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$\varepsilon > 0$, согласно лемме 2 найдем n_1 такое, что при всех n выполнено неравенство $\mathbf{P}(\tau_n \leq n - n_1 | M_n \geq \theta n) \leq \varepsilon/2$. Рассматривая для каждого $l \leq n_1$ компакт $L_{l,\varepsilon} := \sqrt{(n_1 + 1)/(n_1 + 1 - l)} K_{C_l \varepsilon} \supseteq (\sqrt{n}/\sqrt{n-l}) K_{C_l \varepsilon}$, где $C_l := \lim_{n \rightarrow \infty} \mathbf{P}(M_n \geq \theta n) \mathbf{P}(S_{n-l} \geq \theta n)^{-1}/2^{l+4}$, имеем:

$$\begin{aligned} \mathbf{P}(\tau_n = n - l, Y^{(n)} \notin L_{l,\varepsilon} | M_n \geq \theta n) &\leq \frac{\mathbf{P}(S_{n-l} \geq \theta n, X^{(n-l)} \notin K_{C_l \varepsilon/2^{l+4}})}{\mathbf{P}(M_n \geq \theta n)} \\ &= \frac{\mathbf{P}(S_{n-l} \geq \theta(n-l))}{\mathbf{P}(M_n \geq \theta n)} C_l \varepsilon. \end{aligned}$$

Выберем такое N , что $\mathbf{P}(S_{n-l} \geq \theta(n-l))/\mathbf{P}(M_n \geq \theta n) \leq 2C_l$ при всех $n > N$. Имеем $\mathbf{P}(\tau_n = n - l, Y^{(n)} \notin L_{l,\varepsilon} | M_n \geq \theta n) \leq \varepsilon/2^{l+3}$. Вводя компакт $L_\varepsilon = \bigcup_{0 \leq l \leq n_1} L_{l,\varepsilon}$, при всех $n \geq N$ получаем

$$\begin{aligned} \mathbf{P}(Y^{(n)} \notin L_\varepsilon | M_n \geq \theta n) &\leq \mathbf{P}(\tau_n \leq n - n_1, | M_n \geq \theta n) \\ &+ \sum_{l=0}^{n_1} \mathbf{P}(\tau_n = n - l, Y^{(n)} \notin L_l | M_n \geq \theta n) \leq 3\varepsilon/4. \end{aligned}$$

Выбирая компакт $K_\varepsilon \supset L_\varepsilon$ так, чтобы при всех $n \leq N$ выполнялось неравенство $\mathbf{P}(Y^n \notin K_\varepsilon) \leq \varepsilon/4$, выводим, что семейство мер $Q^{(n)}$ плотно. Теорема 7 доказана.

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СПИСОК ЛИТЕРАТУРЫ

1. Петров В. В. О вероятностях больших уклонений сумм независимых случайных величин. — Теория вероятн. и ее примен., 1965, т. 10, в. 2, с. 310–322.
2. Боровков А. А., Коршунов Д. А. Вероятности больших уклонений одномерных цепей Маркова. Часть 2. Дастационарные распределения в экспоненциальном случае. — Теория вероятн. и ее примен., 2000, т. 45, в. 3, с. 437–468.
3. Komlos J., Tusnady G. On sequence of «pure heads». — Ann. Probab., 1975, v. 3, № 4, p. 608–617.
4. Боровков А. А. О преобразовании Крамера, больших уклонениях в граничных задачах и условном принципе инвариантности. — Сиб. матем. журн., 1995, т. 36, № 3, с. 493–509.

