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**ABSOLUTE CONTINUITY CONDITIONS FOR
MULTIVARIATE INFINITELY DIVISIBLE DISTRIBUTIONS
AND THEIR APPLICATIONS¹⁾**

Приводятся два достаточных условия абсолютной непрерывности относительно меры Лебега многомерных безгранично делимых распределений. Эти условия применяются для доказательства абсолютной непрерывности безгранично делимых распределений, меры Леви которых имеют обобщенное полярное разложение с абсолютно непрерывной радиальной частью указанного вида. Более того, одно из этих условий применяется для доказательства необходимого и достаточного условия абсолютной непрерывности переходных вероятностей процессов типа Орнштейна–Уленбека, управляемых процессами Леви.

Ключевые слова и фразы: абсолютная непрерывность, многомерные безгранично делимые распределения, процессы типа Орнштейна–Уленбека.

1. Introduction and main results. Let μ be an infinitely divisible distribution on $\mathbf{R}^d$ with characteristic function

$$\hat{\mu}(\theta) = \int e^{i\langle \theta, x \rangle} \mu(dx) = \exp \left\{ \int_{\mathbf{R}^d} (e^{i\langle \theta, x \rangle} - 1 - i\langle x, \theta \rangle 1_{\{|x| \leq 1\}}(x)) \nu(dx) \right\}, \quad (1)$$

where $\theta \in \mathbf{R}^d$ and ν is a measure on $\mathbf{R}^d$ with $\int (|x|^2 \wedge 1) \nu(dx) < \infty$ and $\nu(\{0\}) = 0$. The measure ν is called the Lévy measure of μ . For a Lévy measure ν , define a finite measure $\tilde{\nu}$ by $\tilde{\nu}(dx) = |x|^2/(1 + |x|^2) \nu(dx)$. We denote by ρ^{r*} the r -fold convolution of a finite measure ρ .

In this paper, we are interested in under what condition, μ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^d$. In the above representation, there is no Gaussian part. In order to consider the absolute continuity problem, the case with Gaussian part is not so interesting. So, we omit this case. It is known [3] that if $\nu(\mathbf{R}^d) = \infty$ and $\tilde{\nu}^{r*}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^d$ for some positive integer r , then μ is absolutely continuous. If ν satisfies GPDACR property

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stated in Section 4 and there is no $(d-2)$ -dimensional linear subspace having positive or infinite mass and $\nu(H^c) = \infty$ for every $(d-1)$ -dimensional linear subspace H , then $\tilde{\nu}^{d*}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^d$. Also, in case that there are finitely many proper linear subspaces on which the Lévy measure ν has positive or infinite masses, we can easily conclude whether μ is absolutely continuous or not. However, if there are infinitely many $(d-1)$ -dimensional linear subspaces on which ν has finite positive masses, then the situation become very complicated. So many cases should be considered. Therefore, we concentrate ourselves on such cases.

For a linear subspace W of $\mathbf{R}^d$, denote by Λ_W the Lebesgue measure on W . For two measures μ_1 and μ_2 , we use the notation $\mu_1 \ll \mu_2$ if μ_1 is absolutely continuous with respect to μ_2 .

Theorem 1.1. *Let $d \geq 1$. Assume that there is an integer $r \geq 1$ such that*

- 1) *ν is represented as $\nu = \sum_{k=1}^{\infty} \nu_k$, where each ν_k is concentrated on a linear subspace H_k and it has no mass on any proper linear subspace of H_k and $(\tilde{\nu}_k)^{r*} \ll \Lambda_{H_k}$;*
- 2) *$\sum_k (\nu_k(K^c))^r = \infty$ for any $(d-1)$ -dimensional linear subspace K of $\mathbf{R}^d$.*

Then $\mu \ll \Lambda_{\mathbf{R}^d}$.

Let ν^ℓ be a measure defined by removing masses of ν on all $(\ell-1)$ -dimensional linear subspaces. We denote by $\{H_k^{(\ell)}\}$ the totality of ℓ -dimensional linear subspaces satisfying $0 < \nu^\ell(H_k^{(\ell)}) \leq \infty$ and define $\nu^{(\ell)} = \nu^\ell|_{\bigcup_k H_k^{(\ell)}}$.

