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**TRACE APPROXIMATIONS OF PRODUCTS
OF TRUNCATED TOEPLITZ OPERATORS**

В статье устанавливаются порядки ошибок при интегральных предельных аппроксимациях следов произведений усеченных тёплицевых операторов, порожденных интегрируемыми вещественными четными функциями, определенными на вещественной прямой. Эти аппроксимации и оценки соответствующих ошибок имеют большое значение в статистическом анализе стационарных процессов с непрерывным временем (асимптотические распределения и большие отклонения тёплицевых квадратичных функционалов, оценивание спектральных параметров и функционалов и т.д.). Найдено асимптотическое разложение второго порядка в явном виде для следа произведения двух усеченных тёплицевых операторов, порожденных спектральными плотностями стационарных дробных движений Рисса–Бесселя с непрерывным временем. Показано, что порядок величины второго члена в этом разложении зависит от параметров долговременной памяти этих процессов. Показано также, что особенность первого члена компенсируется вторым членом разложения, что гарантирует существенно лучшее приближение исходного функционала.

Ключевые слова и фразы: аппроксимация следов, усеченный тёплицев оператор, стационарный процесс с непрерывным временем, долговременная память, спектральная плотность, особенность, дробное движение Рисса–Бесселя.

1. Introduction. Truncated Toeplitz operators (respectively, matrices) arise commonly in many topics of the statistical analysis of continuous-time (respectively, discrete-time) stationary processes: asymptotic distributions and large deviations of Toeplitz type quadratic functionals, estimation of the spectral parameters and functionals, asymptotic expansions of the estimators, hypotheses testing about the spectrum, prediction of the future

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value based on the observed past of the process, etc. (see, e.g., [4], [7], [10], [12]–[19], [21]–[26], [28], and references therein).

The present paper is devoted to the problem of approximation of the traces of products of truncated Toeplitz operators generated by integrable real symmetric functions defined on the real line, and estimation of the corresponding errors.

This problem is of particular importance in the cases, where the underlying model is a continuous-time stationary process with possibly unbounded or vanishing spectral density. Such cases arise in the study of long-memory (the spectral density is unbounded) and anti-persistent (the spectral density has zeros) continuous-time stationary processes.

Let $f(\lambda)$ and $g(\lambda)$ be integrable real symmetric functions on $\mathbf{R} = (-\infty, \infty)$, and let $B_T(f)$ and $B_T(g)$ be the truncated Toeplitz operators generated by functions $f(\lambda)$ and $g(\lambda)$, respectively (see [17], [21], [23]): for $h(\lambda) \in L^1(\mathbf{R})$ and $u(\lambda) \in L^2[0, T]$ we define

$$[B_T(h)u](\lambda) = \int_0^T \hat{h}(\lambda - \mu)u(\mu) d\mu, \quad (1.1)$$

where

$$\hat{h}(t) = \int_{-\infty}^{+\infty} e^{i\lambda t} h(\lambda) d\lambda, \quad t \in \mathbf{R}, \quad (1.2)$$

is the Fourier transform of $h(\lambda)$.

Let ν be an arbitrary fixed positive integer. Define

$$\begin{aligned} S_{T,\nu} &:= S_{T,\nu}(f, g) = \frac{1}{T} \operatorname{tr} [B_T(f)B_T(g)]^\nu, \\ M_\nu &:= M_\nu(f, g) = (2\pi)^{2\nu-1} \int_{-\infty}^{\infty} [f(\lambda)g(\lambda)]^\nu d\lambda, \end{aligned}$$

and set

$$\Delta_{T,\nu} := \Delta_{T,\nu}(f, g) = |S_{T,\nu} - M_\nu|.$$

The problem is to approximate $S_{T,\nu}$ by M_ν and estimate the error rate for $\Delta_{T,\nu}$. More precisely, find conditions on generating functions $f(\lambda)$ and $g(\lambda)$ ensuring the relations

- (A): $\Delta_{T,\nu}(f, g) = o(1)$ as $T \rightarrow \infty$, or
 (B): $\Delta_{T,\nu}(f, g) = O(T^{-\gamma})$ for some $\gamma > 0$ as $n \rightarrow \infty$.

The trace approximation problem for Toeplitz matrices goes back to the classical monograph [21], and has been extensively studied in the literature (see, e.g., [4], [7], [10], [12], [16], [18], [19], [23], [25], [26], [28], and references therein). For Toeplitz operators the above problem was partially studied

in [13], [15], [17], [23]. In Theorem A below we summarize some results concerning Problems (A) (cf. [17]).

Theorem A. *Each of the following conditions is sufficient for*

$$\Delta_{T,\nu}(f, g) = o(1) \quad \text{as } T \rightarrow \infty. \quad (1.3)$$

(A1) $f \in L^1(\mathbf{R}) \cap L^p(\mathbf{R})$ ($p \geq 1$) and $g \in L^1(\mathbf{R}) \cap L^q(\mathbf{R})$ ($q \geq 1$) with $1/p + 1/q \leq 1/\nu$.

(A2) For $\mathbf{u} := (u_1, u_2, \dots, u_{2\nu-1}) \in \mathbf{R}^{2\nu-1}$, the function

$$\varphi(\mathbf{u}) := \int_{-\infty}^{+\infty} f(\lambda)g(\lambda - u_1)f(\lambda - u_2)g(\lambda - u_3) \cdots f(\lambda - u_{2\nu-2})g(\lambda - u_{2\nu-1}) d\lambda \quad (1.4)$$

belongs to $L^2(\mathbf{R}^{2\nu-1})$ and is continuous at $\mathbf{0} = (0, \dots, 0)$.

(A3) $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, $g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, $fg \in L^\nu(\mathbf{R})$ and

$$\int_{-\infty}^{+\infty} |f(\lambda)g(\lambda - \mu)|^\nu d\lambda \longrightarrow \int_{-\infty}^{+\infty} |f(\lambda)g(\lambda)|^\nu d\lambda \quad \text{as } \mu \rightarrow 0.$$

(A4) The functions f and g are integrable on $\mathbf{R}$, bounded on $\mathbf{R} \setminus (-1, 1)$, and satisfy

$$f(\lambda) \leq |\lambda|^{-\alpha_1} L_1(\lambda), \quad |g(\lambda)| \leq |\lambda|^{-\alpha_2} L_2(\lambda), \quad \lambda \in [-1, 1],$$

where $\alpha_1 < 1$, $\alpha_2 < 1$, $\alpha_1 + \alpha_2 \leq 1/\nu$, $L_i \in SV(\mathbf{R})$, $\lambda^{-(\alpha_1 + \alpha_2)} L_i(\lambda) \in L^\nu(\mathbf{R})$, $i = 1, 2$, and $SV(\mathbf{R})$ is the class of slowly varying at zero functions $u(\lambda) \in L^\infty(\mathbf{R})$ satisfying $\lim_{\lambda \rightarrow 0} u(\lambda) = 0$, $u(\lambda) = u(-\lambda)$, $0 < u(\lambda) < u(\mu)$ for $0 < \lambda < \mu$.