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MAXIMUM OF CONTINUOUS VERSIONS OF POISSON AND NEGATIVE BINOMIAL TYPE DISTRIBUTIONS

Пусть M_n — максимум случайной выборки из $F(x)$ на R . Известно, что существует функция $g_n: R \rightarrow R$ такая, что $g_n(M_n) \xrightarrow{\mathcal{L}} Y$, $n \rightarrow \infty$, где случайная величина Y невырождена тогда и только тогда, когда $\bar{F}(x)/\bar{F}(x-) \rightarrow 1$ при $x \rightarrow \infty$, где $\bar{F}(x) = 1 - F(x)$. Это условие выполнено для непрерывных F , но не выполнено, например, когда F —

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пуассоновское или отрицательно биномиальное распределение. Мы рассматриваем классы распределений $\bar{F}(x) = cx^\beta \exp(\alpha x)x^{-x}\{1 + o(1)\}$ and $\bar{F}(x) = dx^c \exp(-ax)\{1 + o(1)\}$ при $x \rightarrow \infty$. Эти классы включают в себя пуассоновское и отрицательно биномиальное распределение при целых x , но не включают при произвольных x . Мы показываем, что $(M_n - a_n)/b_n \xrightarrow{\mathcal{L}} Y$ при $n \rightarrow \infty$ для некоторых a_n и b_n , где Y — случайная величина Гумбеля с функцией распределения $\exp\{-\exp(-y)\}$, $-\infty < y < \infty$.

Ключевые слова и фразы: экстремальные значения, распределение Гумбеля, отрицательно биномиальное распределение, пуассоновское распределение.

1. Introduction and summary. Let $X_1, \dots, X_n$ be independent and identically distributed with distribution function $F(x)$ on $\mathbf{R}$ with maximum M_n . Set $\bar{F}(x) = 1 - F(x)$. By [7, Theorem 1.7.13] there exists a transformation $g_n: \mathbf{R} \rightarrow \mathbf{R}$ such that $g_n(M_n) \xrightarrow{\mathcal{L}} Y$, nondegenerate as $n \rightarrow \infty$ if and only if

$$\frac{\bar{F}(x)}{\bar{F}(x-)} \rightarrow 1 \quad (1.1)$$

as $x \rightarrow \infty$. So, every continuous maximum can be transformed to a random variable with nondegenerate limit. The classical *Extremal Types Theorem* [5], [6] states that when a nondegenerate limit exists one can have three possible forms. Details about the Extremal Types Theorem can be found in [7].

However, (1.1) is not satisfied for most discrete distributions, including the Poisson and negative binomial distributions. Recently, several authors have obtained nondegenerate limits by «tuning» parameters of the distributions to depend on n , see [3], [4], [8], [9]. We feel that the approach of these papers is somewhat «artificial» because one assumes that the population parameter depends on the size of the sample drawn. The purpose of this note is to suggest an alternative approach for obtaining nondegenerate limits. We consider the classes of continuous distributions specified by

$$\bar{F}(x) = cx^\beta \exp(\alpha x)x^{-x}\{1 + o(1)\} \quad (1.2)$$

and

$$\bar{F}(x) = dx^c \exp(-ax)\{1 + o(1)\} \quad (1.3)$$

as $x \rightarrow \infty$. The Poisson distribution satisfies (1.2) on $\mathbf{N} = \{0, 1, 2, \dots\}$. The negative binomial distribution satisfies (1.3) on $\mathbf{N} = \{0, 1, 2, \dots\}$. Set $n_1 = \ln n$ and $n_r = \ln n_{r-1}$ for $r \geq 2$. In Section 2, we show that for (1.2) there exist a_n and b_n such that $(M_n - a_n)/b_n \xrightarrow{\mathcal{L}} Y$ as $n \rightarrow \infty$, where Y is a Gumbel random variable with distribution function $\exp\{-\exp(-y)\}$, $-\infty < y < \infty$. In Section 3, we show the same for (1.3), and, in Section 4, we show that these results cannot be extended for general x or for lattice random variables.

2. Poisson case.

Theorem 2.1. Suppose that F satisfies (1.2). Set $c_1 = \ln c$, $b_n = n_2^{-1}$, and $a_n = n_1(n_2 - n_3 - \alpha)^{-1} + \beta + (\alpha\beta + c_1)n_2^{-1}$. Then $M'_n = (M_n - a_n)/b_n \xrightarrow{\mathcal{L}} Y$ as $n \rightarrow \infty$, where Y is a Gumbel random variable with distribution function $\exp\{-\exp(-y)\}$, $-\infty < y < \infty$.