Remark 1.1. 1) In the assumptions in Theorem 1.1, H_k are not restricted to proper linear subspaces and not assumed to be distinct. We rearrange $\{H_k\}$ as $\{H_{\ell,k}\}$, where ℓ is the dimension of $H_{\ell,k}$ for each $\ell \leq 1$, and define $\nu_{\ell,k} = \nu_j$ for $H_j = H_{\ell,k}$. Then $\nu^{(\ell)} = \sum_k \nu_{\ell,k}$.

2) Assumption 1 in Theorem 1.1 is weaker than the assumption that $(\tilde{\nu}^{(\ell)}|_{H_k^{(\ell)}})^{r*} \ll \Lambda_{H_k^{(\ell)}}$ since for $H_k^{(\ell)} = H_{k_1} = H_{k_2} = \dots$, $(\tilde{\nu}^{(\ell)}|_{H_k^{(\ell)}})^{r*} \ll \Lambda_{H_k^{(\ell)}}$ implies that $\tilde{\nu}_{k_1}^{r*}, \tilde{\nu}_{k_2}^{r*}, \dots \ll \Lambda_{H_k^{(\ell)}}$ but not vice versa. Note that Assumption 1 is weaker than $\tilde{\nu}^{r*} \ll \Lambda_{\mathbf{R}^d}$.

3) Tucker [7] shows that if $\nu \ll \Lambda_{\mathbf{R}^d}$ and $\nu(\mathbf{R}^d) = \infty$, then $\mu \ll \Lambda_{\mathbf{R}^d}$. This is equivalent to Theorem 4.1 in $d = 1$ and the assumption of Theorem 1.1 is weaker by the above argument even in one-dimension.

Corollary 1.1. *Assume that for each $1 \leq k \leq d$, $\tilde{\nu}^{(k)}|_H \ll \Lambda_H$ on each k -dimensional linear subspace H with $0 < \nu^{(k)}(H) \leq \infty$. Then $\mu \ll \Lambda_{\mathbf{R}^d}$ if and only if $\nu(K^c) = \infty$ for any $(d-1)$ -dimensional linear subspace K .*

This corollary is straightforward from Theorem 1.1 setting $r = 1$.

Now, we describe another type result. For a linear subspace W of $\mathbf{R}^d$, denote by T_W the orthogonal projection operator onto W . Denote $\prod_{i=1}^r H_i =$

$H_1 \times \cdots \times H_r$ and let

$$K(H_1, \dots, H_r) = \left\{ (x_1, \dots, x_r) \in \prod_{i=1}^r H_i : x_1, \dots, x_r \text{ are linearly independent} \right\}$$

for linear subspaces $H_1, \dots, H_r$.

Theorem 1.2. *Let $d \geq 1$, $1 \leq \ell \leq d$ and let W be an m -dimensional linear subspace of $\mathbf{R}^d$ with $\nu^{(\ell)}(W) > 0$ for $\ell \leq m \leq d$ or $W = \{0\}$. Let $T := T_{W^\perp}$ and $T\nu_\ell(\cdot) := \nu^{(\ell)}(T^{-1}(\cdot) \cap W^c)$. Let $\{V_k\}$ be a sequence of all j -dimensional linear subspaces of $W^\perp$ such that $0 < T\nu_\ell(V_k) \leq \infty$. Regard $m = 0$ and $j = \ell$ if $W = \{0\}$. Suppose that if $\dim(V := \sum_{i=1}^r V_{k_i}) = r$ ($1 \leq j \leq r \leq d - m$), then*

$$\int_{K(V_{k_1}, \dots, V_{k_r})} 1_Z \left(\sum_{i=1}^r x_{k_i} \right) \prod_{i=1}^r (T\nu_\ell)^{(j)}(dx_{k_i}) = 0 \quad (2)$$

for any Borel subset Z of V with $\Lambda_V(Z) = 0$. Then $\mu \ll \Lambda_{\mathbf{R}^d}$ if and only if

$$\nu(K^c) = \infty \quad (3)$$

for any $(d - 1)$ -dimensional linear subspace K of $\mathbf{R}^d$.

