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ON THE SCALING PROPERTY IN FLUCTUATION THEORY FOR STABLE LÉVY PROCESSES

В статье получено выражение для преобразования Лапласа совместного распределения пары $(T_{[x, +\infty[}, X_{T_{[x, +\infty[}})$, где X — процесс Леви, а $T_{[x, +\infty[}$ — момент первого достижения процессом X множества $[x, +\infty[$. В случае, когда X является α -устойчивым процессом Леви с $1 < \alpha < 2$, показано, как по этой формуле восстанавливается распределение случайной величины $X_{T_{[x, +\infty[}}$ — результат, ранее полученный Д. Рэем в симметричном случае и Н. Бинэмом в случае, когда X не является спектрально отрицательным. Затем мы изучаем поведение момента $T_{[x, +\infty[}$ при условии $\{X_{T_{[x, +\infty[}} - x \leq h\}$, когда h стремится к 0. При этом на первый план выходит случайная величина T_x^0 , которая, по-видимому, связана с абсолютной непрерывностью распределения супремума процесса X .

Ключевые слова и фразы: теория флуктуаций, масштабная инвариантность, процессы Леви, устойчивые процессы, момент первого достижения.

1. Introduction. Let $X = (X_t, t \geq 0)$ denote a Lévy process with characteristic exponent ψ .

For $t \geq 0$ we set

$$S_t = \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad I_t = \inf_{0 \leq s \leq t} X_s,$$

and for $x > 0$

$$T_{[x, +\infty[} = \inf\{s > 0 : X_s \geq x\} \quad \text{and} \quad K_x = X_{T_{[x, +\infty[}} - x.$$

We denote by e_γ an exponential variable with parameter γ , independent of X .

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Our interest is focused on the joint law of $(T_{[x,+\infty[}, X_{T_{[x,+\infty[}})$ (see [7]) and on questions concerned with the absolute continuity of the law of S_t .

In Section 2 we present some known results, which will be needed in the following sections. The first result (Proposition 1) is a famous formula obtained by Pecherskii and Rogozin which concerns the joint Laplace transform of $(T_{[\mathbf{e}_\gamma,+\infty[}, X_{T_{[\mathbf{e}_\gamma,+\infty[}})$. The second result (Proposition 1) deals with the asymptotic behavior of the quantity $\mathbf{P}(S_1 < x)$ in the stable case when x tends to 0.

In Section 3 we obtain an expression of the joint Laplace transform of the pair $(T_{[x,+\infty[}, X_{T_{[x,+\infty[}})$. First, we remark that the Laplace transform of this quantity can be expressed by means of the joint Laplace transform of $(T_{[\mathbf{e}_\gamma,+\infty[}, X_{T_{[\mathbf{e}_\gamma,+\infty[}})$, for which we dispose of a formula (Proposition 1). In this way we obtain the result by inverting a Laplace transform.

Then, we pass to the α -stable case, with $1 < \alpha < 2$. By means of the scaling property and of Proposition 2, we prove some asymptotic properties associated to quantities introduced in the previous section. After this, we recover the law of $X_{T_{[x,+\infty[}}$, as a corollary of these properties. This law is already known. In [10] Ray gives an expression of its density in the symmetric stable case. In [3] Bingham generalizes this result for stable processes which are not spectrally negative. However, the interest of this part of the work is to point out the role of the scaling property in the obtention of this result.

Finally, we focus on the asymptotic behavior of the random variable $T_{[x,+\infty[}$ conditioned upon $\{K_x \leq h\}$ when h tends to $0+$. We show that a convergence in law holds towards a random variable which we denote by T_x^0 .

2. Preliminaries. First, we introduce some notation that will be useful afterwards. For $q > 0$, $\operatorname{Re} \lambda \leq 0$ and $\operatorname{Re} \mu \geq 0$, we define

$$\psi_q^+(\lambda) = \exp \left(- \int_0^{+\infty} (e^{\lambda x} - 1) dx \left(\int_0^{+\infty} u^{-1} e^{-qu} \mathbf{P}(X_u > x) du \right) \right), \quad (1)$$

$$\psi_q^-(\mu) = \exp \left(\int_{-\infty}^0 (e^{\mu x} - 1) dx \left(\int_0^{+\infty} u^{-1} e^{-qu} \mathbf{P}(X_u < x) du \right) \right). \quad (2)$$

We use the following result due to Pecherskii and Rogozin.

