrt{\sigma_{22}(\mathbf{r})\sigma_{33}(\mathbf{r})}}, \quad (3.15)$$

$$\psi_4(\mathbf{r}) = \frac{1}{\pi^2 \sqrt{1 - r_4^2}} \arcsin \frac{\sigma_{14}(\mathbf{r})}{\sqrt{\sigma_{11}(\mathbf{r})\sigma_{44}(\mathbf{r})}}, \quad (3.16)$$

$$\psi_5(\mathbf{r}) = \frac{1}{\pi^2 \sqrt{1 - r_5^2}} \arcsin \frac{\sigma_{13}(\mathbf{r})}{\sqrt{\sigma_{11}(\mathbf{r})\sigma_{33}(\mathbf{r})}}. \quad (3.17)$$

P r o o f. Let $i \in \{2, 3, 4, 5\}$. We denote by $\mathbf{I}_h$ the mapping $\mathbf{r} \mapsto \mathbf{I}_h \mathbf{r}$ from $\mathcal{R}$ onto itself. By the definition of Ψ ,

$$\psi_i(\mathbf{r}) = 2 \frac{\partial \Phi}{\partial r_i}(\mathbf{r}) + 2 \frac{\partial(\Phi \circ \mathbf{I}_{-1})}{\partial r_i}(\mathbf{r}) - 4 \frac{\partial(\Phi \circ \mathbf{I}_0)}{\partial r_i}(\mathbf{r}). \quad (3.18)$$

With (3.1) we obtain $(\Phi \circ \mathbf{I}_0)(\mathbf{r}) = (1/4 + (\arcsin r_1)/(2\pi))(1/4 + (\arcsin r_6)/(2\pi))$. Thus, $\mathbf{r} \mapsto (\Phi \circ \mathbf{I}_0)(\mathbf{r})$ is constant in r_i and, consequently, the last term on the right-hand side of (3.18) is equal to 0. Furthermore, because $\frac{\partial \mathbf{I}_{-1}}{\partial r_i}(\mathbf{r}) = -1$, the chain rule of differentiation yields

$$\frac{\partial(\Phi \circ \mathbf{I}_{-1})}{\partial r_i}(\mathbf{r}) = -\frac{\partial \Phi}{\partial r_i}(\mathbf{I}_{-1}(\mathbf{r})).$$

According to (3.6)–(3.13), $\sigma_{ij}(\mathbf{I}_{-1}(\mathbf{r})) = \sigma_{ij}(\mathbf{r})$ if $(i, j) \in \{(1, 1), (2, 2), (3, 3), (4, 4)\}$, and $\sigma_{ij}(\mathbf{I}_{-1}(\mathbf{r})) = -\sigma_{ij}(\mathbf{r})$ if $(i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$. Since $x \mapsto \arcsin x$ is an odd function, inserting $\mathbf{I}_{-1}(\mathbf{r})$ instead of $\mathbf{r}$ into (3.2)–(3.5) yields (3.14)–(3.17). Lemma 3.2 is proved.

3.2. Integral representation. Further we state the main result of this section. Note that a similar representation of $\Psi(\mathbf{r})$ as in (3.19) is used for the proof of the Berman inequality (see above).

Theorem 3.1. *For every $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathcal{R}$,*

$$\Psi(\mathbf{r}) = \sum_{i=2}^5 r_i \int_0^1 \psi_i(\mathbf{I}_h \mathbf{r}) \, dh, \quad (3.19)$$

$$\begin{aligned} \Phi(\mathbf{r}) &= \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin r_1 \right) \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin r_6 \right) \\ &\quad + \frac{1}{8\pi} \sum_{i=2}^5 \arcsin r_i + \frac{\Psi(\mathbf{r})}{4}, \end{aligned} \quad (3.20)$$

$$\Phi(\mathbf{r}) - \Phi(\mathbf{I}_{-1} \mathbf{r}) = \frac{1}{4\pi} \sum_{i=2}^5 \arcsin r_i. \quad (3.21)$$

P r o o f. Let $\mathbf{r} \in \mathcal{R}$. Since $\mathbf{I}_h \mathbf{r} \in \mathcal{R}$ for all $h \in [0, 1]$, the mapping $[0, 1] \ni h \mapsto u(h) := \Psi(\mathbf{I}_h \mathbf{r})$ is well defined, being the concatenation of $h \mapsto \mathbf{I}_h \mathbf{r}$ and Ψ . Clearly, $u(1) = \Psi(\mathbf{r})$ and, by Lemma 3.1, $u(0) = 0$. Since Ψ has continuous partial derivatives (see Lemma 3.2), u is differentiable at h for all $h \in [0, 1]$, and hence, by the Fundamental Theorem of Calculus and the chain rule of differentiation,

$$\Psi(\mathbf{r}) = u(0) + \int_0^1 u'(h) \, dh = \int_0^1 \sum_{i=2}^5 r_i \frac{\partial \Psi}{\partial r_i}(\mathbf{I}_h \mathbf{r}) \, dh = \sum_{i=2}^5 r_i \int_0^1 \psi_i(\mathbf{I}_h \mathbf{r}) \, dh.$$

Analogously, define $v(h) := \Phi(\mathbf{I}_h \mathbf{r})$ for $h \in [0, 1]$. According to (3.2)–(3.5) and (3.14)–(3.17),

$$\frac{\partial \Phi}{\partial r_i}(\mathbf{r}) = \frac{1}{8\pi\sqrt{1-r_i^2}} + \frac{\psi_i(\mathbf{r})}{4}$$

for $i = 2, 3, 4, 5$. Consequently,

$$\begin{aligned} \int_0^1 v'(h) \, dh &= \sum_{i=2}^5 r_i \int_0^1 \frac{\partial \Phi}{\partial r_i}(\mathbf{I}_h \mathbf{r}) \, dh \\ &= \frac{1}{8\pi} \sum_{i=2}^5 r_i \int_0^1 \frac{1}{\sqrt{1-r_i^2 h^2}} \, dh + \frac{1}{4} \sum_{i=2}^5 r_i \int_0^1 \psi_i(\mathbf{I}_h \mathbf{r}) \, dh \\ &= \frac{1}{8\pi} \sum_{i=2}^5 \arcsin r_i + \frac{\Psi(\mathbf{r})}{4}. \end{aligned}$$

According to (3.1),

$$v(0) = \Phi(\mathbf{I}_0 \mathbf{r}) = \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin r_1 \right) \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin r_6 \right),$$

and hence (3.20) follows. Equation (3.21) is an immediate consequence of (3.20), Lemma 3.1 and the fact that $x \mapsto \arcsin x$ is an odd function. Theorem 3.1 is proved.

Note that for $i = 2, 3, 4, 5$, the mappings $h \mapsto \psi_i(\mathbf{I}_h \mathbf{r})$ have bounded derivatives on $[0, 1]$. (The derivatives are easily obtained from (3.14)–(3.17).) Moreover for fixed $\mathbf{r} \in \mathcal{R}$, upper and lower bounds can be given in a closed form which allows us to evaluate the integrals in Theorem 3.1 numerically to any desired precision.