R e m a r k 1.2. 1) Regard $H_1^{(\ell)} = H_2^{(\ell)} = \cdots = \mathbf{R}^d$ if $\ell = d$ in the above theorem. Hence, if $d = 1$, then condition (2) means that $\nu \ll \Lambda_{\mathbf{R}^d}$.

2) The results in Sections 4 and 5 show that the assumption in the above theorem is not empty and useful.

In the following section, we prepare lemmas necessary for the proofs of the above theorems. In Section 3, we give the proofs of Theorems 1.1 and 1.2. In Section 4, we apply the main results to absolute continuity of multivariate infinitely divisible distributions under GPDACR condition for the Lévy measure, which is defined in Section 4. Section 5 is devoted to a proof of the absolute continuity of transition probabilities of the so-called processes of Ornstein–Uhlenbeck type (OU type processes) in [5] as an application of Theorem 1.2.

2. Lemmas.

Lemma 2.1. *Let H_1 and H_2 be linear subspaces of $\mathbf{R}^d$ such that $\dim(H_1 + H_2) = d$. If $\mu_1 \ll \Lambda_{H_1}$ and $\mu_2 \ll \Lambda_{H_2}$, then $\mu_1 * \mu_2 \ll \Lambda_{\mathbf{R}^d}$.*

Lemma 2.2. *Let $H \neq \{0\}$ be a proper linear subspace of $\mathbf{R}^d$ and $H^\perp$ its orthogonal complement in $\mathbf{R}^d$. Let $T = T_{H^\perp}$. Let $\mu_1(dx) = q(dx) + p(x)dx$ be a probability measure on H and μ_2 a probability measure on $\mathbf{R}^d$ such that $T\mu_2 \ll \Lambda_{H^\perp}$, where $T\mu_2(\cdot) = \mu_2(T^{-1}(\cdot))$. Then $\mu(Z) = \mu_1 * \mu_2(Z) \leq q := \mu(H)$ for any Borel set Z in $\mathbf{R}^d$ with $\Lambda_{\mathbf{R}^d}(Z) = 0$.*

P r o o f. For $x \in \mathbf{R}^d$, we denote $x_2 = Tx$ and $x_1 = x - x_2$. Let Y be a random variable with distribution μ_2 . Let $P_1(dy_1|y_2) = \mathbf{P}(Y - TY \in dy_1 | TY = y_2)$ be the conditional probability of $Y - TY$ given $TY = y_2$. Since

$$\int_{H^\perp} dy_2 \int_H 1_Z(x_1, y_2) dx_1 = 0$$

there is a Borel set Z_2 in $H^\perp$ with $\Lambda_{H^\perp}(Z_2) = 0$ such that for every $y_2 \notin Z_2$

$$\int_H 1_Z(x_1, y_2) dx_1 = 0.$$

Let

$$\begin{aligned} h_1(y_1, y_2) &= \int_H 1_Z(x_1 + y_1, y_2) p(x_1) dx_1, \\ h(y_1, y_2) &= h_1(y_1, y_2) + \int_H 1_Z(x_1 + y_1, y_2) q(dx_1), \end{aligned}$$

which are Borel measurable in (y_1, y_2) , where $y_1 \in H$ and $y_2 \in H^\perp$. Then we have

$$h_1(y_1, y_2) = 1_{Z_2}(y_2) h_1(y_1, y_2)$$

and

$$\mu(Z) = \int_{\mathbf{R}^d} \mu_2(dy) h(y_1, y_2) \leq \int_{Z_2} T\mu_2(dy_2) \int_H h_1(y_1, y_2) P_1(dy_1|y_2) + q = q.$$

Lemma 2.2 is proved.