Remark 1. Theorem A is proved in [17] for $\nu = 2$, the proof in the case $\nu > 2$ can be carried out similarly. The details will be given elsewhere.

Remark 2. The analog of assertion (A1) for Toeplitz matrices was obtained in [4], the discrete analogs of (A2) and (A4) for $\nu = 2$ were proved in [16] (the special case of (A4) where $\alpha_1 + \alpha_2 < 1/2$ was considered in [10]), while the discrete analog of (A3) for $\nu = 2$ was proved in [19].

In this paper we study problem (B), and derive conditions in terms of the generating functions $f(\lambda)$ and $g(\lambda)$ ensuring the asymptotic relation

$$\Delta_{T,\nu}(f, g) = O(T^{-\gamma}), \quad \gamma > 0, \quad \text{as } T \rightarrow \infty. \quad (1.5)$$

The obtained results improve the $o(1)$ rates given in Theorem A. We also derive an explicit second-order asymptotic expansion for $S_{T,1}(f, g)$, in the case where f and g are spectral densities of continuous-time stationary fractional Riesz–Bessel motions.

The paper is organized as follows. In Section 2 we state the main results of the paper—error bounds for $\Delta_{T,\nu}$ and an explicit second-order asymptotic expansion for $S_{T,1}(f, g)$. In Section 3 we prove some auxiliary results. Section 4 is devoted to the proofs of results stated in Section 2.

2. Main results.

2.1. Error bounds for $\Delta_{T,\nu}$. For $\psi \in L^p(\mathbf{R})$, $1 \leq p \leq \infty$, we denote by $\omega_p(\psi; \delta)$ the L^p -modulus of continuity of ψ :

$$\omega_p(\psi; \delta) := \sup_{0 < h \leq \delta} \|\psi(x+h) - \psi(x)\|_p, \quad \delta > 0.$$

Given numbers $0 < \gamma \leq 1$ and $1 \leq p \leq \infty$, we denote by $\text{Lip}(p, \gamma) := \text{Lip}(\mathbf{R}; p, \gamma)$ the L^p -Lipschitz class of functions defined on $\mathbf{R}$ (see, e.g., [1], [6]):

$$\text{Lip}(p, \gamma) = \{\psi(\lambda) \in L^p(\mathbf{R}): \omega_p(\psi; \delta) = O(\delta^\gamma), \delta \rightarrow 0\}.$$

Observe that if $\psi \in \text{Lip}(p, \gamma)$, then there exists a constant C such that $\omega_p(\psi; \delta) \leq C \delta^\gamma$ for all $\delta > 0$.

Theorem 1. *Let the function $\varphi(\mathbf{u}) = \varphi(u_1, u_2, \dots, u_{2\nu-1})$ be as in (1.4). Assume that $\varphi(\mathbf{u}) \in L^\infty(\mathbf{R}^{2\nu-1})$ and with some constants $L > 0$ and $\gamma \in (0, 1]$*

$$|\varphi(\mathbf{u}) - \varphi(\mathbf{0})| \leq L|\mathbf{u}|^\gamma, \quad \mathbf{u} = (u_1, u_2, \dots, u_{2\nu-1}) \in \mathbf{R}^{2\nu-1}, \quad (2.1)$$

where $\mathbf{0} = (0, 0, \dots, 0)$ and $|\mathbf{u}| = |u_1| + |u_2| + \dots + |u_{2\nu-1}|$. Then for any $\varepsilon > 0$

$$\Delta_{T,\nu}(f, g) = O(T^{-\gamma+\varepsilon}) \quad \text{as } T \rightarrow \infty. \quad (2.2)$$

Theorem 2. *Let $f(\lambda) \in \text{Lip}(p, \gamma)$ and $g(\lambda) \in \text{Lip}(q, \gamma)$ with $\gamma \in (0, 1]$ and $p, q \geq 1$ such that $1/p + 1/q \leq 1/\nu$. Then (2.2) holds for any $\varepsilon > 0$.*

Theorem 3. *Let $f_i(\lambda)$, $i = 1, 2$, be two differentiable functions defined on $\mathbf{R} \setminus \{0\}$, such that for some constants $M > 0$ and $\sigma_i > 0$, $\delta_i > 1$, $i = 1, 2$, with $\sigma_1 + \sigma_2 < 1/\nu$*

$$|f_i(\lambda)| \leq \begin{cases} M|\lambda|^{-\sigma_i} & \text{if } |\lambda| \leq 1, \\ M|\lambda|^{-\delta_i} & \text{if } |\lambda| > 1, \end{cases} \quad (2.3)$$

$$|f'_i(\lambda)| \leq \begin{cases} M|\lambda|^{-\sigma_i-1} & \text{if } |\lambda| \leq 1, \\ M|\lambda|^{-\delta_i-1} & \text{if } |\lambda| > 1, \end{cases} \quad i = 1, 2.$$

Then for any $\varepsilon > 0$

$$\Delta_{T,\nu}(f_1, f_2) = O(T^{-\gamma+\varepsilon}) \quad \text{as } T \rightarrow \infty \quad (2.4)$$

with

$$\gamma = \frac{1}{2\nu} - \frac{\sigma_1 + \sigma_2}{2}. \quad (2.5)$$

Remark 3. The analogs of Theorems 1–3 for Toeplitz matrices were proved in [18].

2.2. Fractional Riesz–Bessel motions. In this subsection we obtain error bounds for $\Delta_{T,\nu}$ and an explicit second-order expansion for $S_{T,1}$ in the special important case, where the functions $f(\lambda)$ and $g(\lambda)$ are spectral densities of continuous-time stationary fractional Riesz–Bessel motions.

