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G. Mazurkiewicz, On $\mathbf{R}^+$ -weakly stable distribution, *Теория вероятн. и ее примен.*, 2011, том 56, выпуск 1, 197–202

DOI: <https://doi.org/10.4213/tvp4336>

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26 апреля 2023 г., 15:29:53

2. Erdős P., Feller W., Pollard H. A property of power series with positive coefficients. — Bull. Amer. Math. Soc., 1949, v. 55, p. 201–204.
3. Феллер В. Введение в теорию вероятностей и ее приложения, т. 1. М.: Мир, 1964, 498 с.
4. Феллер В. Введение в теорию вероятностей и ее приложения, т. 2. М.: Мир, 1967, 752 с.
5. de Bruijn N. G., Erdős P. On a recursion formula and some Tauberian theorems. — J. Res. Nat. Bur. Stand., 1953, v. 50, p. 161–164.
6. Garsia A., Lamperti J. A discrete renewal theorem with infinite mean. — Comment. Math. Helv., 1963, v. 37, p. 221–234.
7. Williamson J. A. Random walks and Riesz kernels. — Pacific J. Math., 1968, v. 25, № 2, p. 393–415.
8. Рвачева Е. Л. Об области притяжения многомерных распределений. — Ученые записки Львовского гос. ун-та, 1954, сер. мех.-матем., т. 6, № 29, p. 5–44.
9. Darling D. A. The influence of the maximal term in the addition of independent random variables. — Trans. Amer. Math. Soc., 1952, v. 73, № 1, p. 95–107.
10. Нагаев С. В., Вахтель В. И. О суммах независимых случайных величин без степенных моментов. — Сиб. матем. журн., 2008, т. 49, № 6, с. 1369–1380.

Поступила в редакцию
12.VII.2010

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MAZURKIEWICZ G.*

ON $\mathbf{R}^+$ -WEAKLY STABLE DISTRIBUTION

Случайный вектор $\mathbf{X}$ называется $(\mathbf{R}^+)$ -слабо устойчивым, если для любых (неотрицательных) случайных величин Θ_1 и Θ_2 , независимых от $\mathbf{X}$, $\mathbf{X}'$ существует (неотрицательная) случайная величина Θ , независимая от $\mathbf{X}$, такая, что

$$\Theta_1 \mathbf{X} + \Theta_2 \mathbf{X}' \stackrel{d}{=} \Theta \mathbf{X}.$$

В настоящей статье, отвечая на открытый вопрос, поставленный в (Studia Math., 2005, v. 167, № 3, p. 195–213), мы показываем, что $\mathbf{R}^+$ -слабо устойчивые распределения имеют те же свойства и ту же стохастическую структуру, что и слабо устойчивые распределения.

Ключевые слова и фразы: слабо устойчивое распределение, устойчивое распределение, ℓ_α -симметричное распределение, масштабная смесь, сократимые случайные векторы.

1. Introduction. Throughout the paper we denote by $\mathcal{L}(\mathbf{X})$ the distribution of the random vector $\mathbf{X}$. If random vectors $\mathbf{X}$ and $\mathbf{Y}$ have the same distribution, then we write $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$.

Let $\mathbb{E}$ be a separable Banach space. By $\mathcal{P}(\mathbb{E})$ we denote the set of all probability measures on $\mathbb{E}$. We use the simplified notation $\mathbf{R}^+ = [0, \infty)$, $\mathcal{P}^+ = \mathcal{P}([0, \infty))$, $\mathcal{P} = \mathcal{P}(\mathbf{R})$.

For every $a \in \mathbf{R}$ and every probability measure μ we define the rescaling operator $T_a: \mathcal{P}(\mathbb{E}) \rightarrow \mathcal{P}(\mathbb{E})$ by the formula

$$T_a \mu(A) = \begin{cases} \mu(A/a) & \text{for } a \neq 0, \\ \delta_0(A) & \text{for } a = 0, \end{cases}$$

* Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, ul. prof. Z. Szafrana 4a, 65-516 Zielona Góra, Poland; e-mail: G.Mazurkiewicz@wmie.uz.zgora.pl

for every Borel set $A \subseteq \mathbb{E}$. Equivalently $T_a\mu$ is the distribution of the random vector $a\mathbf{X}$ if μ is the distribution of $\mathbf{X}$.