Proposition 1 [9]. *For all $\gamma, \lambda, \mu > 0$, we have*

$$\mathbf{E} \left[\exp \left(-\lambda T_{[\mathbf{e}_\gamma,+\infty[} - \mu K_{\mathbf{e}_\gamma} \right) \right] = \frac{\gamma}{\gamma - \mu} \left(1 - \frac{\psi_\lambda^+(-\gamma)}{\psi_\lambda^+(-\mu)} \right). \quad (3)$$

R e m a r k 1. It follows from the definition of ψ_γ^+ and ψ_γ^- that

$$\psi_\gamma^+(i\lambda) \psi_\gamma^-(i\lambda) = \frac{\gamma}{\gamma + \psi(\lambda)}. \quad (4)$$

On the other hand, it has been shown in [11] that for $\lambda, \mu \in \mathbf{C}$ with $\operatorname{Re} \lambda \leq 0$, $\operatorname{Re} \mu \geq 0$, we have

$$\psi_\gamma^+(\lambda) = \mathbf{E}[\exp(\lambda S_{\mathbf{e}_\gamma})] \quad \text{and} \quad \psi_\gamma^-(\mu) = \mathbf{E}[\exp(\mu I_{\mathbf{e}_\gamma})]. \quad (5)$$

Hence, the representation of $\gamma(\gamma + \psi(\lambda))^{-1}$ in (4) is an infinitely divisible factorization.

R e m a r k 2. We can find formula (3) in [2, Chap. VI, Exercise 1, p. 182] expressed as follows:

$$\int_0^{+\infty} e^{-\lambda a} \mathbf{E} \left[\exp \left(-\beta T_{[a,+\infty[} - \theta X_{T_{[a,+\infty[}} \right) \right] da = \frac{\kappa(\beta, \lambda + \theta) - \kappa(\beta, \theta)}{\lambda \kappa(\beta, \lambda + \theta)}, \quad (6)$$

where $\kappa(\cdot, \cdot)$ is defined by

$$\exp(-\kappa(\beta, \theta)) = \mathbf{E}[\exp(-\beta \bar{S}_1 - \theta S_{\bar{S}_1})], \quad (7)$$

where $\bar{\tau}$ denotes the right-continuous inverse of the local time at level 0 of the reflected process $S - X$.

The relation between formulae (3) and (6) is simple to establish. In fact, using the following formula (see [2, Chap. VI, Corollary 10, p. 165]):

$$\kappa(\beta, \theta) = \kappa(1, 0) \exp \left(\int_0^{+\infty} dt \int_{[0, +\infty[} (e^{-t} - e^{-\beta t - \theta x}) t^{-1} \mathbf{P}(X_t \in dx) \right), \quad (8)$$

we prove that for $q, \lambda > 0$,

$$\psi_q^+(-\lambda) = \frac{\kappa(q, 0)}{\kappa(q, \lambda)}.$$

In the case of a stable process with index α , the following result will play a key role.

Proposition 2 [2, Chap. VIII, Proposition 2, p. 219]. *We set $\rho = \mathbf{P}(X_1 > 0)$. If $\rho \in]0, 1[$, then there exists $k > 0$ such that*

$$\mathbf{P}(S_1 \leq x) \sim kx^{\alpha\rho} \quad \text{as } x \rightarrow 0+. \quad (9)$$

3. About the law of $(T_{[x, +\infty[}, X_{T_{[x, +\infty[}})$.

3.1. The general case. The purpose of this section is to find an expression for the joint Laplace transform of $(T_{[x, +\infty[}, X_{T_{[x, +\infty[}})$ for $x > 0$:

$$\mathbf{E} \left[\exp \left(-\lambda T_{[x, +\infty[} - \mu X_{T_{[x, +\infty[}} \right) \right].$$

Formula (3) gives an expression of the Laplace transform with respect to $x > 0$ of the quantity we are interested in. So the problem is reduced to inverting this Laplace transform.

First, we note that for $\gamma > \mu > 0$, we have

$$\frac{1}{\gamma - \mu} = \int_0^{+\infty} e^{-(\gamma - \mu)x} dx \quad \text{and} \quad \frac{\gamma}{\gamma - \mu} = 1 + \frac{\mu}{\gamma - \mu}.$$

On the other hand, from (5), we obtain for $\gamma, \lambda > 0$

$$\psi_\gamma^+(-\lambda) = \mathbf{E}[\exp(-\lambda S_{\mathbf{e}_\gamma})] = \int_0^{+\infty} \lambda e^{-\lambda z} \mathbf{P}(S_{\mathbf{e}_\gamma} \leq z) dz. \quad (10)$$