Many recent studies indicated that data in a large number of fields (e.g., in economics and finance) display long-range dependence (LRD) and/or intermittency/volatility clustering (see, e.g., [3], [11], and references therein). In the discrete context, a basic model for representing LRD is the ARFIMA(0, d , 0) process $X(t)$:

$$(1 - B)^d X(t) = \varepsilon(t), \quad 0 < d < \frac{1}{2}, \quad (2.6)$$

where B is the backshift operator $BX(t) = X(t - 1)$ and $\varepsilon(t)$ is a white noise. The spectral density of $X(t)$ is given by

$$f(\lambda) = \left(2 \sin \frac{\lambda}{2}\right)^{-2d}, \quad 0 < \lambda \leq \pi. \quad (2.7)$$

In the continuous context, a basic process which has commonly been used to model LRD is fractional Brownian motion (fBm) B_H with Hurst index H . This is a Gaussian process which has stationary increments and spectral density of the form

$$f(\lambda) = \frac{c}{|\lambda|^{2H+1}}, \quad c > 0, \quad 0 < H < 1, \quad \lambda \in \mathbf{R}; \quad (2.8)$$

the form (2.8) can be understood in the sense of time-scale analysis [9] or in a limiting sense [27], since the fBm B_H is a nonstationary process.

A proper stationary model in lieu of fBm is the fractional Riesz–Bessel motion (fRBm), introduced in [2], and then extensively discussed in a number of papers (see, e.g., [3], [11] and references therein). The fRBm is defined to be a continuous-time Gaussian process $X(t)$ with spectral density function of the form

$$f(\lambda) = \frac{c}{|\lambda|^{2\alpha}(1 + \lambda^2)^\beta}, \quad \lambda \in \mathbf{R}, \quad (2.9)$$

where $0 < c < \infty$, $0 < \alpha < 1$ and $\beta > 0$.

It is noted that the process $X(t)$ is stationary if $0 < \alpha < 1/2$ and is nonstationary with stationary increments if $1/2 \leq \alpha < 1$. Observe that the spectral density (2.9) behaves as $O(|\lambda|^{-2\alpha})$ as $|\lambda| \rightarrow 0$ and as $O(|\lambda|^{-2(\alpha+\beta)})$ as $|\lambda| \rightarrow \infty$. Thus, under the conditions $0 < \alpha < 1/2$, $\beta > 0$, and $\alpha + \beta > 1/2$, the function $f(\lambda)$ in (2.9) is well defined for both $|\lambda| \rightarrow 0$ and $|\lambda| \rightarrow \infty$ due to the presence of the component $(1 + \lambda^2)^{-\beta}$, $\beta > 0$, which is the Fourier transform of the Bessel potential. Note that in the spacial case $0 < \alpha < 1/2$, $\beta > 1/2$ the condition $\alpha + \beta > 1/2$ holds automatically.

The exponent α determines the long-range dependence, or self-similarity of fRBm, while the exponent β indicates the second-order intermittency of the process (see, e.g., [3], [11]).

Comparing (2.8) and (2.9), we observe that the spectral density of fBm is the limiting case as $\beta \rightarrow 0$ — that of fRBm with Hurst index $H = \alpha - 1/2$. Thus, the form (2.9) means that fRBm may exhibit both LRD and second-order intermittency.

We consider the stationary case, and assume that $0 < \alpha < 1/2$, $\beta > 0$ and $\alpha + \beta > 1/2$. The following result is an immediate consequence of Theorem (A1).

Theorem 4. *Let $f(\lambda)$ be the spectral density of a fractional Riesz–Bessel motion defined by (2.9), and let $g(\lambda)$ be an integrable real symmetric function on $\mathbf{R}$. If for some $p, q \geq 1$ with $1/p + 1/q \leq 1/\nu$ we have*

$$g(\lambda) \in L^q(\mathbf{R}) \quad \text{and} \quad 0 < \alpha < \frac{1}{2p}, \quad \alpha + \beta > \frac{1}{2}, \quad (2.10)$$

then

$$\Delta_{T,\nu} = o(1) \quad \text{as} \quad T \rightarrow \infty.$$

Theorem 5. *Let $f(\lambda)$ be as in (2.9) with $0 < \alpha < 1/(2p)$ and $\alpha + \beta > 1/2$, and let $g(\lambda)$ be an integrable real symmetric function from the class $\text{Lip}(q, 1/p - 2\alpha)$ with $1/p + 1/q \leq 1/\nu$. Then for any $\varepsilon > 0$ the asymptotic relation (2.2) holds with $\gamma = 1/p - 2\alpha$.*

Theorem 6. *Let $f_i(\lambda)$, $i = 1, 2$, be the spectral density functions of two fractional Riesz–Bessel motions defined as*

$$f_i(\lambda) = \frac{C_i}{|\lambda|^{2\alpha_i}(1 + \lambda^2)^{\beta_i}}, \quad 0 < \alpha_i < 1/2, \quad \alpha_i + \beta_i > 1/2, \quad i = 1, 2. \quad (2.11)$$

If $\alpha_1 + \alpha_2 < 1/(2\nu)$, then for any $\varepsilon > 0$ the asymptotic relation (2.4) holds with

$$\gamma = \frac{1}{2\nu} - (\alpha_1 + \alpha_2). \quad (2.12)$$

The following theorem gives an explicit second-order asymptotic expansion for $S_{T,1}$.

Theorem 7. *Let $f_i(\lambda)$, $i = 1, 2$, be as in (2.11). Then under $\alpha := \alpha_1 + \alpha_2 < 1/2$*

$$\begin{aligned} S_{T,1}(f_1, f_2) &:= \frac{1}{T} \text{tr}[B_T(f_1)B_T(f_2)] \\ &= 2\pi \int_{-\infty}^{+\infty} f_1(\lambda)f_2(\lambda) d\lambda - \frac{C(\alpha_1, \alpha_2)}{T^{1-2\alpha}} + o\left(\frac{1}{T^{1-2\alpha}}\right) \end{aligned} \quad (2.13)$$

as $T \rightarrow \infty$, where

$$C(\alpha_1, \alpha_2) = \frac{2C_1C_2\pi^2}{\cos(\pi\alpha_1)\cos(\pi\alpha_2)\Gamma(2\alpha_1)\Gamma(2\alpha_2)} \frac{1}{2\alpha(1-2\alpha)}. \quad (2.14)$$

R e m a r k 4. The analog of Theorem 6 for discrete-time processes was obtained in [18], while the discrete analog of Theorem 7 was established in [25].