The scale mixture $\mu \circ \lambda$ of a measure $\mu \in \mathcal{P}(\mathbb{E})$ with respect to the measure $\lambda \in \mathcal{P}$ is defined by the formula

$$\mu \circ \lambda(A) \stackrel{\text{def}}{=} \int_{\mathbf{R}} T_s\mu(A) \lambda(ds).$$

It is easy to see that $\mu \circ \lambda$ is the distribution of the random vector $\Theta\mathbf{X}$ if $\mu = \mathcal{L}(\mathbf{X})$, $\lambda = \mathcal{L}(\Theta)$, $\mathbf{X}$ and Θ are independent.

Recall (see, e.g., [4], [8]) that a random vector $\mathbf{X}$ is strictly α -stable, $\alpha \in (0, 2]$, if for every choice of $a, b \in (0, \infty)$ there exists a constant, $c = (a^\alpha + b^\alpha)^{1/\alpha}$, such that

$$a\mathbf{X} + b\mathbf{X} \stackrel{d}{=} c\mathbf{X}.$$

The concept of weakly stable distribution was introduced by Kucharczak and Urbanik in 1974, in the paper [2]. They introduced a quasi-stable random variable X as a symmetric random variable such that for all positive numbers a, b there exists a random variable Θ supported on $[0, \infty)$ and independent of X , such that $aX + bX' \stackrel{d}{=} \Theta X$. But in this paper we follow a definition introduced by Misiewicz, Oleszkiewicz, and Urbanik in [7]. We say that a random vector $\mathbf{X}$ is weakly stable if for all real numbers a, b there exists a random variable Θ independent of $\mathbf{X}$, such that

$$a\mathbf{X} + b\mathbf{X}' \stackrel{d}{=} \Theta\mathbf{X}, \quad (1)$$

where $\mathbf{X}'$ is independent copy of $\mathbf{X}$.

In the paper [7] Misiewicz, Oleszkiewicz, and Urbanik defined also an $\mathbf{R}^+$ -weakly stable distribution. According to their definition a random variable $X \geq 0$ is $\mathbf{R}^+$ -weakly stable if for any $a, b \in \mathbf{R}^+$ there exists $\Theta \geq 0$ such that $aX + bX \stackrel{d}{=} \Theta X$. Since we do not want to restrict this definition to X being nonnegative, we change notation a little bit. We say that a random vector $\mathbf{X}$ is $\mathbf{R}^+$ -weakly stable if for all positive numbers a, b there exists a random variable Θ supported on $[0, \infty)$ independent of $\mathbf{X}$, such that

$$a\mathbf{X} + b\mathbf{X}' \stackrel{d}{=} \Theta\mathbf{X}. \quad (2)$$

Notice that in this definition we can, without loss of generality, assume that $a, b \in [0, \infty)$. If, for example, $a = 0$, then it is enough to take $\Theta = b$ almost everywhere.

It is evident that if a random vector $\mathbf{X}$ is symmetric, then it is weakly stable if and only if it is $\mathbf{R}^+$ -weakly stable. Notice that every strictly stable vector is $\mathbf{R}^+$ -weakly stable, because it is enough to take $\Theta = (a^\alpha + b^\alpha)^{1/\alpha}$ almost everywhere, and every symmetric stable vector is weakly stable and $\mathbf{R}^+$ -weakly stable, because in this case $\Theta = (|a|^\alpha + |b|^\alpha)^{1/\alpha}$ almost everywhere.