4. Asymptotically equivalent expressions. For fixed $n \in \mathbf{N}$, let $(\mathbf{r}(k))_{k \in \mathbf{N}}$, $(\mathbf{s}(k))_{k \in \mathbf{N}}$ be sequences of vectors in $\mathbf{R}^n$ with $\mathbf{r}(k) = (r_1(k), \dots, r_n(k))$ and $\mathbf{s}(k) = (s_1(k), \dots, s_n(k))$ for $k \in \mathbf{N}$. We write

$$\mathbf{r}(k) \sim \mathbf{s}(k)$$

and say that $(\mathbf{r}(k))_{k \in \mathbf{N}}$ and $(\mathbf{s}(k))_{k \in \mathbf{N}}$ are *asymptotically equivalent* if $r_i(k) \sim s_i(k)$ for all $i \in \{1, 2, \dots, n\}$, that is, $\lim_{k \rightarrow \infty} r_i(k)/s_i(k) = 1$, where $0/0 := 1$.

The following theorem relates asymptotics of special sequences in $\mathcal{R}$ and asymptotics of the corresponding values of Ψ . In Corollary 5.1, this result is used for deriving asymptotics of zero crossing covariances.

Theorem 4.1. *Let $(\mathbf{r}(k))_{k \in \mathbf{N}}$ be a sequence in $\mathcal{R}$. If there exist an $f: \mathbf{N} \rightarrow \mathbf{R}$ with $\lim_{k \rightarrow \infty} f(k) = 0$ and an $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in \mathbf{R}^6$ with $|\alpha_1|, |\alpha_6| < 1$ such that $\mathbf{r}(k) \sim (\alpha_1, \alpha_2 f(k), \alpha_3 f(k), \alpha_4 f(k), \alpha_5 f(k), \alpha_6)$, then*

$$\Psi(\mathbf{r}(k)) \sim \frac{(f(k))^2 q(\boldsymbol{\alpha})}{2\pi^2 \sqrt{(1 - \alpha_1^2)(1 - \alpha_6^2)}} + O((f(k))^4),$$

where $q(\boldsymbol{\alpha}) = \alpha_1 \alpha_6 \sum_{i=2}^5 \alpha_i^2 - 2\alpha_1(\alpha_2 \alpha_3 + \alpha_4 \alpha_5) - 2\alpha_6(\alpha_2 \alpha_4 + \alpha_3 \alpha_5) + 2(\alpha_2 \alpha_5 + \alpha_3 \alpha_4)$.

P r o o f. Let $\mathbf{r}(k) = (r_1(k), r_2(k), r_3(k), r_4(k), r_5(k), r_6(k))$ for $k \in \mathbf{N}$. According to Corollary 3.1, the Taylor theorem asserts for each $k \in \mathbf{N}$ the existence of $h_1(k) \in [0, 1]$ and $h_2(k) \in [-1, 0]$ such that

$$\begin{aligned} \Phi(\mathbf{r}(k)) &= \Phi(\mathbf{I}_0 \mathbf{r}(k)) + \sum_{i=2}^5 r_i(k) \frac{\partial \Phi}{\partial r_i}(\mathbf{I}_0 \mathbf{r}(k)) \\ &+ \frac{1}{2} \sum_{i,j=2}^5 r_i(k) r_j(k) \frac{\partial^2 \Phi}{\partial r_i \partial r_j}(\mathbf{I}_0 \mathbf{r}(k)) \\ &+ \frac{1}{6} \sum_{i,j,l=2}^5 r_i(k) r_j(k) r_l(k) \frac{\partial^3 \Phi}{\partial r_i \partial r_j \partial r_l}(\mathbf{I}_0 \mathbf{r}(k)) \\ &+ \frac{1}{24} \sum_{i,j,l,m=2}^5 r_i(k) r_j(k) r_l(k) r_m(k) \\ &\quad \times \frac{\partial^4 \Phi}{\partial r_i \partial r_j \partial r_l \partial r_m}((\mathbf{I}_0 + h_1(k)(\mathbf{I}_1 - \mathbf{I}_0)) \mathbf{r}(k)) \end{aligned}$$

and, using the fact that $\mathbf{I}_0 \mathbf{I}_{-1} = \mathbf{I}_0$,

$$\begin{aligned} \Phi(\mathbf{I}_{-1} \mathbf{r}(k)) &= \Phi(\mathbf{I}_0 \mathbf{r}(k)) - \sum_{i=2}^5 r_i(k) \frac{\partial \Phi}{\partial r_i}(\mathbf{I}_0 \mathbf{r}(k)) \\ &+ \frac{1}{2} \sum_{i,j=2}^5 r_i(k) r_j(k) \frac{\partial^2 \Phi}{\partial r_i \partial r_j}(\mathbf{I}_0 \mathbf{r}(k)) \\ &- \frac{1}{6} \sum_{i,j,l=2}^5 r_i(k) r_j(k) r_l(k) \frac{\partial^3 \Phi}{\partial r_i \partial r_j \partial r_l}(\mathbf{I}_0 \mathbf{r}(k)) \\ &+ \frac{1}{24} \sum_{i,j,l,m=2}^5 r_i(k) r_j(k) r_l(k) r_m(k) \end{aligned}$$

$$\times \frac{\partial^4 \Phi}{\partial r_i \partial r_j \partial r_l \partial r_m} ((\mathbf{I}_0 - h_2(k)(\mathbf{I}_{-1} - \mathbf{I}_0)) \mathbf{r}(k)).$$

Since $\mathbf{I}_0 + h(\mathbf{I}_1 - \mathbf{I}_0) = \mathbf{I}_h$ and $\mathbf{I}_0 + h(\mathbf{I}_{-1} - \mathbf{I}_0) = \mathbf{I}_{-h}$ for $h \in [0, 1]$, we have

$$\begin{aligned} 2\Phi(\mathbf{r}(k)) + 2\Phi(\mathbf{I}_{-1} \mathbf{r}(k)) &= 4\Phi(\mathbf{I}_0 \mathbf{r}(k)) + 2 \sum_{i,j=2}^5 r_i(k)r_j(k) \frac{\partial^2 \Phi}{\partial r_i \partial r_j} (\mathbf{I}_0 \mathbf{r}(k)) \\ &+ \frac{1}{12} \sum_{i,j,l,m=2}^5 r_i(k)r_j(k)r_l(k)r_m(k) \frac{\partial^4 \Phi}{\partial r_i \partial r_j \partial r_l \partial r_m} (\mathbf{I}_{h_1(k)} \mathbf{r}(k)) \\ &+ \frac{1}{12} \sum_{i,j,l,m=2}^5 r_i(k)r_j(k)r_l(k)r_m(k) \frac{\partial^4 \Phi}{\partial r_i \partial r_j \partial r_l \partial r_m} (\mathbf{I}_{h_2(k)} \mathbf{r}(k)). \end{aligned} \quad (4.1)$$