3. Preliminaries. In this section we prove some technical lemmas.

Lemma 1. *Let $D_T(u)$ be the Dirichlet kernel*

$$D_T(u) = \frac{\sin(Tu/2)}{u/2}. \quad (3.1)$$

Then, for any $\delta \in (0, 1)$

$$|D_T(u)| \leq 2 \frac{T^\delta}{|u|^{1-\delta}}, \quad u \in \mathbf{R}. \quad (3.2)$$

P r o o f. For fixed $T > 0$ and $u \in \mathbf{R}$, from (3.1) we have $|D_T(u)| \leq \min\{T, 2/|u|\}$. On the other hand, the function $I(\delta) := T^\delta/|u|^{1-\delta} = (T|u|)^\delta/|u|$ is monotone on δ and hence

$$I(\delta) \geq \min\{I(0), I(1)\} = \min\left\{T, \frac{1}{|u|}\right\} \geq \frac{1}{2}|D_T(u)|,$$

yielding (3.2).

Lemma 2. *Let $0 < \beta < 1$, $0 < \alpha < 1$, and $\alpha + \beta > 1$. Then for any $y \in \mathbf{R}$, $y \neq 0$,*

$$\int_{\mathbf{R}} \frac{1}{|x|^\alpha |x+y|^\beta} dx \leq \frac{M}{|y|^{\alpha+\beta-1}}, \quad (3.3)$$

where M is a constant depending on α and β .

P r o o f. We have

$$\begin{aligned} \int_{\mathbf{R}} \frac{1}{|x|^\alpha |x+y|^\beta} dx &\leq \int_{|x| \leq |y|/2} \frac{2}{|x|^\alpha |y|^\beta} dx + \int_{|y|/2 < |x| < 2|y|} \frac{2}{|y|^\alpha |x+y|^\beta} dx \\ &\quad + \int_{|x| \geq 2|y|} \frac{2}{|x|^{\alpha+\beta}} dx \\ &\leq \frac{M_1}{|y|^{\alpha+\beta-1}} + \frac{M_1}{|y|^\alpha} \int_{0 < t < 3|y|} \frac{1}{t^\beta} dt + \frac{M_1}{|y|^{\alpha+\beta-1}} \leq \frac{M}{|y|^{\alpha+\beta-1}}, \end{aligned}$$

yielding (3.3).

Denote $E = \{(u_1, \dots, u_n) \in \mathbf{R}^n: |u_i| \leq 1, i = 1, \dots, n\}$ and $E^C = \mathbf{R}^n \setminus E$.

Lemma 3. *Let $0 < \alpha \leq 1$ and $n/(n+1) < \beta < (n+\alpha)/(n+1)$. Then*

$$B_i := \int_E \frac{|u_i|^\alpha}{|u_1 \cdots u_n (u_1 + \cdots + u_n)|^\beta} du_1 \cdots du_n < \infty, \quad i = 1, \dots, n. \quad (3.4)$$

P r o o f. Using Lemma 2 we can write

$$\begin{aligned}
B_1 &\leq \int_{|u_1| \leq 1} \frac{1}{|u_1|^{\beta-\alpha}} \int_{\mathbf{R}^{n-2}} \frac{1}{|u_2 \cdots u_{n-1}|^\beta} \\
&\quad \times \int_{\mathbf{R}} \frac{1}{|u_n(u_1 + \cdots + u_n)|^\beta} du_n du_{n-1} \cdots du_1 \\
&\leq M \int_{|u_1| \leq 1} \frac{1}{|u_1|^{\beta-\alpha}} \int_{\mathbf{R}^{n-2}} \frac{1}{|u_2 \cdots u_{n-1}|^\beta (u_1 + \cdots + u_{n-1})^{2\beta-1}} du_{n-1} \cdots du_1 \\
&\leq M^2 \int_{|u_1| \leq 1} \frac{1}{|u_1|^{\beta-\alpha}} \\
&\quad \times \int_{\mathbf{R}^{n-3}} \frac{1}{|u_2 \cdots u_{n-2}|^\beta (u_1 + \cdots + u_{n-2})^{3\beta-2}} du_{n-2} \cdots du_1 \leq \cdots \\
&\leq M^{n-2} \int_{|u_1| \leq 1} \frac{1}{|u_1|^{\beta-\alpha}} \int_{\mathbf{R}} \frac{1}{|u_2|^\beta (u_1 + u_2)^{(n-1)\beta-n+2}} du_2 du_1 \\
&\leq M^{n-1} \int_{|u_1| \leq 1} \frac{1}{|u_1|^{(n+1)\beta-\alpha-n+1}} du_1 < \infty,
\end{aligned}$$

yielding (3.4) for $i = 1$. The quantities $B_2, \dots, B_n$ can be estimated in the same way.

Lemma 4. Let $n/(n+1) < \beta < 1$. Then

$$I := \int_{E^C} \frac{1}{|u_1 \cdots u_n(u_1 + \cdots + u_n)|^\beta} du_1 \cdots du_n < \infty. \quad (3.5)$$

P r o o f. We have

$$I \leq \sum_{i=1}^n \int_{|u_i| > 1} \frac{1}{|u_1 \cdots u_n(u_1 + \cdots + u_n)|^\beta} du_1 \cdots du_n =: I_1 + \cdots + I_n. \quad (3.6)$$

Using Lemma 2 we get

$$\begin{aligned}
I_1 &\leq \int_{|u_1| > 1} \frac{1}{|u_1|^\beta} \int_{\mathbf{R}^{n-2}} \frac{1}{|u_2 \cdots u_{n-1}|^\beta} \int_{\mathbf{R}} \frac{1}{|u_n(u_1 + \cdots + u_n)|^\beta} du_n \cdots du_1 \\
&\leq M \int_{|u_1| > 1} \frac{1}{|u_1|^\beta} \int_{\mathbf{R}^{n-2}} \frac{1}{|u_2 \cdots u_{n-1}|^\beta} \\
&\quad \times \frac{1}{|u_1 + \cdots + u_{n-1}|^{2\beta-1}} du_{n-1} \cdots du_1 \leq \cdots \\
&\leq M^{n-1} \int_{|u_1| > 1} \frac{1}{|u_1|^{(n+1)\beta-n+1}} du_1 < \infty.
\end{aligned}$$

The quantities $I_2, \dots, I_n$ can be estimated in the same way, and by (3.6) the result follows.