The above lemma is an extension of a result in the proof of Theorem 27.10 in [4]. The proof in [4] treats the case $q = 0$. In other words, absolute continuity of μ_1 is assumed.

Lemma 2.3. *Let $\{a_j\}$ and $\{b_j\}$ be sequences of positive numbers. Assume that $r \geq 1$. Then $\sum_j (a_j + b_j)^r < \infty$ if and only if $\sum_j (a_j)^r < \infty$ and $\sum_j (b_j)^r < \infty$.*

P r o o f. By Minkowski's inequality

$$\sum_j (a_j + b_j)^r \leq \left(\left(\sum_j (a_j)^r \right)^{1/r} + \left(\sum_j (b_j)^r \right)^{1/r} \right)^r$$

and the inequality $a^r + b^r < (a + b)^r$ for $a, b > 0$, we have the conclusion.

Lemma 2.4. *Let W be a linear subspace of $\mathbf{R}^d$ and let $T = T_{W^\perp}$. Let $H_1, H_2, \dots$ be distinct r -dimensional linear subspaces of $\mathbf{R}^d$ such that $H_k \not\subset W$ and $W \not\subset H_k$ for $k \geq 1$. Assume that for each $k \leq 1$, $0 < \nu^{(r)}(H_k) < \infty$ and $(\nu^{(r)}|_{H_k})^{r*} \ll \Lambda_{H_k}$. Then $(T(\nu^{(r)}|_{H_k}))^{r*} \ll \Lambda_{T(H_k)}$ for each k .*

P r o o f. Set $H = H_k$. By the assumption, $(\nu^{(r)}|_H)^{r*} \ll \Lambda_H$. Let Z be a Borel set with $\Lambda_{T(H)}(Z) = 0$. We have $(T|_H)^{-1}(0) = H \cap W$ and

$$\dim T|_H(H) + \dim(H \cap W) = \dim T|_H(H) + (T|_H)^{-1}(0) = r.$$

We also have that $\dim(H \cap W) + \dim(H \cap (H \cap W)^\perp) = r$. Let $H' = H \cap (H \cap W)^\perp$. Then $\dim H' = \dim T|_H(H)$. Since $H \cap W$ is a proper linear subspace of H by the assumption, $0 \leq \dim(H \cap W) < r$.

We decompose $x \in H$ as $x = x_1 + x_2$, $x_1 \in H \cap W$ and $x_2 \in H'$. Then $Tx = Tx_2$. Hence $T|_{H'}$ is a linear map on H' onto $T(H)$. Since $\dim H' = \dim T|_H(H) = \dim T|_{H'}(H')$, $T|_{H'}(H')$ is a bijection on H' onto $T(H)$. Hence $\Lambda_{H'}(T|_{H'})^{-1}(Z) = 0$.

For $y \in T(H)$, let x be a uniquely determined element of H' such that $Tx = y$. Assume that for $z \in H \cap W$, $x + z = w_1 + w_2$ with $w_1 \in W$ and $w_2 \in W^\perp$. Then, $y = T(x + z) = T(w_1 + w_2) = w_2$ and hence $x + z = w_1 + y$, i.e., $x + H \cap W \subset (y + W) \cap H$. Conversely, for any $w \in W$ satisfying $y + w \in H$, decompose $y + w$ as $y + w = h_1 + h_2$ with $h_1 \in H \cap W$ and $h_2 \in H'$. Then, $y = T(y + w) = T(h_1 + h_2) = T(h_2)$. Thus, $x = h_2$ and $(y + W) \cap H \subset x + H \cap W$. Hence, we have $(y + W) \cap H = x + H \cap W$. This means that $T^{-1}(Z) \cap H = (T|_{H'})^{-1}(Z) + H \cap W$. Therefore, $\Lambda_H(T^{-1}(Z) \cap H) = 0$ and we have

$$(T(\nu^{(r)}|_H))^{r*}(Z) = T((\nu^{(r)}|_H)^{r*})(Z) = (\nu^{(r)})^{r*}(T^{-1}(Z) \cap H) = 0.$$

Lemma 2.4 is proved.