Under the assumptions, $r_{i_1}(k) \cdots r_{i_n}(k) \sim \alpha_{i_1} \cdots \alpha_{i_n} (f(k))^n$ for all $i_1, \dots, i_n \in \{2, 3, 4, 5\}$ with $n \in \mathbf{N}$. According to the definition of Ψ , inserting these asymptotically equivalent expressions into (4.1) yields

$$\Psi(\mathbf{r}(k)) \sim 2(f(k))^2 \sum_{i,j=2}^5 \alpha_i \alpha_j \frac{\partial^2 \Phi}{\partial r_i \partial r_j} (\mathbf{I}_0 \mathbf{r}(k)) + (f(k))^4 R(k) \quad (4.2)$$

with

$$\begin{aligned} R(k) &= \frac{1}{12} \sum_{i,j,l,m=2}^5 \alpha_i \alpha_j \alpha_l \alpha_m \\ &\times \left(\frac{\partial^4 \Phi}{\partial r_i \partial r_j \partial r_l \partial r_m} (\mathbf{I}_{h_1(k)} \mathbf{r}(k)) + \frac{\partial^4 \Phi}{\partial r_i \partial r_j \partial r_l \partial r_m} (\mathbf{I}_{h_2(k)} \mathbf{r}(k)) \right). \end{aligned}$$

Note that $|\alpha_1|, |\alpha_6| < 1$ implies $\mathbf{I}_0 \boldsymbol{\alpha} \in \mathcal{R}$. Furthermore, according to Corollary 3.1, the second derivatives of Φ are continuous in $\mathbf{I}_0 \boldsymbol{\alpha}$. Since $\lim_{k \rightarrow \infty} \mathbf{I}_0 \mathbf{r}(k) = \mathbf{I}_0 \boldsymbol{\alpha}$, we obtain

$$\lim_{k \rightarrow \infty} \frac{\partial^2 \Phi}{\partial r_i \partial r_j} (\mathbf{I}_0 \mathbf{r}(k)) = \frac{\partial^2 \Phi}{\partial r_i \partial r_j} (\mathbf{I}_0 \boldsymbol{\alpha})$$

for all $i, j \in \{2, 3, 4, 5\}$. Inserting (4.3)–(4.6) from Lemma 4.1 in what follows with $\mathbf{r} = \boldsymbol{\alpha}$ into (4.2), we obtain

$$\Psi(\mathbf{r}(k)) \sim \frac{2(f(k))^2 q(\boldsymbol{\alpha})}{4\pi^2 \sqrt{(1 - \alpha_1^2)(1 - \alpha_6^2)}} + (f(k))^4 R(k),$$

with $q(\boldsymbol{\alpha})$ as given above.

In order to prove that $(f(k))^4 R(k) = O((f(k))^4)$, we show that $\sup_{k \in \mathbf{N}} R(k) < \infty$. Since $\lim_{k \rightarrow \infty} \mathbf{r}(k) = \mathbf{I}_0 \boldsymbol{\alpha} = \lim_{k \rightarrow \infty} \mathbf{I}_{-1} \mathbf{r}(k)$, the set

$$\mathcal{S} := \{\mathbf{I}_0 \boldsymbol{\alpha}\} \cup \bigcup_{k \in \mathbf{N}} \{\mathbf{r}(k), \mathbf{I}_{-1} \mathbf{r}(k)\}$$

is closed in $\mathbf{R}^6$. Since $\mathcal{S} \subset [-1, 1]^6$, the convex hull of $\mathcal{S}$ is compact. Now, since

$$\widetilde{\mathcal{S}} := \bigcup_{k \in \mathbf{N}} \{\mathbf{I}_h \mathbf{r}(k) : h \in [-1, 1]\}$$

is a subset of the convex hull of $\mathcal{S}$ and the fourth partial derivatives of Φ are continuous at every point of $\mathcal{R}$ (see Corollary 3.1),

$$\sup \frac{\partial^4 \Phi}{\partial r_i \partial r_j \partial r_l \partial r_m}(\widetilde{\mathcal{S}}) < \infty$$

for all $i, j, l, m \in \{2, 3, 4, 5\}$, and hence the result follows. Theorem 4.1 is proved.

Lemma 4.1. *For $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathcal{R}$, the second partial derivatives of Φ with respect to r_2, r_3, r_4, r_5 at $\mathbf{I}_0 \mathbf{r}$ are given by*

$$\frac{\partial^2 \Phi}{\partial r_i^2}(\mathbf{I}_0 \mathbf{r}) = \frac{r_1 r_6}{4\pi^2 \sqrt{(1-r_1^2)(1-r_6^2)}} \quad \text{for } i = 2, 3, 4, 5, \quad (4.3)$$

$$\frac{\partial^2 \Phi}{\partial r_2 \partial r_3}(\mathbf{I}_0 \mathbf{r}) = \frac{\partial^2 \Phi}{\partial r_4 \partial r_5}(\mathbf{I}_0 \mathbf{r}) = \frac{-r_1}{4\pi^2 \sqrt{(1-r_1^2)(1-r_6^2)}}, \quad (4.4)$$

$$\frac{\partial^2 \Phi}{\partial r_2 \partial r_4}(\mathbf{I}_0 \mathbf{r}) = \frac{\partial^2 \Phi}{\partial r_3 \partial r_5}(\mathbf{I}_0 \mathbf{r}) = \frac{-r_6}{4\pi^2 \sqrt{(1-r_1^2)(1-r_6^2)}}, \quad (4.5)$$

$$\frac{\partial^2 \Phi}{\partial r_2 \partial r_5}(\mathbf{I}_0 \mathbf{r}) = \frac{\partial^2 \Phi}{\partial r_3 \partial r_4}(\mathbf{I}_0 \mathbf{r}) = \frac{1}{4\pi^2 \sqrt{(1-r_1^2)(1-r_6^2)}}. \quad (4.6)$$

P r o o f. Let $k \in \{2, 3, 4, 5\}$. According to (3.2)–(3.5), there exist unique numbers $i, j \in \{1, 2, 3, 4\}$ such that

$$\frac{\partial \Phi}{\partial r_k}(\mathbf{r}) = \frac{1}{2\pi} f(\mathbf{r}) \left(\frac{1}{4} + \frac{1}{2\pi} g(\mathbf{r}) \right)$$

with

$$f(\mathbf{r}) = \frac{1}{\sqrt{1-r_k^2}} \quad \text{and} \quad g(\mathbf{r}) = \arcsin \frac{\sigma_{ij}(\mathbf{r})}{\sqrt{\sigma_{ii}(\mathbf{r})\sigma_{jj}(\mathbf{r})}}.$$

Clearly, $f(\mathbf{I}_0 \mathbf{r}) = 1$ and $\frac{\partial f}{\partial r_l}(\mathbf{I}_0 \mathbf{r}) = 0$ for $l = 2, 3, 4, 5$ and hence

$$\frac{\partial^2 \Phi}{\partial r_k \partial r_l}(\mathbf{I}_0 \mathbf{r}) = \frac{1}{4\pi^2} \frac{\partial g}{\partial r_l}(\mathbf{I}_0 \mathbf{r}).$$

Since $\sigma_{ij}(\mathbf{I}_0 \mathbf{r}) = 0$ and the first derivative of arcsin in 0 is 1, we obtain

$$\frac{\partial^2 \Phi}{\partial r_k \partial r_l}(\mathbf{I}_0 \mathbf{r}) = \frac{1}{4\pi^2 \sqrt{\sigma_{ii}(\mathbf{I}_0 \mathbf{r})\sigma_{jj}(\mathbf{I}_0 \mathbf{r})}} \frac{\partial \sigma_{ij}}{\partial r_l}(\mathbf{I}_0 \mathbf{r}).$$

Now, the result follows by (3.10)–(3.13). Lemma 4.1 is proved.