Lemma 5. *Let $p > 1$, $0 < \sigma < 1/p$, $\delta > 1/p$ and let $f(\lambda)$ be differentiable function defined on $\mathbf{R} \setminus \{0\}$, such that for some constant $M > 0$*

$$|f(\lambda)| \leq \begin{cases} M|\lambda|^{-\sigma} & \text{if } |\lambda| \leq 1, \\ M|\lambda|^{-\delta} & \text{if } |\lambda| > 1, \end{cases} \quad |f'(\lambda)| \leq \begin{cases} M|\lambda|^{-\sigma-1} & \text{if } |\lambda| \leq 1, \\ M|\lambda|^{-\delta-1} & \text{if } |\lambda| > 1. \end{cases} \quad (3.7)$$

Then $f \in \text{Lip}(p, 1/p - \sigma)$.

P r o o f. Let $h \in (0, 1/2)$ be fixed. Then

$$\int_{|\lambda| \leq 2h} |f(\lambda + h) - f(\lambda)|^p d\lambda \leq (2M)^p \int_0^{3h} |\lambda|^{-p\sigma} d\lambda \leq Ch^{1-p\sigma}. \quad (3.8)$$

Next, for $|\lambda| > 2h$ we have $f(\lambda + h) - f(\lambda) = f'(\xi)h$ with some $\xi \in (\lambda, \lambda + h)$. Hence

$$\int_{2h < |\lambda| < 1/2} |f(\lambda + h) - f(\lambda)|^p d\lambda \leq M^p h^p \int_h^{1/2} |\lambda|^{-p(\sigma+1)} d\lambda \leq Ch^{1-p\sigma} \quad (3.9)$$

and

$$\begin{aligned} & \int_{1/2 < |\lambda| < \infty} |f(\lambda + h) - f(\lambda)|^p d\lambda \\ & \leq M^p h^p \left\{ \int_{1/2}^1 |\lambda|^{-p(\sigma+1)} d\lambda + \int_1^\infty |\lambda|^{-p(\delta+1)} d\lambda \right\} \\ & \leq Ch^p \leq Ch^{1-p\sigma}. \end{aligned} \quad (3.10)$$

From (3.8)–(3.10) we get

$$\|f(\lambda + h) - f(\lambda)\|_p \leq Ch^{1/p-\sigma},$$

implying $f \in \text{Lip}(p, 1/p - \sigma)$.

The following lemma gives an asymptotic formula for covariance function of a fRBm process.

Lemma 6. *Let $f(\lambda)$ be as in (2.9) with $0 < \alpha < 1/2$ and $\beta > 1/2$, and let $r(t) := \hat{f}(t)$ be the Fourier transform of $f(\lambda)$. Then as $t \rightarrow \infty$*

$$r(t) = t^{2\alpha-1} \frac{\pi C}{\cos(\pi\alpha)\Gamma(2\alpha)} (1 + o(1)). \quad (3.11)$$

P r o o f. We use the technique of [24], where (3.11) was proved for $\beta = 1$. Since the underlying process $X(t)$ is real-valued, we have for $t > 0$

$$r(t) := \int_{-\infty}^{+\infty} e^{it\lambda} f(\lambda) d\lambda = 2 \int_0^\infty \frac{C}{\lambda^{2\alpha}(1 + \lambda^2)^\beta} \cos(t\lambda) d\lambda. \quad (3.12)$$

Using the change of variable $t\lambda = 1/u$, $\lambda = 1/(tu)$, $d\lambda = -du/(tu^2)$, from (3.12) we obtain

$$\begin{aligned} r(t) &= 2C \int_0^\infty \frac{1}{[1/(tu)]^{2(\alpha+\beta)} [1 + (tu)^2]^\beta} \cos \frac{1}{u} \frac{du}{tu^2} \\ &= 2C t^{2\alpha-1} \int_0^\infty \left(\frac{(tu)^2}{1 + (tu)^2} \right)^\beta u^{2\alpha-2} \cos \frac{1}{u} du \\ &= 2C t^{2\alpha-1} \int_0^\infty L(tu) k(u) du, \end{aligned} \quad (3.13)$$

where

$$L(u) = \left(\frac{u^2}{1 + u^2} \right)^\beta \quad \text{and} \quad k(u) = u^{2\alpha-2} \cos \frac{1}{u}.$$

Choose $\delta > 0$ such that $\delta < \min(1 - 2\alpha, 2\alpha)$. Then the improper integrals

$$\int_{0+}^1 u^{-\delta} k(u) du \quad \text{and} \quad \int_1^{\infty-} u^\delta k(u) du$$

exist. Therefore, by the Bojanic–Karamata theorem (see [5, Theorem 4.1.5]) we have

$$\int_0^\infty k(u) L(tu) du \longrightarrow \int_{0+}^{\infty-} k(u) du \quad \text{as } t \rightarrow \infty. \quad (3.14)$$

Further, using the change of variable $1/u = v$ and the formula (see, e.g., [8])

$$\int_0^{\infty-} x^{-p} \cos(mx) dx = \frac{\pi m^{p-1}}{2 \cos(p\pi/2) \Gamma(p)}, \quad 0 < p < 1, \quad m > 0,$$

with $p = 2\alpha$ and $m = 1$, we obtain

$$\begin{aligned} \int_{0+}^{\infty-} k(u) du &= \int_0^{\infty-} u^{2\alpha-2} \cos \frac{1}{u} du = \int_0^{\infty-} v^{-2\alpha} \cos v dv \\ &= \frac{\pi}{2 \cos(\pi\alpha) \Gamma(2\alpha)}. \end{aligned} \quad (3.15)$$

From (3.12)–(3.15) we get (3.11).

R e m a r k 5. Taking into account the reflection formula $\Gamma(2\alpha)\Gamma(1 - 2\alpha) = \pi/\sin(2\pi\alpha)$, from (3.11) we obtain the following asymptotic relation for covariance function $r(t)$:

$$r(t) = C t^{2\alpha-1} \sin(\pi\alpha) \Gamma(1 - 2\alpha) (1 + o(1)) \quad \text{as } t \rightarrow \infty. \quad (3.16)$$

The following lemma contains an explicit formula for $S_{T,1}$ in general case.