In the literature we can find different examples of $\mathbf{R}^+$ -weakly stable distributions. Very well-known example are an n -dimensional symmetric random vector, which is uniformly distributed on the unit sphere in $\mathbf{R}^n$, and all its k -dimensional marginal distributions, $k \leq n$ (see [10]). In [1] Cambanis, Keener, and Simons describe a very interesting, but little-known, symmetric distribution in $\mathbf{R}^n$, which is (up to rescaling) the only one extreme point in the family of ℓ_1 -symmetric distributions. It turns out that this probability measure is weakly stable (see [5]). Paper [5] contains explicit formulas of densities for projections of this distribution, which are very interesting multidimensional examples of weakly stable measures. We can find other interesting examples of one-dimensional $\mathbf{R}^+$ -weakly stable measures: symmetric in [11], and supported on $[0, \infty)$ in [3]. A lot of properties of weakly stable random vectors one can find in [7]. Weak stability is also the subject of many papers of Vol'kovich (see, e.g., [12]–[14]). Recently there appeared a paper written by Misiewicz (see [6]).

The relationship between the classes of weakly and $\mathbf{R}^+$ -weakly stable probability measures is still unknown except a symmetric case, when these classes are equivalent. Notice

that a strictly stable random vector, which is not symmetrically distributed, is $\mathbf{R}^+$ -weakly stable, but we do not know whether this vector is weakly stable or not. We also do not know if every weakly stable measure must be $\mathbf{R}^+$ -weakly stable. It seems to be natural, but is not proved, that a corresponding random variable $\Theta = \Theta(a, b)$ from definition is nonnegative for a, b nonnegative. This property holds for one-sided (supported either on $[0, \infty)$ or on $(-\infty, 0]$) or symmetric weakly stable measures.

The next important problem is a connection of $\mathbf{R}^+$ -weak stability with Urbanik's general convolution (see, e.g., [9]). $\mathbf{R}^+$ -weakly stable measure μ supported on $[0, \infty)$ induces some general convolution if and only if there exists a probability measure $\lambda \in \mathcal{P}$ such that $\mu \circ \lambda$ belongs to the domain of attraction of some stable measure (see [3, Proposition 1.2]). But the relationship in general, namely when $\mathbf{R}^+$ -weakly stable measure is not supported on $[0, \infty)$, is still an open problem.

All these problems are the reasons why we decide to investigate $\mathbf{R}^+$ -weak stability. Our aim is to compare the behaviour of weakly stable and $\mathbf{R}^+$ -weakly stable distributions. We follow the methods and results from [7], which concern weakly stable distributions, and this way we indicate analogy and differences between weakly stable and $\mathbf{R}^+$ -weakly stable random vectors. Since definitions of weak stability and $\mathbf{R}^+$ -weak stability are very similar, most properties can be proved using the same methods or only slight changes are required. In such situations either we simply describe these changes instead of repeating the whole proofs or we omit the proof. Only Lemma 1, Lemma 2, and case 1 in Theorem 3 require new ideas.

2. Some general properties of $\mathbf{R}^+$ -weakly stable distributions. Since definitions of weak stability and $\mathbf{R}^+$ -weak stability are equivalent for symmetric measures, symmetric $\mathbf{R}^+$ -weakly stable vectors have the same properties as weakly stable ones. Thus in this section we assume that considered $\mathbf{R}^+$ -weakly stable distribution is not symmetric.

It turns out that stochastic behavior of weakly stable and $\mathbf{R}^+$ -weakly stable measures is identical. The following theorems confirm it.

Theorem 1 (analog of Theorem 1 in [7]). *Let $\mathbf{X}'$ be independent copy of random vector $\mathbf{X}$ and assume that all considered variables and vectors are independent. Then the following conditions are equivalent:*

$$\begin{aligned} \forall \Theta_1 \geq 0, \Theta_2 \geq 0 \exists \Theta \geq 0 \quad \text{independent of } \mathbf{X}: \Theta_1 \mathbf{X} + \Theta_2 \mathbf{X}' \stackrel{d}{=} \Theta \mathbf{X}, \\ \forall a \geq 0, b \geq 0 \exists \Theta \geq 0 \quad \text{independent of } \mathbf{X}: a\mathbf{X} + b\mathbf{X}' \stackrel{d}{=} \Theta \mathbf{X}. \end{aligned}$$