As a consequence, (3) can be expressed as

$$\begin{aligned} & \int_0^{+\infty} e^{-\gamma x} \mathbf{E} \left[\exp \left(-\lambda T_{[x, +\infty[} - \mu K_x \right) \right] dx \\ &= \frac{1}{\gamma - \mu} - \frac{1}{\psi_\lambda^+(-\mu)} \left(\frac{\gamma}{\gamma - \mu} \int_0^{+\infty} e^{-\gamma z} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq z) dz \right) \\ &= \int_0^{+\infty} e^{-\gamma x} \left(e^{\mu x} - \frac{\mathbf{P}(S_{\mathbf{e}_\lambda} \leq x)}{\psi_\lambda^+(-\mu)} \right) dx - \frac{I(\gamma, \lambda, \mu)}{\psi_\lambda^+(-\mu)}, \end{aligned} \quad (11)$$

where

$$I(\gamma, \lambda, \mu) = \mu \int_0^{+\infty} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq z) \left(\int_0^{+\infty} e^{-\gamma(y+z)} e^{\mu y} dy \right) dz.$$

Note that, thanks to the change of variables $v = z$, $x = y + z$ and Fubini's theorem, we get

$$I(\gamma, \lambda, \mu) = \mu \int_0^{+\infty} e^{-(\gamma - \mu)x} \left(\int_0^x e^{-\mu v} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq v) dv \right) dx.$$

If we substitute this expression for $I(\gamma, \lambda, \mu)$ in (11), we obtain using (10) that

$$\begin{aligned} & \int_0^{+\infty} e^{-\gamma x} \mathbf{E} \left[\exp \left(-\lambda T_{[x, +\infty[} - \mu K_x \right) \right] dx \\ &= \frac{1}{\psi_\lambda^+(-\mu)} \int_0^{+\infty} e^{-(\gamma-\mu)x} J(\mu, \lambda; x) dx, \end{aligned} \quad (12)$$

where

$$J(\mu, \lambda; x) = \int_x^{+\infty} \mu e^{-\mu v} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq v) dv - e^{-\mu x} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq x). \quad (13)$$

Therefore,

$$\mathbf{E} \left[\exp \left(-\lambda T_{[x, +\infty[} - \mu K_x \right) \right] = \frac{e^{\mu x}}{\mathbf{E}[e^{-\mu S_{\mathbf{e}_\lambda} }]} J(\mu, \lambda; x). \quad (14)$$

Observe that

$$\begin{aligned} J(\mu, \lambda; x) &= \mathbf{E} \left[\int_{x \vee S_{\mathbf{e}_\lambda}}^{+\infty} \mu e^{-\mu v} dv \right] - e^{-\mu x} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq x) \\ &= \mathbf{E} \left[e^{-\mu(x \vee S_{\mathbf{e}_\lambda})} \right] - e^{-\mu x} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq x), \end{aligned}$$

and thus

$$J(\mu, \lambda; x) = \mathbf{E} \left[e^{-\mu S_{\mathbf{e}_\lambda}} 1_{\{S_{\mathbf{e}_\lambda} \geq x\}} \right]. \quad (15)$$

Theorem 1. For all $\lambda, \mu > 0$ and $x \geq 0$,

$$\mathbf{E} \left[\exp \left(-\lambda T_{[x, +\infty[} - \mu X_{T_{[x, +\infty[}} \right) \right] = \frac{\mathbf{E} [e^{-\mu S_{\mathbf{e}_\lambda}} 1_{\{S_{\mathbf{e}_\lambda} \geq x\}}]}{\mathbf{E}[e^{-\mu S_{\mathbf{e}_\lambda}}]}]. \quad (16)$$

P r o o f. It follows immediately from (14) and (15).

R e m a r k 3. This identity was obtained for random walks in [2]. On the other hand, this result was proved already in [1] by using the lack of memory property of the exponential random variables.

The law of $S_{\mathbf{e}_\lambda}$ is discussed in the following section.

3.2. The stable case. Henceforth, X denotes a real-valued stable Lévy process with index $\alpha \in]1, 2]$. We denote

$$\rho = \mathbf{P}(X_1 \geq 0).$$

We know that in this case, $\rho \in [1 - 1/\alpha, 1/\alpha]$ (see [2]), in particular $\rho \in]0, 1[$. The cases $\rho = 1 - 1/\alpha$ and $\rho = 1/\alpha$ correspond to the cases when X has no negative jumps (spectrally positive case) and no positive jumps (spectrally negative case), respectively.

3.2.1 Scaling for $S_{\mathbf{e}_\lambda}$ and some asymptotic results. In Theorem 1, we see a link between the random variable $S_{\mathbf{e}_\lambda}$ and the joint distribution of $(T_{[x, +\infty[}, X_{T_{[x, +\infty[}})$. In this section, we show how the scaling property allows one to study the absolute continuity of the law of $S_{\mathbf{e}_\lambda}$ and the asymptotic behavior of some quantities associated to this random variable.