P r o o f. A constructive proof is more informative. We seek a sequence $u_n = u_n(x) > 1$ such that $Q_n = -\ln \mathbf{P}(M_n \leq u_n(x)) \rightarrow \exp(-x) = Q$ as $n \rightarrow \infty$. Using the Taylor series expansion, we can rewrite $Q_n = -\ln \mathbf{P}(M_n \leq u_n(x)) \rightarrow Q$ in the form

$$Q_n = -n \ln(1 - \bar{F}(u_n)) = n\bar{F}(u_n) + \frac{n\bar{F}(u_n)^2}{2} + \dots \rightarrow Q.$$

Set $L_1 = \ln u_n$, $L_2 = \ln L_1$ and $\gamma = c_1 - \ln Q$. So, $\ln Q_n = n_1 + \ln \bar{F}(u_n) + O(\bar{F}(u_n))$. By (1.2), $\ln \bar{F}(u_n) = -u_n(L_1 - \alpha) + \beta L_1 + c_1 + o(1)$ and so $u_n = (L_1 - \alpha)^{-1}\{n_1 + \beta L_1 + \gamma + o(1)\}$. Taking logs,

$$L_1 = n_2 - L_2 + \delta_1 + n_1^{-1}(\beta L_1 + \gamma + o(1)) = n_2 - L_2 + o(1),$$

where $\delta_1 = -\ln(1 - \alpha L_1^{-1}) \approx \alpha n_2^{-1}$. So, $L_1 = n_2(1 - \delta_2)$, where $\delta_2 = n_2^{-1}\{L_2 - \delta_1 - n_1^{-1}(\beta L_1 + \gamma + o(1))\}$ and $L_2 = n_3 + O(\delta_2) = n_3 + O(n_2^{-1}n_3)$. So, $L_1 - \alpha = n_2 - n_3 - \alpha + O(n_2^{-1}n_3)$, $(L_1 - \alpha)^{-1} = (n_2 - n_3 - \alpha)^{-1}\{1 + O(n_2^{-2}n_3)\}$, and

$$\begin{aligned} u_n &= n_1(n_2 - n_3 - \alpha)^{-1}\{1 + O(n_2^{-2}n_3)\}\{1 + n_1^{-1}[\beta(n_2 - n_3) + \gamma + o(1)]\} \\ &= n_1(n_2 - n_3 - \alpha)^{-1} + \{1 - n_2^{-1}(n_3 + \alpha)\}^{-1}(\beta - \beta n_2^{-1}n_3 + \gamma n_2^{-1}) + o(n_2^{-1}) \\ &= n_1(n_2 - n_3 - \alpha)^{-1} + \beta + \alpha\beta n_2^{-1} + \gamma n_2^{-1} + o(n_2^{-1}) = a_n - b_n \ln Q + o(n_2^{-1}). \end{aligned}$$

So, $\mathbf{P}(M'_n \leq -\ln Q + o(1)) \rightarrow \exp(-Q)$. The theorem follows by setting $x = -\ln Q$.

E x a m p l e 2.1. Suppose $X \sim \text{Poisson}(\lambda)$ with the probability mass function $p(n) = \mathbf{P}(X = n) = \exp(-\lambda)\lambda^n/n!$. By [7, Example 1.7.14], $\bar{F}(n) = p(n+1)(1 + o(1))$. By Stirling's formula,

$$\begin{aligned} \{(n+1)!\}^{-1} &= \exp(n+1)(n+1)^{-n-3/2}(2\pi)^{-1/2}\{1 + o(1)\} \\ &= \exp(n)n^{-n-3/2}(2\pi)^{-1/2}\{1 + o(1)\}. \end{aligned}$$

So, (1.2) holds for x in N with $c = \exp(-\lambda)\lambda(2\pi)^{-1/2}$, $\alpha = \ln(\lambda e) = 1 + \ln \lambda$ and $\beta = -3/2$.