5. Bounds and approximations. Further, we apply the previous results to vectors $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6) \in \mathcal{R}$ with $r_1 = r_6$ and $r_2 = r_5$. Let $\pi^*(\mathbf{r}) := (r_1, r_2, r_3, r_4, r_2, r_1)$ for $\mathbf{r} = (r_1, r_2, r_3, r_4) \in (-1, 1)^4$, and

$$\mathcal{R}^* := \{\mathbf{r} \in (-1, 1)^4 \mid \pi^*(\mathbf{r}) \in \mathcal{R}\}.$$

Clearly, $\mathbf{r} = (r_1, r_2, r_3, r_4) \in \mathcal{R}^*$ if and only if

$$\Sigma(\pi^*(\mathbf{r})) = \begin{pmatrix} 1 & r_1 & r_2 & r_3 \\ r_1 & 1 & r_4 & r_2 \\ r_2 & r_4 & 1 & r_1 \\ r_3 & r_2 & r_1 & 1 \end{pmatrix}$$

is strictly positive definite. Since $\mathcal{R}^* \subset \mathcal{R}$ and $\mathcal{R}$ is convex, $\mathbf{r}, \mathbf{s} \in \mathcal{R}^*$ implies $\pi^*((1-h)\mathbf{r} + h\mathbf{s}) \in \mathcal{R}$ and hence $(1-h)\mathbf{r} + h\mathbf{s} \in \mathcal{R}^*$ for all $h \in [0, 1]$. Thus, $\mathcal{R}^*$ is convex. Now, define $\Psi^*(\mathbf{r}) := \Psi(\pi^*(\mathbf{r}))$ for $\mathbf{r} \in \mathcal{R}^*$. According to (2.8),

$$\gamma_k = \Psi^*(\rho_1, \rho_k, \rho_{k+1}, \rho_{k-1}) \quad (5.1)$$

for $k > 1$. The following corollary is a special case of Theorem 4.1.

Corollary 5.1. *Let $(\mathbf{r}(k))_{k \in \mathbf{N}}$ be a sequence in $\mathcal{R}^*$ and assume $f: \mathbf{N} \rightarrow \mathbf{R}$ is a function with $\lim_{k \rightarrow \infty} f(k) = 0$.*

(i) *If $\mathbf{r}(k) \sim (\alpha_1, \alpha_2 f(k), \alpha_3 f(k), \alpha_4 f(k))$, for some vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $|\alpha_1| < 1$, then*

$$\Psi^*(\mathbf{r}(k)) \sim \frac{(f(k))^2 q(\boldsymbol{\alpha})}{2\pi^2(1 - \alpha_1^2)} + O((f(k))^4),$$

where $q(\boldsymbol{\alpha}) = \alpha_1^2(2\alpha_2^2 + \alpha_3^2 + \alpha_4^2) - 4\alpha_1\alpha_2(\alpha_3 + \alpha_4) + 2(\alpha_2^2 + \alpha_3\alpha_4)$.

(ii) *If $f(k+1) \sim \beta f(k)$ for some $\beta \neq 0$ and there exists an α with $|\alpha| < 1$ such that $\mathbf{r}(k) \sim (\alpha, f(k), f(k+1), f(k-1))$, then*

$$\Psi^*(\mathbf{r}(k)) \sim \frac{(f(k))^2 (2 - \alpha(\beta + \beta^{-1}))^2}{2\pi^2(1 - \alpha^2)} + O((f(k))^4).$$

(iii) *If the assumptions of (ii) hold with $\beta = 1$, then*

$$\Psi^*(\mathbf{r}(k)) \sim \frac{2(f(k))^2 (1 - \alpha)}{\pi^2(1 + \alpha)} + O((f(k))^4).$$

P r o o f. (i) Follows by Theorem 4.1 and the fact that $\pi^*(\mathbf{r}(k))$ is asymptotically equivalent to $(\alpha_1, \alpha_2 f(k), \alpha_3 f(k), \alpha_4 f(k), \alpha_2 f(k), \alpha_1)$.

(ii) is a special case of (i), where $\mathbf{r}(k) \sim (\alpha, f(k), \beta f(k), f(k)/\beta)$ and thus

$$q(\alpha, 1, \beta, \beta^{-1}) = \alpha^2(2 + \beta^2 + \beta^{-2}) - 4\alpha(\beta + \beta^{-1}) + 4 = (2 - \alpha(\beta + \beta^{-1}))^2.$$

Now, (iii) is obvious.

Corollary 5.1 is proved.

5.1. Lower and upper bounds. Theorem 5.1 in what follows gives sufficient conditions on $\mathbf{r} \in \mathcal{R}^*$ to obtain lower and upper bounds for $\Psi^*(\mathbf{r})$ by setting r_2, r_3, r_4 all equal to r_3 and r_4 , respectively. We first prove the following lemma.

Lemma 5.1. *For every $\mathbf{r} = (r_1, r_2, r_3, r_4) \in \mathcal{R}^*$,*

$$\frac{\partial \Psi^*}{\partial r_2}(\mathbf{r}) = \frac{2}{\pi^2 \sqrt{1 - r_2^2}} \arcsin \frac{\sigma_{13}(\pi^*(\mathbf{r}))}{\sqrt{\sigma_{11}(\pi^*(\mathbf{r}))\sigma_{22}(\pi^*(\mathbf{r}))}}, \quad (5.2)$$

$$\frac{\partial \Psi^*}{\partial r_3}(\mathbf{r}) = \frac{1}{\pi^2 \sqrt{1 - r_3^2}} \arcsin \frac{\sigma_{23}(\pi^*(\mathbf{r}))}{\sigma_{22}(\pi^*(\mathbf{r}))}, \quad (5.3)$$

$$\frac{\partial \Psi^*}{\partial r_4}(\mathbf{r}) = \frac{1}{\pi^2 \sqrt{1 - r_4^2}} \arcsin \frac{\sigma_{14}(\pi^*(\mathbf{r}))}{\sigma_{11}(\pi^*(\mathbf{r}))}, \quad (5.4)$$

and

$$\sigma_{13}(\pi^*(\mathbf{r})) = r_2 - r_1 r_3 + r_2 r_3 r_4 - r_2^3 - r_1 r_4 + r_2 r_1^2, \quad (5.5)$$

$$\sigma_{14}(\pi^*(\mathbf{r})) = r_3 - 2r_1 r_2 + r_4 r_2^2 - r_3 r_4^2 + r_4 r_1^2, \quad (5.6)$$

$$\sigma_{23}(\pi^*(\mathbf{r})) = r_4 - 2r_1 r_2 + r_3 r_2^2 - r_4 r_3^2 + r_3 r_1^2. \quad (5.7)$$

P r o o f. The validity of (5.5)–(5.7) directly follows from (3.10)–(3.12). Furthermore,

$$\frac{\partial \Psi^*}{\partial r_2}(\mathbf{r}) = \frac{\partial \Psi}{\partial r_2}(\pi^*(\mathbf{r})) + \frac{\partial \Psi}{\partial r_5}(\pi^*(\mathbf{r})), \quad \frac{\partial \Psi^*}{\partial r_i}(\mathbf{r}) = \frac{\partial \Psi}{\partial r_i}(\pi^*(\mathbf{r})), \quad i = 3, 4.$$

Since $\sigma_{11}(\pi^*(\mathbf{r})) = \sigma_{44}(\pi^*(\mathbf{r}))$, $\sigma_{22}(\pi^*(\mathbf{r})) = \sigma_{33}(\pi^*(\mathbf{r}))$, and $\sigma_{13}(\pi^*(\mathbf{r})) = \sigma_{24}(\pi^*(\mathbf{r}))$ (compare to (3.6)–(3.9), (3.10), and (3.13)), we obtain (5.2)–(5.4) by equations (3.14)–(3.17) in Lemma 3.2. Lemma 5.1 is proved.