Lemma 2.5. Let $\{c_i^{(n)}\}_{i,n=1,2,\dots}$ be a sequence of positive numbers such that $\sup_{i,n} c_i^{(n)} < \infty$. Let

$$a_i^{(n)} = \sum_{j=1}^{r-1} \frac{(c_i^{(n)})^j}{j!} \quad \text{and} \quad b_i^{(n)} = \sum_{j=r}^{\infty} \frac{(c_i^{(n)})^j}{j!}$$

for some $r \geq 1$. If $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} (c_i^{(n)})^r = \infty$, then

$$\lim_{n \rightarrow \infty} \lim_{q \rightarrow \infty} \exp\left(-\sum_{k=1}^q c_k^{(n)}\right) \left(\prod_{k=1}^q (1 + a_k^{(n)})\right) \left(\sum_{1 \leq j_1 < \dots < j_p \leq q} \prod_{k=1}^p \frac{b_{j_k}^{(n)}}{1 + a_{j_k}^{(n)}}\right) = 0$$

for $p \geq 1$.

P r o o f. We have

$$\begin{aligned} & \frac{\exp(\sum_{i=1}^q c_i^{(n)})}{\prod_{i=1}^q (1 + a_i^{(n)}) (\sum_{1 \leq j_1 < \dots < j_p \leq q} \prod_{k=1}^p b_{j_k}^{(n)} / (1 + a_{j_k}^{(n)}))} \\ & \geq \prod_{i=1}^q \left(1 + \frac{b_i^{(n)}}{1 + a_i^{(n)}}\right) \left(\sum_{k=1}^q \frac{b_k^{(n)}}{1 + a_k^{(n)}}\right)^{-p}. \end{aligned} \quad (4)$$

Taking logarithm, we have

$$\begin{aligned} & \sum_{i=1}^q \ln \left(1 + \frac{b_i^{(n)}}{1 + a_i^{(n)}} \right) - p \ln \left(\sum_{k=1}^q \frac{b_k^{(n)}}{1 + a_k^{(n)}} \right) \\ & \geq \sum_{i=1}^q c \frac{b_i^{(n)}}{1 + a_i^{(n)}} - p \ln \left(\sum_{i=1}^q \frac{b_i^{(n)}}{1 + a_i^{(n)}} \right) \geq c' \sum_{i=1}^q b_i^{(n)} - p \ln \left(\sum_{i=1}^q b_i^{(n)} \right), \end{aligned}$$

where c and c' are positive constants and we used $a_k^{(n)}, b_k^{(n)} \leq e^{\sup_k c_k} < \infty$ in the above estimates. Hence the left-hand side of (4) tends to ∞ as $q \rightarrow \infty$ and $n \rightarrow \infty$ since

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} b_i^{(n)} \geq \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \frac{(c_i^{(n)})^r}{r!} = \infty.$$

3. Proofs of theorems. In this section, we give proofs of theorems described in Section 1.

Proof of Theorem 1.1. We prove the theorem by induction in d . Let $d = 1$. Let $\nu = \sum_k \nu_k$, where $(\tilde{\nu}_k)^{r*} \ll \Lambda_{\mathbf{R}^1}$ and $\sum_k \nu_k^{r*}(\mathbf{R}^1) = \infty$. Let Z be a Borel set with $\Lambda_{\mathbf{R}^1} = 0$. Define $\rho_k^s = \sum_{i=1}^{r-1} (\nu_k)^{i*}/i!$ and $\rho_k^a = \sum_{i=r}^{\infty} (\nu_k)^{i*}/i!$. Let $a_k = \rho_k^s(\mathbf{R}^1)$, $b_k = \rho_k^a(\mathbf{R}^1)$ and $e^{c_k} = 1 + a_k + b_k$. If $c_k = \infty$ for some k , then $\mu \ll \Lambda_{\mathbf{R}^1}$. Suppose that $c_k < \infty$ for every k . Define a probability measure μ_n by

$$\begin{aligned} \mu &= e^{-\sum c_k} (\delta_0 + \rho_1^s + \rho_1^a) * \cdots * (\delta_0 + \rho_n^s + \rho_n^a) * \mu_n, \\ \mu(Z) &\leq e^{-\sum c_k} \prod_{k=1}^n (1 + a_k) = \left[\prod_{k=1}^n \left(1 + \frac{b_k}{1 + a_k} \right) \right]^{-1}. \end{aligned}$$