Theorem 2 (analog of Theorem 4 in [7]). *Assume that a random vector $\mathbf{X}$ taking values in a separable Banach space $\mathbb{E}$ and having distribution μ is such that $\mathbf{E}\|\mathbf{X}\| < \infty$ and $\mathbf{E}\mathbf{X} = a \neq 0$. Then μ is $\mathbf{R}^+$ -weakly stable if and only if $\mu = \delta_a$.*

Theorem 3 (analog of Theorem 5 in [7]). *Let μ be $\mathbf{R}^+$ -weakly stable probability measure on a separable Banach space $\mathbb{E}$. Then either there exists $a \in \mathbb{E}$ such that $\mu = \delta_a$, or there exists $a \in \mathbb{E} \setminus \{0\}$ such that $\mu = \delta_{-a}/2 + \delta_a/2$, or $\mu(\{a\}) = 0$ for all $a \in \mathbb{E}$.*

P r o o f o f T h e o r e m 1 follows from the proof of Theorem 1 in [7], because it is enough to define two classes of measures

$$\mathcal{M}^+(\mu) := \{\mu \circ \lambda: \lambda \in \mathcal{P}^+\}$$

and

$$K_\mu^+(\nu_1, \nu_2) := \{\lambda \in \mathcal{P}^+: (\mu \circ \nu_1) * (\mu \circ \nu_2) = \mu \circ \lambda\},$$

where $\nu_1, \nu_2 \in \mathcal{P}^+$, and to notice that proofs of Lemmas 1–5 and Theorem 1 in [7] holds for the class $\mathcal{M}(\mu)$ replaced by $\mathcal{M}^+(\mu)$ and the class $K_\mu(\nu_1, \nu_2)$ replaced by $K_\mu^+(\nu_1, \nu_2)$.

P r o o f o f T h e o r e m 2 is the same as the proof of Theorem 4 in [7].

Theorem 3 is a crucial one and it is analogous to Theorem 6 from [7]. It indicates similar stochastic structure of $\mathbf{R}^+$ -weakly stable and weakly stable probability measures.

In order to show it we must know the following properties and lemmas. We focus our attention only on the proof of Lemma 1, Lemma 2, and case 1 in Theorem 3 since they require new ideas.

Property 1 (analog of Lemma 8 in [7]). *If a measure μ on $\mathbf{R}$ is $\mathbf{R}^+$ -weakly stable, then $\mu(\{0\}) = 0$ or 1.*

Property 2 (analog of Lemma 9 in [7]). *Assume that $\mathbf{R}^+$ -weakly stable probability measure $\mu \neq \delta_0$ on $\mathbf{R}$ has at least one atom. Then the discrete part of μ (normalized to a probability measure) is also $\mathbf{R}^+$ -weakly stable.*

Lemma 1 (analog of Lemma 10 in [7]). *Let X be a real random variable with distribution μ . If μ is $\mathbf{R}^+$ -weakly stable and supported on a finite set, then either there exists $a \in \mathbf{R}$ such that $\mu = \delta_a$ or there exists $a \neq 0$ such that $\mu = \delta_{-a}/2 + \delta_a/2$.*

P r o o f. A sense of this lemma is analogous to Lemma 10 in [7] but the proof had to be substantially modified.

Let X' be an independent copy of X . Assume that $\mu \neq \delta_a$ for all $a \in \mathbf{R}$. Theorem 2 implies that X must take negative and positive values with positive probability. Let $V = \{x \in \mathbf{R}: \mu(\{x\}) > 0\}$. By Property 1 we have $0 \notin V$. Let b be the greatest positive, w be the least positive, $-a$ the least negative, and $-u$ the greatest negative elements of V . Clearly $a, b, w, u > 0$.

We will prove first that $a = u$.