Theorem 5.1. *Let $\mathbf{r} = (r_1, r_2, r_3, r_4) \in \mathcal{R}^*$ with $r_4, r_2 \geq r_3 \geq 0$. For $h \in [0, 1]$, define $\mathbf{s}_h := (1 - h) \cdot \mathbf{r} + h \cdot (r_1, r_3, r_3, r_3)$ and $\mathbf{t}_h := (1 - h) \cdot \mathbf{r} + h \cdot (r_1, r_4, r_4, r_4)$.*

1. *If $1 + r_1 - 2r_3 > 0$ and $\sigma_{13}(\pi^*(\mathbf{s}_h)), \sigma_{14}(\pi^*(\mathbf{s}_h)) \geq 0$ for all $h \in [0, 1]$, then*

$$\Psi^*(\mathbf{r}) \geq \Psi^*(r_1, r_3, r_3, r_3). \quad (5.8)$$

2. *If $1 + r_1 - 2r_4 > 0$ and $\sigma_{13}(\pi^*(\mathbf{t}_h)), \sigma_{23}(\pi^*(\mathbf{t}_h)) \geq 0$ for all $h \in [0, 1]$, then*

$$\Psi^*(r_1, r_4, r_4, r_4) \geq \Psi^*(\mathbf{r}). \quad (5.9)$$

P r o o f. 1. First, note that the set of eigenvalues of $\Sigma(\pi^*(r_1, r_3, r_3, r_3))$ is given by $\{1 - r_1, 1 + r_1 - 2r_3, 1 + r_1 + 2r_3\}$. Under the assumptions, each eigenvalue is strictly larger than 0, so $(r_1, r_3, r_3, r_3) \in \mathcal{R}^*$. Since $\mathcal{R}^*$ is convex,

we have $\mathbf{s}_h \in \mathcal{R}^*$ for all $h \in [0, 1]$. Hence, $f(h) := \Psi^*(\mathbf{s}_h)$ is well defined for all $h \in [0, 1]$. Since $f(0) = \Psi^*(\mathbf{r})$ and $f(1) = \Psi^*(r_1, r_3, r_3, r_3)$, it is sufficient to show that $h \mapsto f(h)$ is monotonically decreasing on $[0, 1]$, or, equivalently,

$$f'(h) = (r_3 - r_2) \frac{\partial \Psi^*(\mathbf{s}_h)}{\partial r_2} + (r_3 - r_4) \frac{\partial \Psi^*(\mathbf{s}_h)}{\partial r_4} \leq 0$$

for all $h \in [0, 1]$. With the assumptions $r_3 - r_2 \leq 0$ and $r_3 - r_4 \leq 0$, a sufficient condition for this inequality to be satisfied is $\sigma_{13}(\pi^*(\mathbf{s}_h)) \geq 0$ and $\sigma_{14}(\pi^*(\mathbf{s}_h)) \geq 0$ for all $h \in [0, 1]$ (compare to (5.2) and (5.4)).

2. Analogously, define $g(h) := \Psi^*(\mathbf{t}_h)$, and note that a sufficient condition for

$$g'(h) = (r_4 - r_2) \frac{\partial \Psi(\mathbf{t}_h)}{\partial r_2} + (r_4 - r_3) \frac{\partial \Psi(\mathbf{t}_h)}{\partial r_3} \geq 0$$

is given by $\sigma_{13}(\pi^*(\mathbf{t}_h)) \geq 0$ and $\sigma_{23}(\pi^*(\mathbf{t}_h)) \geq 0$ for all $h \in [0, 1]$. Theorem 5.2 is proved.

As the proof of Theorem 5.1 shows, a sufficient condition for strict inequality in (5.8) is given by $r_4 > r_3$ and $\sigma_{14}(\pi^*(\mathbf{s}_h)) > 0$ for some $h \in [0, 1]$, or $r_2 > r_3$ and $\sigma_{13}(\pi^*(\mathbf{s}_h)) > 0$ for some $h \in [0, 1]$. Analogously, a sufficient condition for strict inequality in (5.9) is given by $r_4 > r_3$ and $\sigma_{23}(\pi^*(\mathbf{t}_h)) > 0$ for some $h \in [0, 1]$, or $r_4 > r_2$ and $\sigma_{13}(\pi^*(\mathbf{t}_h)) > 0$ for some $h \in [0, 1]$.

The following lemma gives easily verifiable conditions for the assumptions of Theorem 5.1.

Lemma 5.2. *Let $\mathbf{r} = (r_1, r_2, r_3, r_4) \in \mathcal{R}^*$ with $r_1 \leq 0$ and $r_2, r_3, r_4 \geq 0$. Then $\sigma_{13}(\mathbf{s}_h), \sigma_{14}(\mathbf{s}_h) > 0$ and $\sigma_{13}(\mathbf{t}_h), \sigma_{23}(\mathbf{t}_h) > 0$ for all $h \in [0, 1]$.*

P r o o f. For fixed $h \in [0, 1]$, let $\mathbf{s}_h = (s_1, s_2, s_3, s_4)$. Clearly, $s_1 \leq 0$ and $s_2, s_3, s_4 \in [0, 1]$. Since $\sigma_{13}(\pi^*(\mathbf{s}_h)) \geq s_2 - s_2^2$ and $\sigma_{14}(\pi^*(\mathbf{s}_h)) \geq s_3 - s_3^2$, we obtain $\sigma_{13}(\mathbf{s}_h), \sigma_{14}(\mathbf{s}_h) > 0$. Analogously, let $\mathbf{t}_h = (t_1, t_2, t_3, t_4)$, and note that $\sigma_{23}(\pi^*(\mathbf{t}_h)) \geq t_4 - t_4^2$. Lemma 5.2 is proved.