Lemma 7. Let $f_i(\lambda)$, $i = 1, 2$, be integrable real symmetric functions on $\mathbf{R}$ such that $f_1 f_2 \in L^1(\mathbf{R})$, and let $B_T(f_i)$ be the truncated Toeplitz operators generated by $f_i(\lambda)$, $i = 1, 2$. Then

$$\begin{aligned} S_{T,1} &:= \frac{1}{T} \operatorname{tr} [B_T(f_1) B_T(f_2)] = 2\pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_T(\lambda - \mu) f_1(\lambda) f_2(\mu) d\lambda d\mu \\ &= \int_{-T}^T \left(1 - \frac{|t|}{T}\right) r_1(t) r_2(t) dt, \end{aligned} \quad (3.17)$$

where $r_i(t)$ are the Fourier transforms of $f_i(\lambda)$, $i = 1, 2$, and

$$F_T(t) = \frac{1}{2\pi T} \left[\frac{\sin(Tt/2)}{t/2} \right]^2 \quad (3.18)$$

is the Fejer kernel.

P r o o f. The operators $B_T(f_i)$, $i = 1, 2$, can be realized as integral operators in $L^2(\mathbf{R})$ with kernels

$$\Gamma_T(f_i; \lambda, \mu) = \int_{-\infty}^{+\infty} G_T(\lambda, t) G_T(t, \mu) f_i(t) dt, \quad i = 1, 2,$$

where

$$G_T(\lambda, \mu) = e^{iT(\lambda - \mu)/2} \frac{\sin[T(\lambda - \mu)/2]}{(\lambda - \mu)/2}. \quad (3.19)$$

Taking into account the equality

$$\int_{-\infty}^{+\infty} G_T(u, t) G_T(t, v) dt = G_T(u, v), \quad (3.20)$$

we observe that the operator $B_T(f_1) B_T(f_2)$ can be realized as an integral operator in $L^2(\mathbf{R})$ with kernel

$$\begin{aligned} \Gamma(f_1, f_2; \lambda, \mu) &= \int_{-\infty}^{+\infty} \Gamma_T(f_1; \lambda, t) \Gamma_T(f_2; t, \mu) dt \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_T(\lambda, u) G_T(u, v) G_T(v, \mu) f_1(u) f_2(v) du dv. \end{aligned} \quad (3.21)$$

Therefore, using the formula for traces of integral operators (see, e.g., [20]) and (3.20), we find

$$\begin{aligned} S_{T,1} &= \frac{1}{T} \operatorname{tr} [B_T(f_1) B_T(f_2)] = \int_{-\infty}^{+\infty} \Gamma(f_1, f_2; \lambda, \lambda) d\lambda \\ &= \frac{1}{T} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_T(\lambda, u) G_T(u, v) G_T(v, \lambda) f_1(u) f_2(v) du dv d\lambda \\ &= \frac{1}{T} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_T(u, v) G_T(v, u) f_1(u) f_2(v) du dv \\ &= 2\pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_T(u - v) f_1(u) f_2(v) du dv, \end{aligned} \quad (3.22)$$

yielding the first equality in (3.17). The second equality in (3.17) follows from the Parseval–Plancherel theorem.

4. Proofs. **P r o o f o f T h e o r e m 1.** For $\vec{\mathbf{u}} := (u_1, \dots, u_{2\nu-1}) \in \mathbf{R}^{2\nu-1}$ we denote

$$\Phi(\mathbf{u}) := \frac{1}{(2\pi)^{2\nu-1}T} D_T(u_1) \cdots D_T(u_{2\nu-1}) D_T(u_1 + \cdots + u_{2\nu-1})$$

and

$$\Psi(\mathbf{u}) := \varphi(u_1, u_1 + u_2, \dots, u_1 + \cdots + u_{2\nu-1}), \quad (4.1)$$

where $D_T(u)$ is as in (3.1), and $\varphi(u_1, \dots, u_{2\nu-1})$ is defined by (1.4). We have (see [17] for $\nu = 2$)

$$\begin{aligned} \Delta_T &= \frac{1}{T} \operatorname{tr} [B_T(f)B_T(g)]^\nu - (2\pi)^{2\nu-1} \int_{-\infty}^{\infty} [f(\lambda)g(\lambda)]^\nu d\lambda \\ &= \int_{\mathbf{R}^{2\nu-1}} [\Psi(\mathbf{u}) - \Psi(\mathbf{0})] \Phi_T(\mathbf{u}) d\mathbf{u}, \quad \mathbf{0} = (0, \dots, 0). \end{aligned} \quad (4.2)$$

It follows from (2.1) and (4.1) that for $\mathbf{u} = (u_1, \dots, u_{2\nu-1}) \in \mathbf{R}^{2\nu-1}$

$$|\Psi(\mathbf{u}) - \Psi(\mathbf{0})| \leq (2\nu - 1)L(|u_1|^\gamma + \cdots + |u_{2\nu-1}|^\gamma). \quad (4.3)$$

Let $\varepsilon \in (0, \gamma)$. Then, applying Lemma 1 with $\delta = (1 + \varepsilon - \gamma)/(2\nu)$, and using (4.2) and (4.3), we have

$$\begin{aligned} |\Delta_T| &\leq \int_E |\Psi(\mathbf{u}) - \Psi(\mathbf{0})| |\Phi_T(\mathbf{u})| d\mathbf{u} + \int_{E^c} |\Psi(\mathbf{u}) - \Psi(\mathbf{0})| |\Phi_T(\mathbf{u})| d\mathbf{u} \\ &\leq \frac{C_\nu}{T^{1-2\nu\delta}} \sum_{i=1}^{2\nu-1} \int_E \frac{|u_i|^\gamma}{|u_1 \cdots u_{2\nu-1}(u_1 + \cdots + u_{2\nu-1})|^{1-\delta}} du_1 \cdots du_{2\nu-1} \\ &\quad + 2\|\varphi\|_\infty \frac{C_\nu}{T^{1-2\nu\delta}} \\ &\quad \times \int_{E^c} \frac{1}{|u_1 \cdots u_{2\nu-1}(u_1 + \cdots + u_{2\nu-1})|^{1-\delta}} du_1 \cdots du_{2\nu-1}. \end{aligned} \quad (4.4)$$

Since

$$\frac{2\nu - 1}{2\nu} < 1 - \delta < \frac{2\nu - 1 + \gamma}{2\nu},$$

we can apply Lemmas 3 and 4 with $\alpha = \gamma$, $n = 2\nu - 1$, and $\beta = 1 - \delta$ to conclude that all the integrals in (4.4) are finite. Since $1 - 2\nu\delta = \gamma - \varepsilon$, from (4.4) follows the statement of Theorem 1.