Proposition 3. For all $\lambda > 0$, the law of the random variable $S_{\mathbf{e}_\lambda}$ is absolutely continuous. Its density f_λ can be expressed as

$$f_\lambda(x) = \frac{\lambda\alpha}{x} \mathbf{E} \left[T_{[x, +\infty[} \exp \left(-\lambda T_{[x, +\infty[} \right) \right], \quad x > 0. \quad (17)$$

P r o o f. If we take the limit when μ goes to $0+$ in (16), we get

$$\mathbf{P}(S_{\mathbf{e}_\lambda} \leq x) = 1 - \mathbf{E} \left[\exp \left(-\lambda T_{[x, +\infty[} \right) \right].$$

On the other hand, the scaling property yields

$$\mathbf{E} \left[\exp \left(-\lambda T_{[x, +\infty[} \right) \right] = \mathbf{E} \left[\exp \left(-\lambda x^\alpha T_{[1, +\infty[} \right) \right].$$

The result follows.

Lemma 1 (Scaling). *For all $\lambda, \mu > 0$ and $x \geq 0$*

$$\begin{aligned} \text{(i)} \quad \mathbf{P}(S_{\mathbf{e}_\lambda} \leq x) &= \mathbf{P}(S_{\mathbf{e}} \leq x\lambda^{1/\alpha}) = \int_0^{+\infty} e^{-v} \mathbf{P}\left(S_1 \leq \frac{x\lambda^{1/\alpha}}{v^{1/\alpha}}\right) dv. \\ \text{(ii)} \quad \mathbf{E}[e^{-\mu S_{\mathbf{e}_\lambda}}] &= \mathbf{E}\left[\exp\left\{-\frac{\mu}{\lambda^{1/\alpha}} S_{\mathbf{e}}\right\}\right] = \int_0^{+\infty} e^{-v} \mathbf{P}\left(S_{\mathbf{e}} \leq \frac{v\lambda^{1/\alpha}}{\mu}\right) dv. \end{aligned}$$

P r o o f. (i) We have for any $\lambda, \mu > 0$ and $x \geq 0$

$$\begin{aligned} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq x) &= \int_0^{+\infty} \lambda e^{-\lambda u} \mathbf{P}(S_u \leq x) du \quad (\text{change of variables } \lambda u = v) \\ &= \int_0^{+\infty} e^{-v} \mathbf{P}(S_{v/\lambda} \leq x) dv \quad (\text{scaling}) \\ &= \int_0^{+\infty} e^{-v} \mathbf{P}\left(S_1 \leq \frac{x\lambda^{1/\alpha}}{v^{1/\alpha}}\right) dv, \end{aligned}$$

which proves (i).

(ii) From (10), assertion (i), and the change of variables $\mu z = v$ we get

$$\mathbf{E}[e^{-\mu S_{\mathbf{e}_\lambda}}] = \int_0^{+\infty} \mu e^{-\mu z} \mathbf{P}(S_{\mathbf{e}} \leq z\lambda^{1/\alpha}) dz = \int_0^{+\infty} e^{-v} \mathbf{P}\left(S_{\mathbf{e}} \leq \frac{v\lambda^{1/\alpha}}{\mu}\right) dv,$$

which completes the proof.

Proposition 4. *There exists a constant $k^* > 0$ such that*

$$\begin{aligned} \text{(i)} \quad \mathbf{P}(S_{\mathbf{e}_\lambda} \leq x) &\sim k^* x^{\alpha\rho} \lambda^\rho \quad \text{as } \lambda \rightarrow 0+; \\ \text{(ii)} \quad \int_x^{+\infty} \mu e^{-\mu v} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq v) dv &\sim k^* \left(\int_{\mu x}^{+\infty} e^{-y} y^{\alpha\rho} dy \right) \mu^{-\alpha\rho} \lambda^\rho \quad \text{as } \lambda \rightarrow 0+; \\ \text{(iii)} \quad \mathbf{E}[e^{-\mu S_{\mathbf{e}_\lambda}}] &\sim k^* \Gamma(1 + \alpha\rho) \mu^{-\alpha\rho} \lambda^\rho \quad \text{as } \lambda \rightarrow 0+; \\ \text{(iv)} \quad \mathbf{E}[e^{-\mu S_{\mathbf{e}_\lambda}} 1_{\{S_{\mathbf{e}_\lambda} \geq x\}}] &\sim k^* \alpha\rho \left(\int_{\mu x}^{+\infty} e^{-y} y^{\alpha\rho-1} dy \right) \mu^{-\alpha\rho} \lambda^\rho \quad \text{as } \lambda \rightarrow 0+. \end{aligned}$$

P r o o f. (i) Thanks to Proposition 2 and Lemma 1 (i), the dominated convergence theorem yields

$$\frac{\mathbf{P}(S_{\mathbf{e}_\lambda} \leq x)}{\lambda^\rho} = \int_0^{+\infty} e^{-v} \frac{\mathbf{P}(S_1 \leq x\lambda^{1/\alpha}/v^{1/\alpha})}{\lambda^\rho} dv \xrightarrow{\lambda \rightarrow 0+} k\Gamma(1 - \rho)x^{\alpha\rho}.$$

It suffices to choose $k^* = k\Gamma(1 - \rho)$ to finish the proof of (i).