3. Negative binomial case.

Theorem 3.1. Suppose that F satisfies (1.3). Set $b_n = 1/a$ and $a_n = (n_1 + cn_2 + \ln d - c \ln a)/a$. Then $(M_n - a_n)/b_n \xrightarrow{\mathcal{L}} Y$ as $n \rightarrow \infty$, where Y is a Gumbel random variable with distribution function $\exp\{-\exp(-y)\}$, $-\infty < y < \infty$.

P r o o f. The proof is similar to that of Theorem 2.1. We seek a sequence $u_n = u_n(x) > 1$ such that $Q_n = -\ln \mathbf{P}(M_n \leq u_n(x)) \rightarrow \exp(-x) = Q$ as $n \rightarrow \infty$. Using the Taylor series expansion, we can rewrite $Q_n = -\ln \mathbf{P}(M_n \leq u_n(x)) \rightarrow Q$ in the form

$$\ln Q_n \approx n_1 + \ln \bar{F}(u_n) \approx -au_n + cL_1 + n_1 + d_1 \rightarrow \ln Q$$

for $d_1 = \ln d$ and $L_1 = \ln u_n$. So, $au_n \approx n_1 + cL_1 + g = n_1(1 + (cL_1 + g)/n_1)$ for $g = d_1 - \ln Q$. Taking logs, $\ln a + L_1 \approx n_2 + (cL_1 + g)/n_1 + O(n_2^2/n_1^2)$, so

$$L_1 \approx \frac{1}{1 - c/n_1} \left[n_2 - \ln a + \frac{g}{n_1} + O\left(\frac{n_2^2}{n_1^2}\right) \right] = n_2 - \ln a + \frac{g'}{n_1} + \frac{cn_2}{n_1} + O\left(\frac{n_2^2}{n_1^2}\right)$$

for $g' = g - c \ln a$. So,

$$u_n = n_1 \exp \left(-\ln a + \frac{cn_2}{n_1} + \frac{g'}{n_1} \right) \left[1 + O\left(\frac{n_2^2}{n_1^2}\right) \right] = a_n - b_n \ln Q + O\left(\frac{n_2^2}{n_1^2}\right).$$

The proof is complete.

E x a m p l e 3.1. Suppose $X \sim \text{NegBin}(k, p)$ with the probability mass function $p(n) = \binom{c+n}{n} q^n p^k$. Set $c = k - 1$ and $z = n + 1$. Then $(c + n)!/n! \approx n^c$. So, $p(n) \approx n^c q^n p^k/c!$, $p(n+1) \approx dn^c \exp(-an)$ and (1.3) holds with $d = p^k q/c!$ and $a = -\ln q > 0$.

E x a m p l e 3.2. If $X \sim \text{Geom}(p)$ with the probability mass function $p(n) = q^n p$, then (1.3) holds with $c = 0$, $d = pq$, and $a = -\ln q > 0$. This corresponds to the example on p. 45 of [10].

4. Lattice random variables. Here, we show that the above results cannot be extended for general x or for lattice random variables. We only consider the Poisson case. The treatment for the negative binomial case is similar.

Example 4.1. Consider $X \sim \text{Poisson}(\lambda)$ again. To see that (1.2) fails outside $\mathbf{N}$, let $\bar{G}(x)$ equal the right-hand side of (1.2), $\theta = x - [x]$, and $y = [x]$. Then it is easy to show that for fixed $\theta \neq 0$, $\bar{F}(x)/\bar{G}(x) = \bar{F}(y)/\bar{G}(x)$ can be arbitrarily large.