For $0 < \beta < \min\{w/a, u/b\}$ consider the set of values of the variable $X + \beta X'$: $V_\beta = \{x + \beta y: x, y \in V\}$. Since μ is $\mathbf{R}^+$ -weakly stable, there exists a nonnegative random variable Y_β independent of X and such that $X + \beta X' \stackrel{d}{=} Y_\beta X$. Let $\nu = \mathcal{L}(Y_\beta)$ and $T_\beta = \{x \in [0, \infty): \nu(\{x\}) > 0\}$. One can easily see that the set T_β is finite. Let $c > 0$ be the least positive element of T_β .

Since $\beta < \min\{w/a, u/b\}$, we have $w - \beta a > 0$ and $-u + \beta b < 0$ and moreover $w - \beta a$ is the least positive element and $-u + \beta b$ is the greatest negative element of V_β . On the other hand, we have that $V_\beta = \{yx: y \in T_\beta, x \in V\}$ and that cw is the least positive element and $-cu$ is the greatest negative element of V_β . This means

$$w - \beta a = cw \quad \text{and} \quad -u + \beta b = -cu.$$

If $c = 1$, then $a = b = 0$, which implies $\mu = \delta_0$ and contradicts our assumption. Hence $c \neq 1$ and then

$$\frac{\beta}{1-c} = \frac{w}{a} = \frac{u}{b},$$

and therefore

$$wb = au, \tag{3}$$

Since $c = (w - \beta a)/w \in T_\beta$ and $-a \in V$, we have $-ca \in V_\beta$. This means there exist elements $v, z \in V$ such that $-ca = v + \beta z$ and hence $\beta(z - a^2/w) = -a - v$. Assume that $a^2/w \notin V$. Then the last equation may be satisfied for finitely many values of the parameter β only (because v and z can be chosen from a finite set only). In fact, this was proved for all elements $\beta \in (0, \min\{w/a, u/b\})$. We have proved then that $a^2/w \in V$. Clearly $a^2/w \leq b$ hence $a^2 \leq wb$. From (3) we obtain $a^2 \leq au$ which means $-a \geq -u$. But from the assumption we have $-a \leq -u$, thus $a = u$.

Now we will prove that $a = b$.

Since $c = (-u + \beta b)/(-u) = 1 - \beta b/a \in T_\beta$ and $b \in V$, we find that $cb \in V_\beta$. This means there exist elements $r, s \in V$ such that $cb = r + \beta s$ and hence $\beta(s + b^2/a) = b - r$. This equality implies $-b^2/a \in V$. Notice that $-a$ is the only negative element of V , hence $-b^2/a = -a$ and $a = b$.

We have $a = u = b$ and from (3) we obtain $a = u = b = w$. This means $V = \{-a, a\}$. Therefore $\mu = p\delta_{-a} + (1-p)\delta_a$ for some $p \in (0, 1)$. Theorem 2 implies that μ is symmetric. Lemma 1 is proved.

Property 3 (analog of Lemma 11 in [7]). *Let X be a real random variable with distribution $\mu \neq \delta_0$ and let X' be its independent copy. Assume that μ is nonsymmetric $\mathbf{R}^+$ -weakly stable, so that for any $\beta \in [0, \infty)$ there exists a nonnegative random variable Y_β independent of X such that $X + \beta X' \stackrel{d}{=} Y_\beta X$. Then the map*

$$\beta \mapsto \mathcal{L}(Y_\beta)$$

is well defined and continuous on $[0, \infty)$.

Lemma 2 (a simple modification of Property 3). *Let X be a real random variable with distribution $\mu \neq \delta_0$, X' its independent copy and $\mathcal{L}(\Theta_\beta) = \delta_\beta/2 + \delta_1/2$. Assume that μ is nonsymmetric $\mathbf{R}^+$ -weakly stable, so that for any $\beta \in [0, \infty)$ there exists a nonnegative random variable Z_β independent of X such that $\Theta_0 X + \Theta_\beta X' \stackrel{d}{=} Z_\beta X$, where X , X' , Θ_0 , Θ_β are independent. Then the mapping*

$$\beta \mapsto \mathcal{L}(Z_\beta)$$

is well defined and continuous on $[0, \infty)$.