(ii) Applying the dominated convergence theorem, we deduce from (i) that

$$\frac{\int_x^{+\infty} \mu e^{-\mu v} \mathbf{P}(S_{\mathbf{e}_\lambda} \leq v) dv}{\lambda^\rho} \xrightarrow{\lambda \rightarrow 0+} k^* \int_x^{+\infty} \mu e^{-\mu v} v^{\alpha\rho} dv.$$

We obtain (ii) by the change of variables $y = \mu v$.

(iii) It suffices to consider $x = 0$ in assertion (ii).

(iv) Using formulae (13) and (15), assertion (iv) is a consequence of (i) and (ii). Proposition 4 is proved.

In [10], Ray gives an expression for the density of the law of $X_{T_{[x, +\infty[}}$ in the symmetric case. In [3], Bingham generalizes this result to the case when X is not spectrally negative. Now, we recover this result as a corollary of Theorem 1 and Proposition 4 (see also Lemma 4.1 in [12]).

Corollary 1 (see [10], [3]). *If $\alpha\rho < 1$, then*

$$X_{T_{[x, +\infty[}} \stackrel{\text{law}}{=} \frac{x}{\beta_{\alpha\rho, 1-\alpha\rho}}. \quad (18)$$

Consequently,

$$\begin{aligned} \mathbf{P}(X_{T_{[x, +\infty[}} \in dy) &= \rho(x, y) dy, \quad \text{when} \\ \rho(x, y) &= \frac{\sin(\pi\alpha\rho)}{\pi} \frac{1}{y} \left(\frac{x}{y-x} \right)^{\alpha\rho} 1_{]x, \infty[}(y). \end{aligned}$$

P r o o f. If we let λ tend to $0+$ in (16), we get with the help of Proposition 4,

$$\mathbf{E} \left[\exp \left(-\mu X_{T_{[x, +\infty[}} \right) \right] = \frac{1}{\Gamma(\alpha\rho)} \int_{\mu x}^{+\infty} e^{-y} y^{\alpha\rho-1} dy,$$

and by using the identity $y^{\alpha\rho-1} = (\Gamma(1-\alpha\rho))^{-1} \int_0^{+\infty} e^{-yv} v^{-\alpha\rho} dv$, we obtain

$$\begin{aligned} \mathbf{E} \left[\exp \left(-\mu X_{T_{[x, +\infty[}} \right) \right] &= \frac{\sin(\pi\alpha\rho)}{\pi} \int_{\mu x}^{+\infty} dy \int_0^{+\infty} e^{-y(v+1)} v^{-\alpha\rho} dv \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^{+\infty} \frac{e^{-\mu x(v+1)}}{v+1} v^{-\alpha\rho} dv \quad (\text{Fubini's theorem}) \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^1 e^{-\mu x/z} z^{\alpha\rho-1} (1-z)^{-\alpha\rho} dz \\ &\quad (\text{change of variables } z = (v+1)^{-1}), \end{aligned}$$

which proves the Corollary 1.

R e m a r k 4. Similarly, the law of K_x is absolutely continuous. Its density is given by

$$\rho_K(x, y) = \rho(x, x+y) = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{1}{(y+x)} \left(\frac{x}{y} \right)^{\alpha\rho} 1_{]0, \infty[}(y).$$

R e m a r k 5. This result has been generalized in [6] and [8].

3.2.2 An asymptotic law. Note first that a standard application of the Markov property allows one to show that

$$\begin{aligned} \mathbf{P}(S_t \in]x, x+h]) \\ = \iint_{[0, t[\times]x, x+h]} \mathbf{P}(T_{[x+h-y, +\infty[} \geq t-s) \mathbf{P}(T_{[x, +\infty[} \in ds, X_{T_{[x, +\infty[}} \in dy). \end{aligned}$$

Passing to the limit in this equality when h tends to 0, we may expect to find links between the absolute continuity of the law of S_t and the asymptotic behavior of the random variable $T_{[x, +\infty[}$ conditioned to $\{K_x \leq h\}$.

With this motivation, we introduce for each $x, h > 0$ the random variable T_x^h whose law is given by

$$\mathbf{P}(T_x^h \in \cdot) = \mathbf{P}(T_{[x, +\infty[} \in \cdot \mid K_x \leq h).$$

Thus the aim in this section is to study the asymptotic behavior of the variables T_x^h when h goes to $0+$. For this, we start by computing the asymptotic probability of the events $\{K_x \leq h\}$.