Suppose now that X is a lattice random variable satisfying (1.2) for x in $\mathbf{N}$. An adaption of Theorem 2.1 shows that

$$\sup_x \{|\mathbf{P}(M'_n \leq x) - G(x)| : a_n + b_n x \text{ in } \mathbf{N}\} \rightarrow 0 \quad (4.1)$$

as $n \rightarrow \infty$, where $G(x) = \exp\{-\exp(-x)\}$ is the Gumbel distribution function. However, this result does not allow us to find x_n^* so that $\mathbf{P}(M'_n \leq x_n^*) \rightarrow G(x)$ for some given x . For, set $u_n = [a_n + b_n x]$ and $x_n = (u_n - a_n)/b_n$. The obvious choices of x_n^* are x_n or $(u_n + 1 - a_n)/b_n = x_n + n_2$. Since $x - n_2 \leq x_n < x \leq x_n + n_2 < x + n_2$, we have $G(x - n_2) \leq G(x_n) < G(x) \leq G(x_n + n_2) < G(x + n_2)$. For example, by (4.1), $\mathbf{P}(M'_n \leq x_n) - G(x) = o(1) - \delta_n(x)$, where $0 \leq \delta_n(x) = G(x) - G(x_n) < G(x) - G(x - n_2) = \exp(-\lambda) - n^{-\lambda}$, where $\lambda = \exp(-x)$. But this bound tends to $G(x)$ as $n \rightarrow \infty$. Also the bound is attained whenever $a_n + b_n x$ is an integer, although for most fixed x , for example, $x = G^{-1}(0.9)$, the sequence $\{a_n + b_n x\}$ will contain no integers.

Remark 4.1. Condition (1.2) restricted to x in N is a special case of the condition that $\bar{F}(n)/\bar{F}(n+1) \rightarrow \infty$ as $n \rightarrow \infty$. Anderson [1] shows that for random variables with support $\mathbf{N}$, this condition is necessary and sufficient that there exists a sequence of integers k_n such that $\mathbf{P}(M_n = k_n \text{ or } k_n + 1) \rightarrow 1$ as $n \rightarrow \infty$. By Anderson [2], if F has support $\mathbf{N}$ and $\bar{F}(n)/\bar{F}(n+1) \rightarrow 4^\alpha > 1$, then there exists b_n such that $\alpha(M_n - b_n)$ oscillates between Y_α and $Y_\alpha + 1$, where Y_α has the Fréchet distribution function $\exp(-x^{-\alpha})$ on $(0, \infty)$. If $\bar{F}(n)/\bar{F}(n+1) \rightarrow 1$ as $n \rightarrow \infty$ then he gives a local limit theorem.

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REFERENCES

1. Anderson C. W. Extreme value theory for a class of discrete distributions with applications to some stochastic processes. — J. Appl. Probab., 1970, v. 7, p. 99–113.
2. Anderson C. W. Local limit theorems for the maxima of discrete random variables. — Math. Proc. Cambridge Philos. Soc., 1980, v. 88, № 1, p. 161–165.
3. Anderson C. W., Coles S. G., Hüslér J. Maxima of Poisson-like variables and related triangular arrays. — Ann. Appl. Probab., 1997, v. 7, № 4, p. 953–971.
4. Coles S., Pauli F. Extremal limit laws for a class of bivariate Poisson vectors. — Statist. Probab. Lett., 2001, v. 54, № 4, p. 373–379.
5. Fisher R. A., Tippett L. H. C. Limiting forms of the frequency distribution of the largest or smallest member of a sample. — Proc. Cambridge Philos. Soc., 1928, v. 24, p. 180–190.
6. Гнеденко Б. В. О локальной предельной теореме вероятностей. — Успехи матем. наук, 1948, т. 3, № 3, с. 187–194.
7. Лидбеттер М., Лундгрен Х., Ротсен Х. Экстремумы случайных последовательностей и процессов. М.: Мир, 1989, 391 с.
8. Mitov K. V., Nadarajah S. Limit distributions for the bivariate geometric maxima. — Extremes, 2005, v. 8, № 4, p. 357–370.
9. Nadarajah S., Mitov K. V. Asymptotics of maxima of discrete random variables. — Extremes, 2002, v. 5, № 3, p. 287–294.
10. Resnick S. I. Extreme Values, Regular Variation, and Point Processes. New York: Springer-Verlag, 1987, 320 p.

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