5.2. Approximations of the bounds. Further, we analyze approximations of the lower and upper bounds of $\Psi^*(\mathbf{r})$ given by Theorem 5.1. Let $\mathcal{R}^{**}$ be the set of $\mathbf{r} = (r_1, r_2) \in (-1, 1)^2$ such that $\pi^{**}(\mathbf{r}) := (r_1, r_2, r_2, r_2) \in \mathcal{R}^*$ or, equivalently,

$$\Sigma(\pi^{**}(\mathbf{r})) = \begin{pmatrix} 1 & r_1 & r_2 & r_2 \\ r_1 & 1 & r_2 & r_2 \\ r_2 & r_2 & 1 & r_1 \\ r_2 & r_2 & r_1 & 1 \end{pmatrix}$$

is strictly positive definite. Since the set of eigenvalues of $\Sigma(\pi^{**}(\mathbf{r}))$ is given by $\{1 - r_1, 1 + r_1 + 2r_2, 1 + r_1 - 2r_2\}$,

$$\mathcal{R}^{**} = \{(r_1, r_2) \in (-1, 1)^2 \mid 2|r_2| < 1 + r_1\}. \quad (5.10)$$

For $\mathbf{r} \in \mathcal{R}^{**}$, define $\Phi^{**}(\mathbf{r}) := \Phi(\pi^{**}(\mathbf{r}))$ and $\Psi^{**}(\mathbf{r}) := \Psi(\pi^{**}(\mathbf{r}))$. Note that $\sigma_{ii}(\pi^{**}(\mathbf{r})) = (1 - r_1)(1 + r_1 - 2r_2^2)$ for $i = 1, 2, 3, 4$ and $\sigma_{13}(\pi^{**}(\mathbf{r})) = \sigma_{14}(\pi^{**}(\mathbf{r})) = \sigma_{23}(\pi^{**}(\mathbf{r})) = \sigma_{24}(\pi^{**}(\mathbf{r})) = r_2(1 - r_1)^2$ (compare to (3.6)–(3.13)). Hence, according to (3.2)–(3.5),

$$\begin{aligned} \frac{\partial \Phi^{**}}{\partial r_2}(\mathbf{r}) &= \frac{\partial \Phi}{\partial r_2}(\pi^{**}(\mathbf{r})) + \frac{\partial \Phi}{\partial r_3}(\pi^{**}(\mathbf{r})) + \frac{\partial \Phi}{\partial r_4}(\pi^{**}(\mathbf{r})) + \frac{\partial \Phi}{\partial r_5}(\pi^{**}(\mathbf{r})) \\ &= \frac{2}{\pi \sqrt{1 - r_2^2}} \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{r_2(1 - r_1)}{1 + r_1 - 2r_2^2} \right). \end{aligned} \quad (5.11)$$

By (3.19), we obtain the integral representation

$$\begin{aligned} \Psi^{**}(\mathbf{r}) &= \frac{4r_2}{\pi^2} \int_0^1 \frac{1}{\sqrt{1 - r_2^2 h^2}} \arcsin \frac{r_2(1 - r_1)h}{1 + r_1 - 2r_2^2 h^2} dh \\ &= \frac{4}{\pi^2} \int_0^{r_2} \frac{1}{\sqrt{1 - t^2}} \arcsin \frac{(1 - r_1)t}{1 + r_1 - 2t^2} dt. \end{aligned}$$

As the following theorem shows, $\Psi^{**}(\mathbf{r})$ can be approximated monotonically from below by successively adding further terms of the Taylor expansion of $\Psi^{**}(\mathbf{r})$ in $(r_1, 0)$.

Theorem 5.2. *For every $\mathbf{r} = (r_1, r_2) \in \mathcal{R}^{**}$,*

$$\frac{\partial^l \Phi^{**}}{\partial^l r_2}((r_1, 0)^T) \geq 0 \quad \text{for } l \in \mathbf{N}_0, \quad (5.12)$$

$$\Psi^{**}(\mathbf{r}) = 4 \sum_{l=1}^{\infty} \frac{r_2^{2l}}{(2l)!} \frac{\partial^{2l} \Phi^{**}}{\partial^{2l} r_2}(r_1, 0). \quad (5.13)$$

P r o o f. Let $\mathbf{r} = (r_1, r_2) \in \mathcal{R}^{**}$. We define $f(x) := 2\pi^{-1} \arcsin x$ for $x \in (-1, 1)$, and $g_1(x) := x(1 - r_1)$, $g_2(x) := (1 + r_1 - 2x^2)^{-1}$, $g(x) := g_1(x)g_2(x)$ and $h(x) := f(g(x))$ for $x \in (-(1 + r_1)/2, (1 + r_1)/2)$. Clearly, $\Phi^{**}(r_1, 0) \geq 0$, hence (5.12) is true for $l = 0$. According to (5.10), $|r_2| < (1 + r_1)/2$, so (5.11) yields $\frac{\partial \Phi^{**}}{\partial r_2}(\mathbf{r}) = f'(r_2) + 4f'(r_2)h(r_2)$. Applying Leibniz's rule gives

$$\begin{aligned} \frac{\partial^l \Phi^{**}}{\partial^l r_2}((r_1, 0)^T) &= f^{(l)}(0) + 4 \sum_{k=0}^{l-1} \binom{l-1}{k} f^{(k+1)}(0) h^{(l-1-k)}(0) \\ &= f^{(l)}(0) + 4 \sum_{k=1}^l \binom{l-1}{k-1} f^{(k)}(0) h^{(l-k)}(0) \end{aligned}$$

for $l \in \mathbf{N}$. Note that

$$\arcsin x = \sum_{n=0}^{\infty} \frac{3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots (2n) \cdot (2n+1)} x^{2n+1},$$

so $f^{(l)}(0) \geq 0$. Therefore, in order to prove (5.12), it is sufficient to show that $h^{(l)}(0) \geq 0$ for all $l \in \mathbf{N}$.

Let $g_2(x) = f_2(f_1(x))$ with $f_1(x) := 1 + r_1 - 2x^2$, $f_2(x) := 1/x$. Note that $f_1^{(l)}(0) \neq 0$ only if $l \in \{0, 2\}$. For each $l \in \mathbf{N}$, we can write $g_2^{(l)}(0) = (f_2 \circ f_1)^{(l)}(0)$ as the sum of terms $f_2^{(k)}(f_1(0)) \cdot f_1^{(i_1)}(0) \cdots f_1^{(i_k)}(0)$ with $k, i_1, \dots, i_k \in \mathbf{N}$ which satisfy $i_1 + \cdots + i_k = l$. Each term can only be nonzero if $i_1 = \cdots = i_k = 2$, hence a necessary condition for $g_2^{(l)}(0) \neq 0$ is that l is even. Moreover, $g_2^{(l)}(0) > 0$ in this case, since $f_2^{(k)}(f_1(0)) = (-1)^k k! (1 + r_1)^{-(k+1)}$ and $f_1^{(2)}(0) = -4$. Note that $g_1^{(k)}(0) \neq 0$ only if $k = 1$, consequently, by Leibniz's rule, $g^{(l)}(0) = l g_1^{(1)}(0) g_2^{(l-1)}(0) = l(1 - r_1) g_2^{(l-1)}(0)$ for all $l \in \mathbf{N}$, and hence $g^{(l)}(0) \geq 0$ for all $l \in \mathbf{N}$.

Now, similarly to above discussion, we can write $h^{(l)}(0) = (f \circ g)^{(l)}(0)$ for each $l \in \mathbf{N}$ as the sum of products consisting of factors of the form $f^{(k)}(g(0)) = f^{(k)}(0)$ and $g^{(m)}(0)$ with $k, m \in \mathbf{N}$, implying $h^{(l)}(0) \geq 0$.

In order to prove (5.13), first note that g_2 and $g = g_1 g_2$ have power series expansions at 0 with the radius of convergence $(1 + r_1)/2$, and $g((-1 + r_1)/2, (1 + r_1)/2) \subset (-1, 1)$. Since f has a power series expansion at 0 with the radius of convergence 1, according to elementary properties of power series, the mapping $\cdot \mapsto \frac{\partial \Phi^{**}}{\partial r_2}(r_1, \cdot) = f'(\cdot) + 4f'(\cdot)f(g(\cdot))$ has a power series expansion at 0 with the radius of convergence $(1 + r_1)/2$, and hence it also holds for the mapping $\cdot \mapsto \Phi^{**}(r_1, \cdot)$.