P r o o f o f T h e o r e m 2. According to Theorem 1 it is enough to prove that for $n = 2\nu - 1$, $\mathbf{t} = (t_1, \dots, t_n) \in \mathbf{R}^n$ the function

$$\varphi(\mathbf{t}) := \int_{\mathbf{R}} f_0(u) f_1(u - t_1) \cdots f_n(u - t_n) du \quad (4.5)$$

belongs to $L^\infty(\mathbf{R}^n)$, and with some positive constant L and $\gamma \in (0, 1]$

$$|\varphi(\mathbf{t}) - \varphi(\mathbf{0})| \leq L|\mathbf{t}|^\gamma, \quad \mathbf{t} \in \mathbf{R}^n, \quad (4.6)$$

provided that

$$f_i \in \text{Lip}(p_i, \gamma), \quad 1 \leq p_i \leq \infty, \quad i = 0, 1, \dots, n, \quad \text{and} \quad \sum_{i=0}^n \frac{1}{p_i} \leq 1. \quad (4.7)$$

It follows from the Hölder inequality and (4.7) that

$$|\varphi(\mathbf{t})| \leq \prod_{i=0}^n \|f_i\|_{L^{p_i}(\mathbf{R})} < \infty, \quad \mathbf{t} \in \mathbf{R}^n.$$

Hence $\varphi \in L^\infty(\mathbf{R}^n)$. To prove (4.6) we fix $\mathbf{t} = (t_1, \dots, t_n) \in \mathbf{R}^n$ and denote

$$\bar{f}_i(u) = f_i(u - t_i) - f_i(u), \quad u \in \mathbf{R}, \quad i = 1, \dots, n. \quad (4.8)$$

Since $f_i \in \text{Lip}(p_i, \gamma)$, we have

$$\|\bar{f}_i\|_{p_i} \leq L_i |\mathbf{t}|^\gamma, \quad i = 1, \dots, n. \quad (4.9)$$

By (4.5) and (4.8)

$$\varphi(\mathbf{t}) = \int_{\mathbf{R}} f_0(u) \prod_{i=1}^n (\bar{f}_i(u) + f_i(u)) du = \varphi(\mathbf{0}) + W.$$

Each of the $2n - 1$ integrals constituting W contains at least one function $\bar{f}_i$, and in view of (4.9), can be estimated as follows:

$$\begin{aligned} & \left| \int_{\mathbf{R}} f_0(u) \bar{f}_1(u) f_2(u) \cdots f_n(u) du \right| \\ & \leq \|f_0\|_{L^{p_0}} \|\bar{f}_1\|_{L^{p_1}} \|f_2(u)\|_{L^{p_2}} \cdots \|f_n\|_{L^{p_n}} \leq L|\mathbf{t}|^\gamma. \end{aligned}$$

This completes the proof of Theorem 2.

P r o o f o f T h e o r e m 3. For given σ_i and δ_i satisfying $\sigma_i > 0$, $\delta_i > 1$, $i = 1, 2$, and $\sigma_1 + \sigma_2 < 1/\nu$ we set

$$\frac{1}{p} = \frac{1}{2\nu} + \frac{\sigma_1 - \sigma_2}{2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{2\nu} + \frac{\sigma_2 - \sigma_1}{2}.$$

Then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{\nu} \quad \text{and} \quad \frac{1}{p} - \sigma_1 = \frac{1}{q} - \sigma_2 = \frac{1}{2\nu} - \frac{\sigma_1 + \sigma_2}{2} =: \gamma > 0$$

and

$$p > 1, \quad q > 1, \quad \sigma_1 < \frac{1}{p}, \quad \sigma_2 < \frac{1}{q}.$$

Hence, according to Lemma 5, $f_1 \in \text{Lip}(p, \gamma)$ and $f_2 \in \text{Lip}(q, \gamma)$. Applying Theorem 2 with $f = f_1$ and $g = f_2$, we get (2.4) with γ as in (2.5).

P r o o f o f T h e o r e m 5. We have

$$f'(\lambda) = -\frac{2\alpha + 2(\alpha + \beta)\lambda^2}{\lambda^{2\alpha+1}(1 + \lambda)^{\beta+1}}. \quad (4.10)$$

It follows from (2.9) and (4.10) that the functions f and f' satisfy conditions (3.7) with $\sigma = 2\alpha$. Hence by Lemma 5, $f(\lambda) \in \text{Lip}(p, 1/p - 2\alpha)$. Therefore the result follows from Theorem 2.

P r o o f o f T h e o r e m 6. It follows from (2.9) and (4.10) that the functions f_1 and f_2 satisfy conditions of Theorem 3 with $\sigma_i = 2\alpha_i$ and $\delta_i = 2\alpha_i + 2\beta_i$, $i = 1, 2$. Therefore the result follows from Theorem 3.

P r o o f o f T h e o r e m 7. By Lemma 7 and the Parseval–Plancherel theorem

$$S_{T,1} = \int_{-T}^T \left(1 - \frac{|t|}{T}\right) r_1(t) r_2(t) dt = 2\pi \int_{-\infty}^{+\infty} f_1(\lambda) f_2(\lambda) d\lambda - I_1 - I_2, \quad (4.11)$$

where

$$I_1 = \int_{|t|>T} r_1(t) r_2(t) dt \quad (4.12)$$

and

$$I_2 = \frac{1}{T} \int_{-T}^T |t| r_1(t) r_2(t) dt. \quad (4.13)$$

Hence, for $\alpha := \alpha_1 + \alpha_2 < 1/2$ from Lemma 6 and (4.12), we have as $T \rightarrow \infty$

$$\begin{aligned} I_1 &= \int_{|t|>T} r_1(t) r_2(t) dt = 2 \int_{t>T} r_1(t) r_2(t) dt \\ &= \frac{2C_1 C_2 \pi^2}{\cos(\pi\alpha_1) \cos(\pi\alpha_2) \Gamma(2\alpha_1) \Gamma(2\alpha_2)} \int_{t>T} t^{2(\alpha-1)} dt (1 + o(1)) \\ &= \frac{2C_1 C_2 \pi^2}{\cos(\pi\alpha_1) \cos(\pi\alpha_2) \Gamma(2\alpha_1) \Gamma(2\alpha_2)} \frac{1}{1 - 2\alpha} T^{2\alpha-1} (1 + o(1)). \end{aligned} \quad (4.14)$$