Property 4 (analog of Lemma 12 in [7]). *Let X be a real random variable with distribution μ . If μ is discrete $\mathbf{R}^+$ -weakly stable measure, then there exists $a \in \mathbf{R}$ such that $\mu = \delta_a$ or there exists $a \neq 0$ such that $\mu = \delta_{-a}/2 + \delta_a/2$.*

P r o o f o f T h e o r e m 3. Assume first that $\mathbb{E} = \mathbf{R}$. One can express μ as $p\mu_1 + (1-p)\mu_2$, where $p \in [0, 1]$, μ_1 is a discrete probability measure, and $\mu_2(\{x\}) = 0$ for every $x \in \mathbf{R}$. The case $p = 0$ is trivial, so assume $p > 0$. Property 2 implies that μ_1 is $\mathbf{R}^+$ -weakly stable, and therefore by Property 4, $\mu_1 = \delta_a$ for some $a \in \mathbf{R}$ or $\mu_1 = \delta_{-a}/2 + \delta_a/2$ for some $a \neq 0$. If $a = 0$, then by Property 1 we have $p = 1$. If $a \neq 0$, then we have to consider two cases. Without loss of generality we can assume that $a = 1$.

Case 1: $\mu_1 = \delta_1$. Let $\mathcal{L}(\Theta_\beta) = \delta_\beta/2 + \delta_1/2$ for $\beta \geq 0$ and let X , X' , Θ_0 , Θ_β be independent. Notice that for $\beta > 0$ the random variable $X\Theta_0 + X'\Theta_\beta \stackrel{d}{=} Z_\beta X$ has exactly four atoms with mass $p^2/4$ at 1, 2, β , $1 + \beta$. Hence

$$\frac{1}{4}p^2 = \mathbf{P}\{Z_\beta X = \beta\} = \mathbf{P}\{Z_\beta = \beta\}\mathbf{P}\{X = 1\} = p\mathbf{P}\{Z_\beta = \beta\},$$

which implies $\mathbf{P}\{Z_\beta = \beta\} = p/4$. From Lemma 2 we have $Z_\beta \xrightarrow{d} Z_0$ as $\beta \rightarrow 0^+$. Thus $\mathbf{P}\{Z_0 = 0\} \geq p/4$. On the other hand,

$$\mathbf{P}\{Z_0 = 0\} = \mathbf{P}\{XZ_0 = 0\} = \mathbf{P}\{X\Theta_0 + X'\Theta_0 = 0\} = \frac{1}{4}p^2$$

because X has only one atom, at 1. Hence $p^2 \geq p$ and $p = 1$.

Case 2: $\mu_1 = \delta_{-1}/2 + \delta_1/2$. For $\beta \in (0, 1)$ the random variable $X + \beta X' \stackrel{d}{=} Y_\beta X$ has exactly four atoms, with mass $p^2/4$ each, at points $\pm(1 - \beta)$ and $\pm(1 + \beta)$. Since $\mathbf{P}\{X = -1\} = \mathbf{P}\{X = 1\} = p/2$, we find that

$$\mathbf{P}\{Y_\beta = \pm(1 - \beta)\} = \mathbf{P}\{Y_\beta = \pm(1 + \beta)\} = \frac{1}{2}p.$$

Property 3 implies $Y_\beta \xrightarrow{d} Y_1$ as $\beta \rightarrow 1$, so we have $\mathbf{P}\{Y_1 = 0\} \geq p/2$. On the other hand,

$$\mathbf{P}\{Y_1 = 0\} = \mathbf{P}\{Y_1 X = 0\} = \mathbf{P}\{X + X' = 0\} = \frac{1}{2}p^2$$

so that $p^2/2 \geq p/2$ and $p = 1$.