Now, note that $r_2 \in (-(1 + r_1)/2, (1 + r_1)/2)$ (see (5.10)). Therefore, according to the definition of Ψ ,

$$\begin{aligned} \Psi^{**}(\mathbf{r}) &= 2\Phi^{**}(\mathbf{r}) + 2\Phi^{**}(r_1, -r_2) - 4\Phi^{**}(r_1, 0) \\ &= 2 \sum_{l=0}^{\infty} \frac{r_2^l}{l!} \frac{\partial^l \Phi^{**}}{\partial r_2} (r_1, 0) + 2 \sum_{l=0}^{\infty} \frac{(-r_2)^l}{l!} \frac{\partial^l \Phi^{**}}{\partial r_2} (r_1, 0) - 4\Phi^{**}(r_1, 0) \\ &= 4 \sum_{l=1}^{\infty} \frac{r_2^{2l}}{(2l)!} \frac{\partial^{2l} \Phi^{**}}{\partial^{2l} r_2} (r_1, 0). \end{aligned}$$

Theorem 5.3 is proved.

6. The variance of the empirical zero crossing rate. In this section, we apply the previous results to the analysis of the variance of empirical zero crossing rates. Recall formula (2.1),

$$\mathbf{D}\hat{c}_n = \frac{1}{n^2} \left(n\gamma_0 + 2 \sum_{k=1}^{n-1} (n-k)\gamma_k \right).$$

In order to evaluate $\mathbf{D}(\hat{c}_n)$ numerically, we can use formulas (2.4) and (2.5) for the computation of γ_0 and γ_1 . For $k > 1$, formula (5.1) yields

$$\gamma_k = \Psi^*(\rho_1, \rho_k, \rho_{k+1}, \rho_{k-1}),$$

and the right-hand side can be evaluated numerically using the integral representation of Ψ given in (3.19).

When n is large, an «exact» numerical evaluation of γ_k for every $k = 0, 1, \dots, n-1$ is time-consuming. A quick way for getting approximate values of $\mathbf{D}\hat{c}_n$ is to use approximations of γ_k in terms of the function Ψ^{**} . If the assumptions of Theorem 5.1 are satisfied, this yields upper and lower bounds for γ_k . A further speed-up can be achieved by using the finite-order approximations of Ψ^{**} provided by Theorem 5.2. For instance, when the autocorrelations of $\mathbf{Y}$ are not too large, one can use the first-order approximation

$$\gamma_{\vartheta}(k) \approx \frac{2(1 - \rho_{\vartheta}(1))}{\pi^2(1 + \rho_{\vartheta}(1))} (\rho_{\vartheta}(k))^2.$$

An alternative method for computing approximate values of $\mathbf{D}\hat{c}_n$ is to use the exact values of γ_k for $k = 2, 3, \dots$ until the relative error of the approximations falls below a given threshold $\varepsilon > 0$, and then to use the approximations of γ_k . If the relative error does not get larger than ε anymore, then also the relative error of the resulting approximation of $\mathbf{D}_{\vartheta}\hat{c}_n$ is not larger than ε . For the calculations behind Fig. 7.1–7.3, we have used this method with the threshold $\varepsilon = 0.001$.

The following theorem establishes asymptotics of $\mathbf{D}\hat{c}_n$.

Theorem 6.1. *Suppose there exists a mapping $f: \mathbf{N} \rightarrow \mathbf{R}$ such that $\rho_k \sim f(k)$.*

(i) *If $|f(k)| = o(k^{-\beta})$ with $\beta > 1/2$, then $\sigma^2 := \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k < \infty$ and*

$$\mathbf{D}\hat{c}_n \sim \sigma^2 n^{-1}.$$

(ii) *If $f(k) = \alpha k^{-1/2}$ for some $\alpha \in (-1, 1) \setminus \{0\}$, then*

$$\mathbf{D}\hat{c}_n \sim \frac{4\alpha^2(1 - \rho_1)}{\pi^2(1 + \rho_1)} \frac{\ln n}{n}.$$

(iii) *If $f(k) = \alpha k^{-\beta}$ for some $\alpha \in (-1, 1) \setminus \{0\}$ and $\beta \in (0, \frac{1}{2})$, then*

$$\mathbf{D}\hat{c}_n \sim \frac{4\alpha^2(1 - \rho_1)}{\pi^2(1 + \rho_1)(1 - 2\beta)} n^{-2\beta}.$$

P r o o f. (i) According to Corollary 5.1 (i), we have $\gamma_k = O((f(k))^2)$, which shows that $\sum_{k=1}^{\infty} |\gamma_k| < \infty$. By the Dominated Convergence Theorem, we obtain

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{n-k}{n} \gamma_k &\sim \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \max \left\{ \frac{n-k}{n}, 0 \right\} \gamma_k \\ &= \sum_{k=1}^{\infty} \lim_{n \rightarrow \infty} \max \left\{ \frac{n-k}{n}, 0 \right\} \gamma_k = \sum_{k=1}^{\infty} \gamma_k. \end{aligned}$$

Now, with formula (2.1), the result follows.

(ii) Note that $f(k) \sim f(k+1)$ and thus, according to Corollary 5.1 (iii),

$$\gamma_k \sim \frac{2\alpha^2(1-\rho_1)}{\pi^2(1+\rho_1)} k^{-1}.$$

Using the fact that $\sum_{k=1}^{n-1} k^{-1} \sim \ln n$, we obtain

$$\sum_{k=1}^{n-1} \gamma_k \sim \frac{2\alpha^2(1-\rho_1)}{\pi^2(1+\rho_1)} \ln n. \quad (6.1)$$

Furthermore, we have

$$\sum_{k=1}^{n-1} \gamma_k - \sum_{k=1}^{n-1} \frac{n-k}{n} \gamma_k = \sum_{k=1}^{n-1} \frac{k}{n} \gamma_k \sim \frac{1}{n} \sum_{k=1}^{n-1} \frac{2\alpha^2(1-\rho_1)}{\pi^2(1+\rho_1)} = o(\ln n),$$

which shows that

$$\sum_{k=1}^{n-1} \gamma_k \sim \sum_{k=1}^{n-1} \frac{n-k}{n} \gamma_k.$$

According to formula (2.1), we obtain

$$\mathbf{D}\hat{c}_n \sim \frac{2}{n} \sum_{k=1}^{n-1} \gamma_k,$$

and together with (6.1) the statement follows.

(iii) The proof is similar to (ii), using the fact $\sum_{k=1}^{n-1} k^{-2\beta} \sim (1-2\beta)^{-1} n^{1-2\beta}$. Theorem 6.1 is proved.

7. Examples. In this section, we apply the previous results to empirical zero crossing rates in AR(1) processes, fractional Gaussian noise and FARIMA(0, d , 0) processes.

7.1. AR(1) processes. Assume that $\mathbf{Y}$ is an AR(1) process with autoregressive coefficient $a \in (-1, 1)$, that is, $\mathbf{Y}$ is stationary, nondegenerate and zero-mean Gaussian with the autocorrelations $\rho_k = a^k$ for $k \in \mathbf{N}_0$ (where $0^0 := 1$). According to formula (2.3),

$$\mathbf{P}(C_0 = 1) = \frac{1}{2} - \frac{1}{\pi} \arcsin a, \quad (7.1)$$

hence the higher the autoregressive coefficient, the lower the probability of a zero crossing.