For I_2 , from Lemma 6 and (4.13), for $\alpha := \alpha_1 + \alpha_2 < 1/2$ we have as $T \rightarrow \infty$

$$\begin{aligned} I_2 &= \frac{1}{T} \int_{-T}^T |t| r_1(t) r_2(t) dt = \frac{2}{T} \int_0^T t r_1(t) r_2(t) dt \\ &= \frac{2}{T} \frac{C_1 C_2 \pi^2}{\cos(\pi\alpha_1) \cos(\pi\alpha_2) \Gamma(2\alpha_1) \Gamma(2\alpha_2)} \int_0^T t^{2\alpha-1} dt (1 + o(1)) \\ &= \frac{2C_1 C_2 \pi^2}{\cos(\pi\alpha_1) \cos(\pi\alpha_2) \Gamma(2\alpha_1) \Gamma(2\alpha_2)} \frac{1}{2\alpha} T^{2\alpha-1} (1 + o(1)). \end{aligned} \quad (4.15)$$

From (4.11), (4.14), and (4.15) the result follows. Theorem 7 is proved.

R e m a r k s. (a) We analyze the behavior of the approximations as $\alpha_1, \alpha_2 \rightarrow 1/4$. First observe that the first-order asymptotic formula has a pole when $\alpha := \alpha_1 + \alpha_2 = 1/2$. In particular, denoting $\beta := \beta_1 + \beta_2$, and using the change of variable $\lambda^2 = u$, we have

$$\begin{aligned} 2\pi \int_{-\infty}^{+\infty} f_1(\lambda) f_2(\lambda) d\lambda &= 2\pi C_1 C_2 \int_{-\infty}^{+\infty} \frac{1}{|\lambda|^{2\alpha} (1 + \lambda^2)^\beta} d\lambda \\ &= 2\pi C_1 C_2 \int_0^\infty \frac{u^{-1/2-\alpha}}{(1+u)^\beta} du. \end{aligned} \quad (4.16)$$

Applying the formula (see, e.g., [8])

$$\int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}} du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m > 0, \quad n > 0, \quad (4.17)$$

with $m = 1/2 - \alpha$ and $n = \beta - m = \alpha + \beta - 1/2$, from (4.16) we find

$$2\pi C_1 C_2 \int_{-\infty}^{+\infty} f_1(\lambda) f_2(\lambda) d\lambda = 2\pi C_1 C_2 \frac{\Gamma(1/2 - \alpha)\Gamma(\alpha + \beta - 1/2)}{\Gamma(\beta)}. \quad (4.18)$$

Then, using the Laurent expansion of the gamma function $\Gamma(1/2 - \alpha)$ around the pole $\alpha = 1/2$, from (4.18) we obtain

$$2\pi \int_{-\infty}^{+\infty} f_1(\lambda) f_2(\lambda) d\lambda = \frac{4\pi C_1 C_2}{1 - 2\alpha} + O(1) \quad \text{as} \quad \alpha = \alpha_1 + \alpha_2 \rightarrow \frac{1}{2}. \quad (4.19)$$

However, the asymptotic behavior of the second-order term $C(\alpha_1, \alpha_2)$ (see (2.14)) as $\alpha_1, \alpha_2 \rightarrow 1/4$ is readily seen to be

$$\begin{aligned} C(\alpha_1, \alpha_2) &= \frac{2\pi^2 C_1 C_2}{\cos(\pi\alpha_1) \cos(\pi\alpha_2) \Gamma(2\alpha_1) \Gamma(2\alpha_2)} \frac{1}{2\alpha(1 - 2\alpha)} \\ &= \frac{2\pi^2 C_1 C_2}{\cos^2(\pi/4) [\Gamma(1/2)]^2} \frac{1}{1 - 2\alpha} + O(1) = \frac{4\pi C_1 C_2}{1 - 2\alpha} + O(1). \end{aligned}$$

Thus, the pole in the first-order approximation is removed by the second-order approximation, so that the approximation

$$2\pi \int_{-\infty}^{+\infty} f_1(\lambda) f_2(\lambda) d\lambda - \frac{C(\alpha_1, \alpha_2)}{T^{1-2(\alpha_1+\alpha_2)}} \quad (4.20)$$

is bounded as $\alpha_1, \alpha_2 \rightarrow 1/4$.

This good behavior explains why the second-order approximation produces a good approximation that does uniformly well over $\alpha_1, \alpha_2 \in [0, 1/4]$, including the limits of the domain.

(b) The second-order equivalence holds along an arbitrary ray for which $\alpha = \alpha_1 + \alpha_2 \rightarrow 1/2$. Indeed, let $\alpha_1^0 \in [0, 1/2]$ be any fixed number such that

$\alpha_1 \rightarrow \alpha_1^0$ and $\alpha = \alpha_1 + \alpha_2 \rightarrow 1/2$, then representation (4.19) for the first-order asymptotic term continues to apply.

On the other hand, using the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ with $z = 2\alpha_1^0$, we have from (2.14) as $\alpha_1 \rightarrow \alpha_1^0$ and $\alpha = \alpha_1 + \alpha_2 \rightarrow 1/2$

$$\begin{aligned} C(\alpha_1, \alpha_2) &= \frac{2\pi^2 C_1 C_2}{\cos(\alpha_1 \pi) \cos(\alpha_2 \pi) \Gamma(2\alpha_1) \Gamma(2\alpha_2)} \frac{1}{2\alpha(1-2\alpha)} \\ &= \frac{2\pi^2 C_1 C_2}{\cos(\alpha_1^0 \pi) \cos[(1-2\alpha_1^0)\pi/2] \Gamma(2\alpha_1^0) \Gamma(1-2\alpha_1^0)} \frac{1}{1-2\alpha} + O(1) \\ &= \frac{2\pi^2 C_1 C_2}{\cos(\alpha_1^0 \pi) \sin(\alpha_1^0 \pi)} \frac{\sin(2\alpha_1^0 \pi)}{\pi} \frac{1}{1-2\alpha} + O(1) = \frac{4\pi C_1 C_2}{1-2\alpha} + O(1), \end{aligned}$$

and again the second-order equivalence holds.

Thus, in this special case, the second-order asymptotic expansion removes the singularity in the first-order approximation, and provides a substantially improved approximation to the original functional.

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