Lemma 2. For all $x > 0$,

$$\frac{\mathbf{P}(K_x \leq h)}{h^{1-\alpha\rho}} \xrightarrow{h \rightarrow 0+} \frac{\sin(\pi\alpha\rho)}{\pi(1-\alpha\rho)} x^{\alpha\rho-1}. \quad (19)$$

P r o o f. By Corollary 1, we have

$$\begin{aligned} \mathbf{P}(X_{T_{[x,+\infty[}} \leq x+h) &= \frac{\sin(\pi\alpha\rho)}{\pi} \int_x^{x+h} \frac{1}{y} \left(\frac{x}{y-x} \right)^{\alpha\rho} dy \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} x^{\alpha\rho} h^{1-\alpha\rho} \int_0^1 \frac{1}{(uh+x)u^{\alpha\rho}} du \end{aligned}$$

(in the second equality we used the change of variables $u = (y-x)/h$). The result follows by dividing both sides by $h^{1-\alpha\rho}$ and applying the dominated convergence theorem.

Now, we define the probability measure $\mathbf{P}_\lambda^{(x)}$ by

$$\mathbf{P}_\lambda^{(x)}(A) = \frac{\mathbf{E}[1_A \exp(-\lambda T_{[x,+\infty[})]}{\mathbf{E}[\exp(-\lambda T_{[x,+\infty[})]}, \quad A \in \mathcal{F}_\infty.$$

The following lemma is the analogue of Lemma 2 for the probability measure $\mathbf{P}_\lambda^{(x)}$.

Lemma 3. For all $\lambda, x > 0$, we have

$$\frac{\mathbf{P}_\lambda^{(x)}(X_{T_{[x,+\infty[}} - x \leq h)}{h^{1-\alpha\rho}} \xrightarrow{h \rightarrow 0+} \frac{\sin(\pi\alpha\rho)}{k^* \pi \alpha \rho (1-\alpha\rho)} \frac{\lambda^{-\rho} f_\lambda(x)}{\mathbf{P}(S_{\mathbf{e}_\lambda} \geq x)}, \quad (20)$$

where k^* is the constant appearing in Proposition 4.

P r o o f. We consider the function $U: [0, +\infty[\rightarrow [0, +\infty[$ defined by

$$U(h) = \mathbf{P}_\lambda^{(x)}(K_x \leq h).$$

By the Tauberian theorem (see [2, p. 10]), the behavior of U around $0+$ is related to the behavior of its Laplace transform at infinity.

We remark that

$$\int_0^{+\infty} e^{-\mu y} U(dy) = \mathbf{E}_\lambda^{(x)}[\exp(-\mu K_x)] = \frac{\mathbf{E}[\exp(-\lambda T_{[x,+\infty[} - \mu K_x)]}{\mathbf{E}[\exp(-\lambda T_{[x,+\infty[})]}. \quad (21)$$

On the other hand, thanks to (13), (14), and an obvious change of variables, we obtain

$$\mathbf{E}[\exp(-\lambda T_{[x,+\infty[} - \mu K_x)] = \frac{\int_0^{+\infty} e^{-y} (\mathbf{P}(S_{\mathbf{e}_\lambda} \leq y/\mu + x) - \mathbf{P}(S_{\mathbf{e}_\lambda} \leq x)) dy}{\mathbf{E}[e^{-\mu S_{\mathbf{e}_\lambda}}]}. \quad (22)$$

Using Proposition 3 and applying the dominated convergence theorem, we show that

$$\mu \int_0^{+\infty} e^{-y} \left(\mathbf{P}\left(S_{\mathbf{e}_\lambda} \leq \frac{y}{\mu} + x\right) - \mathbf{P}(S_{\mathbf{e}_\lambda} \leq x) \right) dy \xrightarrow{\mu \rightarrow +\infty} f_\lambda(x). \quad (23)$$

We know from part (ii) of Lemma 1 that

$$\mathbf{E}[e^{-\mu S_{\mathbf{e}_\lambda}}] = \mathbf{E}\left[\exp\left\{-\frac{1}{\lambda^{1/\alpha}} S_{\mathbf{e}_\lambda}[\mu^{-\alpha}]\right\}\right],$$

and therefore, from part (iii) of Proposition 4, we obtain that

$$\mu^{\alpha\rho} \mathbf{E}[e^{-\mu S_{\mathbf{e}_\lambda}}] \xrightarrow{\mu \rightarrow +\infty} k^* \Gamma(1+\alpha\rho) \lambda^\rho. \quad (24)$$

Thus, from (21)–(24), we get

$$\mu^{1-\alpha\rho} \int_0^{+\infty} e^{-\mu y} U(dy) \xrightarrow{\mu \rightarrow +\infty} \frac{1}{k^* \Gamma(1+\alpha\rho)} \frac{\lambda^{-\rho} f_\lambda(x)}{\mathbf{P}(S_{\mathbf{e}_\lambda} \geq x)}, \quad (25)$$

and then, thanks to the Tauberian theorem (see [2, p. 10]), we obtain

$$\frac{1}{h^{1-\alpha\rho}} U(h) \xrightarrow{h \rightarrow 0+} \frac{1}{k^* \Gamma(1+\alpha\rho) \Gamma(2-\alpha\rho)} \frac{\lambda^{-\rho} f_\lambda(x)}{\mathbf{P}(S_{\mathbf{e}_\lambda} \geq x)}, \quad (26)$$

which completes the proof.