Fig. 7.1. Variance of $\hat{c}_n$ in AR(1) processes with $a \in (-1, 1)$ and $n = 10, 11, \dots, 100$

By using the method explained in Section 6., we can evaluate the variance of $\hat{c}_n$. Fig. 7.1 displays the values of $\mathbf{D}\hat{c}_n$ for $n = 10, 11, \dots, 100$ and a ranging in $(-1, 1)$. For fixed n , the variance of $\hat{c}_n$ tends to 0 as a tends to -1 and 1 , respectively. According to (7.1), the probability of a zero crossing is equal to 1 and 0 in these limit cases, and thus $\hat{c}_n$ is $\mathbf{P}$ -almost surely equal to 1 and 0, respectively.

For fixed a , the variance of $\hat{c}_n$ is decreasing in n . In particular, according to Theorem 6.1 (i),

$$\mathbf{D}\hat{c}_n \sim \sigma^2 n^{-1},$$

where $\sigma^2 := \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k < \infty$. In the case $a = 0$, formulas (2.4) and (2.5) yield $\gamma_0 = 1/4$ and $\gamma_1 = 0$. Furthermore, according to Lemma 3.1,

$$\gamma_k = \Psi^*(0, 0, 0, 0) = 0$$

for all $k > 1$. Therefore, $\mathbf{D}\hat{c}_n = 1/(4n)$ in this case.

Remarkably, $\mathbf{D}\hat{c}_n$ is always identical for a and $-a$. In fact, one can show that γ_k is identical for a and $-a$ for all $k \in \mathbf{Z}$. For $k = 0$ and $k = 1$, this is an immediate consequence of formulas (2.4) and (2.5) and the fact that $(\arcsin a)^2 = (\arcsin(-a))^2$. For $k > 1$, this is true because, according to Lemma 3.1,

$$\Psi^*(a, a^k, a^{k+1}, a^{k-1}) = \Psi^*(-a, (-a)^k, (-a)^{k+1}, (-a)^{k-1}).$$

7.2. Fractional Gaussian noise. Assume that $\mathbf{Y}$ is fractional Gaussian noise with the Hurst parameter $H \in (0, 1)$, that is, $\mathbf{Y}$ is stationary, nondegenerate and zero-mean Gaussian with the autocorrelations

$$\rho_k = \frac{1}{2} (|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H})$$

for $k \in \mathbf{Z}$. With $\rho_1 = 2^{2H-1} - 1$, we obtain that the probability of a zero crossing is given by

$$\mathbf{P}(C_0 = 1) = \frac{1}{2} - \frac{1}{\pi} \arcsin(2^{2H-1} - 1) = 1 - \frac{2}{\pi} \arcsin 2^{H-1}, \quad (7.2)$$

where the second equation follows from $\arcsin x = 2 \arcsin \sqrt{(1+x)/2} - \pi/2$. Thus, the larger the Hurst parameter, the lower the probability of a zero crossing.

Fig. 7.2. Variance of $\hat{c}_n$ in fractional Gaussian noise for $H \in (0, 1)$ and $n = 10, 11, \dots, 100$

Figure 7.2 displays $\mathbf{D}(\hat{c}_n)$ for $n = 10, 11, \dots, 100$ and H ranging in $(0, 1)$. For fixed n , the variance tends to 0 as H tends to 1. Note that the probability of a zero crossing is 0 in the limit case (see (7.2)), and thus $\hat{c}_n$ is almost surely equal to 0.

Further, we derive asymptotics of $\mathbf{D}\hat{c}_n$. It is well known that $\rho_k \sim H(2H-1)k^{2H-2}$ as $k \rightarrow \infty$ (see [3]). By Theorem 6.1 (i), we obtain

$$\mathbf{D}\hat{c}_n \sim \sigma^2 n^{-1}$$

for $H < 3/4$, where $\sigma^2 := \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k < \infty$. In the case $H = 1/2$, where $\rho_k = 0$ for all $k > 0$, we obtain $\mathbf{D}\hat{c}_n = 1/(4n)$ by the same argument as in the case $a = 0$ for AR(1) processes. If $H = 3/4$, then Theorem 6.1 (ii) yields

$$\mathbf{D}\hat{c}_n \sim \frac{9(\sqrt{2}-1)}{16\pi^2} \frac{\ln n}{n}$$

(in particular, $H^2(2H-1)^2(2^{2-2H}-1) = 9(\sqrt{2}-1)/64$ in this case). Finally, if $H > 3/4$, then Theorem 6.1 (iii) yields

$$\mathbf{D}\hat{c}_n \sim \frac{4H^2(2H-1)^2(2^{2-2H}-1)}{\pi^2(4H-3)} n^{4H-4}.$$

7.3. FARIMA(0, d , 0) processes. If $\mathbf{Y}$ is an FARIMA(0, d , 0) process with the fractional differencing parameter $d \in (-1/2, 1/2)$, then $\mathbf{Y}$ is stationary, nondegenerate and zero-mean Gaussian with the autocorrelations

$$\rho_k = \frac{\Gamma(1-d)\Gamma(k+d)}{\Gamma(d)\Gamma(k+1-d)}$$

for $k \in \mathbf{Z}$. With $\rho_1 = d/(1-d)$, we obtain

$$\mathbf{P}(C_0 = 1) = \frac{1}{2} - \frac{1}{\pi} \arcsin \frac{d}{1-d}$$

for the probability of a zero crossing.

Fig. 7.3. Variance of $\hat{c}_n$ in FARIMA(0, d , 0) for $d \in (-1/2, 1/2)$ and $n = 10, 11, \dots, 100$

Figure 7.3 displays the variance of $\hat{c}_n$ for $n = 10, 11, \dots, 100$ and $d \in (-1/2, 1/2)$. The picture is very similar to Fig. 7.2. In particular, the variance is only slowly decreasing for large parameter values and tends to 0 as d tends to $1/2$.

Next, we derive asymptotics of $\mathbf{D}\hat{c}_n$. It is well known that $\rho_k \sim (\Gamma(1-d)/\Gamma(d))k^{2d-1}$ as $k \rightarrow \infty$ (see [3]). According to Theorem 6.1 (i), we obtain

$$\mathbf{D}\hat{c}_n \sim \sigma^2 n^{-1}$$

for $d < 1/4$, where $\sigma^2 := \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k < \infty$. In the case $d = 0$, where $\rho_k = 0$ for all $k > 0$, we obtain $\mathbf{D}\hat{c}_n = 1/(4n)$. If $d = 1/4$, then Theorem 6.1 (ii) yields

$$\mathbf{D}\hat{c}_n \sim \frac{2(\Gamma(3/4))^2}{\pi^2(\Gamma(1/4))^2} \frac{\ln n}{n}.$$

In the case $d > 1/4$, Theorem 6.1 (iii) yields

$$\mathbf{D}\hat{c}_n \sim \frac{4(\Gamma(1-d))^2(1-2d)}{\pi^2(\Gamma(d))^2(4d-1)} n^{4d-2}.$$

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