Now let $\mathbb{E}$ be an arbitrary separable Banach space. By making use of the above result for real random variables $\xi(\mathbf{X})$, where $\xi \in \mathbb{E}^*$, we can easily finish the proof of Theorem 3.

It is worth noticing that the proof of Property 3 and Lemma 2 requires an important property of $\mathbf{R}^+$ -weakly stable measures, namely property of being cancellable in the following sense.

Remark (analog of Remark 1 in [7]). (a) If $\mathbf{X}$, Y , Z are independent, Y , Z are real, and $\mathbf{X}$ is a nonsymmetric $\mathbf{R}^+$ -weakly stable random vector taking values in a separable Banach space $\mathbf{E}$, then $\mathbf{X}Y \stackrel{d}{=} \mathbf{X}Z$ implies $Y \stackrel{d}{=} Z$.

(b) If X , $\mathbf{Y}$, $\mathbf{Z}$ are independent, X is a nonsymmetric $\mathbf{R}^+$ -weakly stable random variable and $\mathbf{Y}$, $\mathbf{Z}$ take values in a separable Banach space $\mathbf{E}$, then $X\mathbf{Y} \stackrel{d}{=} X\mathbf{Z}$ implies $\mathbf{Y} \stackrel{d}{=} \mathbf{Z}$.

Acknowledgments. The author expressed her gratitude to the anonym referee and the editor, who made a very professional editorial correction of her paper.

REFERENCES

1. *Cambanis S., Keener R., Simons G.* On α -symmetric multivariate distributions. — J. Multivariate Anal., 1983, v. 13, № 2, p. 213–233.
2. *Kucharczak J., Urbanik K.* Quasi-stable functions. — Bull. Acad. Polon. Sci. Math., 1974, v. 22, № 3, p. 263–268.
3. *Kucharczak J., Urbanik K.* Transformations preserving weak stability. — Bull. Acad. Polon. Sci. Math., 1986, v. 34, № 7–8, p. 475–486.
4. *Лукач Е.* Характеристические функции. М.: Наука, 1979, 423 с.
5. *Mazurkiewicz G.* Weakly stable vectors and magic distribution of S. Cambanis, R. Keener and G. Simons. — Appl. Math. Sci. (Ruse), 2007, v. 1, № 17–20, p. 975–996.
6. *Misiewicz J. K.* Weak stability and generalized weak convolution for random vectors and stochastic processes. — IMS Lecture Notes Monogr. Ser., 2006, v. 48, p. 109–118.
7. *Misiewicz J. K., Oleszkiewicz K., Urbanik K.* Classes of measures closed under mixing and convolution. Weak stability. — Studia Math., 2005, v. 167, № 3, p. 195–213.
8. *Samorodnitsky G., Taqqu M. S.* Stable Non-Gaussian Random Processes. New York: Chapman & Hall, 1994, 632 p.
9. *Urbanik K.* Generalized convolutions. — Studia Math., 1964, v. 23, p. 217–245.
10. *Urbanik K.* Extreme point method in probability theory. — Lecture Notes in Math., 1975, v. 472, p. 169–194.
11. *Urbanik K.* Remarks on $\mathcal{B}$ -stable probability distributions. — Bull. Acad. Polon. Sci. Math. Astronom. Phys., 1976, v. 24, № 9, p. 783–787.
12. *Волькович В. Э.* Многомерные $\mathcal{B}$ -устойчивые распределения и реализации обобщенных свертков. — Проблемы устойчивости стохастических моделей. М.: ВНИИСИ, 1984, с. 40–54.
13. *Волькович В. Э.* О безгранично делимых распределениях в алгебрах со стохастической сверткой. — Проблемы устойчивости стохастических моделей. М.: ВНИИСИ, 1985, с. 15–24.
14. *Vol'kovich V.* On symmetric stochastic convolutions. — J. Theoret. Probab., 1992, v. 5, № 3, p. 417–430.

Поступила в редакцию
1.X.2008