Proposition 5. *For every $\lambda, x > 0$, we have*

$$\mathbf{E} \left[\exp(-\lambda T_{[x, +\infty[}) \mid K_x \leq h \right] \xrightarrow{h \rightarrow 0+} \frac{1}{k^* \alpha \rho} x^{1-\alpha\rho} \lambda^{-\rho} f_\lambda(x). \quad (27)$$

We remark that

$$\mathbf{E} \left[\exp(-\lambda T_{[x, +\infty[}) \mid K_x \leq h \right] = \frac{\mathbf{P}_\lambda^{(x)}(K_x \leq h)}{\mathbf{P}(K_x \leq h)} \mathbf{E} \left[\exp(-\lambda T_{[x, +\infty[}) \right].$$

Now, the result is a consequence of Lemmas 2 and 3.

Lemma 4. *For all $x > 0$*

$$\lim_{\lambda \rightarrow 0+} \frac{1}{k^* \alpha \rho} x^{1-\alpha\rho} \lambda^{-\rho} f_\lambda(x) = 1. \quad (28)$$

P r o o f. The scaling property entails

$$\lambda \mathbf{E} \left[T_{[x, +\infty[} \exp(-\lambda T_{[x, +\infty[}) \right] = \int_0^{+\infty} e^{-v} (1-v) \mathbf{P} \left(S_1 \leq \frac{x \lambda^{1/\alpha}}{v^{1/\alpha}} \right) dv,$$

and then, by the definition of f_λ we obtain from Proposition 2 and the dominated convergence theorem that

$$\lambda^{-\rho} f_\lambda(x) \xrightarrow{\lambda \rightarrow 0+} k \alpha x^{\alpha\rho-1} \int_0^{+\infty} e^{-v} (1-v) v^{-\rho} dv = k \alpha \rho \Gamma(1-\rho) x^{\alpha\rho-1}.$$

The result follows.

Theorem 2. *For each $x > 0$, the family of random variables $\{T_x^h\}_{h>0}$ converges in law as h tends to $0+$. The limit is denoted by T_x^0 and its law is given by*

$$\mathbf{P}(T_x^0 \leq t) = \frac{\sin(\pi\rho)}{k\pi\rho} x^{-\alpha\rho} \mathbf{E} \left[\frac{T_{[x, +\infty[} \mathbf{1}_{\{T_{[x, +\infty[} \leq t\}}}{(t - T_{[x, +\infty[})^{1-\rho}} \right].$$

P r o o f. For $x, h > 0$, we denote by $\mathcal{L}_h^x$ the Laplace transform of the random variable T_x^h and by $\mathcal{L}^x$ the function defined by

$$\mathcal{L}^x(\lambda) = \frac{1}{k^* \alpha \rho} x^{1-\alpha\rho} \lambda^{-\rho} f_\lambda(x).$$

According to Proposition 5, the Laplace transform $\mathcal{L}_h^x$ converges pointwise to the function $\mathcal{L}^x$. Since we have already shown in Lemma 4 that

$$\lim_{\lambda \rightarrow 0+} \mathcal{L}^x(\lambda) = 1,$$

it appears that the function $\mathcal{L}^x$ is the Laplace transform to some probability measure with support in $\mathbf{R}_+$ (see [4, Theorem 6.6.3, p. 190]). We denote this limit measure by ν_x .

We remark that

$$\frac{\mathcal{L}^x(\lambda)}{\lambda} = \frac{1}{\lambda} \int_{[0, +\infty[} e^{-\lambda y} \nu_x(dy) = \int_0^{+\infty} e^{-\lambda y} \nu_x([0, y]) dy. \quad (29)$$

On the other hand, we have

$$\frac{\mathcal{L}^x(\lambda)}{\lambda} = \frac{1}{k^* \rho} x^{-\alpha\rho} \lambda^{-\rho} \mathbf{E} \left[T_{[x, +\infty[} \exp(-\lambda T_{[x, +\infty[}) \right].$$

By using the identity $\lambda^{-\rho} = (\Gamma(\rho))^{-1} \int_0^{+\infty} e^{-\lambda z} z^{\rho-1} dz$, we get

$$\frac{\mathcal{L}^x(\lambda)}{\lambda} = \frac{\sin(\pi\rho)}{k\pi\rho} x^{-\alpha\rho} \int_0^{+\infty} e^{-\lambda u} \mathbf{E} \left[\frac{T_{[x, +\infty[} \mathbf{1}_{\{T_{[x, +\infty[} \leq u\}}}{(u - T_{[x, +\infty[})^{1-\rho}} \right] du. \quad (30)$$

The result is obtained by comparing (29) and (30).

3.2.3 The laws of $T_{[x,+\infty[}$ and T_x^0 have no point masses. In [9] (Lemma 1), it has been demonstrated in a more general framework that the law of S_t admits no point masses, i.e., for any $x \geq 0$, $\mathbf{P}(S_t = x) = 0$. Since $\{S_t \geq x\} = \{T_{[x,+\infty[} \leq t\} \cup \{S_t = x\}$, we have

$$\mathbf{P}(S_t \geq x) = \mathbf{P}(T_{[x,+\infty[} \leq t).$$

So, by using the scaling property, we obtain in particular that

$$S_1^{-\alpha} \stackrel{\text{law}}{=} T_{[1,+\infty[}.$$

Thus we see that $T_{[1,+\infty[}$ admits no point masses either (the same for $T_{[x,+\infty[}$, by scaling). In the following proposition, we find this result directly from the scaling property.

Proposition 6. *For every $x, t > 0$*

$$\mathbf{P}(T_{[x,+\infty[} = t) = \mathbf{P}(T_x^0 = t) = 0.$$

P r o o f. We first show the result for $T_{[x,+\infty[}$. By scaling property, with no loss of generality, we may suppose that $x = 1$.

Suppose, by contradiction, that there is $t_0 > 0$ such that

$$\delta := \mathbf{P}(T_{[1,+\infty[} = t_0) > 0.$$

Suppose now that there is $h_0 > 0$ satisfying

$$\eta := \mathbf{P}(T_{[1,+\infty[} = t_0, K_1 > h_0) > 0.$$

In this case, we have

$$\mathbf{P}\left(\forall u \in [1, 1 + h_0/2]: T_{[u,+\infty[} = t_0\right) > \eta,$$

and, therefore, for every $u \in [1, 1 + h_0/2]$, we have $\mathbf{P}(T_{[u,+\infty[} = t_0) > \eta$. Thus, the scaling property yields that for each $u \in [1, 1 + h_0/2]$

$$\mathbf{P}\left(T_{[1,+\infty[} = t_0 u^{-\alpha}\right) > \eta.$$

In other words, for all $s \in [(2/(2 + h_0))^\alpha t_0, t_0]$

$$\mathbf{P}(T_{[1,+\infty[} = s) > \eta,$$

which cannot be true. We have, therefore, for any $h > 0$

$$\mathbf{P}(T_{[1,+\infty[} = t_0, K_1 > h) = 0.$$

Thus, for every $h > 0$

$$0 < \delta = \mathbf{P}\left(T_{[1,+\infty[} = t_0\right) = \mathbf{P}\left(T_{[1,+\infty[} = t_0, K_1 \leq h\right) \leq \mathbf{P}(K_1 \leq h),$$

which is a contradiction, since the right-hand side converges to 0 when h tends to 0.

The result for T_x^0 can be obtained from the result for $T_{[x,+\infty[}$, since thanks to Theorem 2 we have

$$\mathbf{P}(t - h < T_x^0 \leq t) \leq \frac{\sin(\pi\rho)}{k\pi\rho} x^{-\alpha\rho} \mathbf{E} \left[\frac{T_{[x,+\infty[} \mathbf{1}_{\{t-h < T_{[x,+\infty[} \leq t\}}}{(t - T_{[x,+\infty[})^{1-\rho}} \right].$$

Comment. In a forthcoming publication, I would like to establish more precise links between the absolute continuity of the law of S_t and the asymptotic random variable T_x^0 .

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FACTORIZATION PROPERTY OF GENERALIZED S-SELFDECOMPOSABLE MEASURES AND CLASS L^F DISTRIBUTIONS¹⁾

Известно, что саморазложимые и s-саморазложимые вероятностные меры допускают представление в виде распределений случайных интегралов. В настоящей заметке мы предлагаем такие интегральные представления для класса обобщенных s-саморазложимых распределений и класса саморазложимых распределений, обладающих так называемым свойством факторизации.

Ключевые слова и фразы: обобщенные s-саморазложимые распределения, саморазложимые распределения, факторизационное свойство, безграничная делимость, формула Леви–Хинчина, процесс Леви, броуновское движение, случайный интеграл.

In the past few decades many classes of probability distributions were characterized as collections of the distributions of certain random integrals. These include selfdecomposable, n -times selfdecomposable, s-selfdecomposable, and type G distributions; cf. [5]–[10]

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