If $\sup_k c_k = \infty$, then we can assume that $c_k \uparrow \infty$ as $k \rightarrow \infty$. Then

$$\left[\prod_{k=1}^n \left(1 + \frac{b_k}{1 + a_k} \right) \right]^{-1} \leq \frac{1 + a_n}{1 + a_n + b_n} \rightarrow 0$$

as $n \rightarrow \infty$. If $\sup_k c_k < \infty$, then we have

$$\left[\prod_{k=1}^n \left(1 + \frac{b_k}{1 + a_k} \right) \right]^{-1} \rightarrow 0$$

as $n \rightarrow \infty$ since $a_k, b_k \leq e^{\sup_k c_k}$ and $\sum_k b_k = \infty$. Hence $\mu \ll \Lambda_{\mathbf{R}^1}$.

Now, we assume that the theorem is valid for dimension less than $d \geq 2$.

Step 1. We prove the following statement: Let $1 \leq \ell \leq d - 1$ and let $\{H_{\ell,k}\}$ be a sequence of not necessarily distinct ℓ -dimensional linear subspaces appearing in assumption 1 as in Remark 1.1. If $\{H_{\ell,k}\}$ satisfies that $0 < \nu_{\ell,k}(H_{\ell,k}) < \infty$ and $\sum_{k=1}^{\infty} \nu_{\ell,k}(H_{\ell,k} \cap K^c)^r = \infty$ for any $(d - 1)$ -dimensional linear subspace K in $\mathbf{R}^d$, then $\mu \ll \Lambda_{\mathbf{R}^d}$.

We denote $\nu_{\ell,k}$ by ν_k and $H_{\ell,k}$ by H_k . Let $D_m = \sup_W \sum_k (\nu_k(H_k \cap W))^r$, where W is an m -dimensional linear subspace of $\mathbf{R}^d$. We divide into the following cases: $D_1 \leq \dots \leq D_{m-1} < \infty = D_m = \dots = D_d$ for $\ell \leq m \leq d$.

1. For $\ell \leq m \leq d-1$, assume that $D_\ell, \dots, D_{m-1} < \infty = D_m$. Fix an m -dimensional linear subspace W of $\mathbf{R}^d$. Let $T = T_{W^\perp}$, where $W^\perp$ is the orthogonal complement of W . Let $\sigma_1 = \sum_k \nu_k|_{H_k \cap W}$ and $\sigma_2 = \sum_k \nu_k|_{H_k \cap W^c}$ and let μ_1 and μ_2 be infinitely divisible distributions determined through (1) by σ_1 and σ_2 , respectively. Note that $\dim W^\perp = d - m \leq d - 1$. By the assumption 2,

$$\sum_{k=1}^{\infty} (T\nu_2(H_k \cap K^c))^r = \sum_k (\nu_k(H_k \cap (T^{-1}(K))^c))^r = \infty$$

for any $(d-m-1)$ -dimensional linear subspace K of $W^\perp$ since $\dim T^{-1}(K) = d-1$. By the assumption 1 and Lemma 2.4, $(T(\nu|_{H_k}))^{r*} \ll \Lambda_{T(H_k)}$ for each k (it may happen that $T(H_j) = T(H_k)$ for distinct H_j and H_k). Hence $T\sigma_2$ satisfies the assumption 1 with $W^\perp$ as $\mathbf{R}^d$. Since $\dim W^\perp < d$, infinitely divisible distribution $T\mu_2$ corresponding to $T\sigma_2$ is absolutely continuous on $W^\perp$ by the induction hypothesis.

2. Now, we consider μ_1 . Rearrange H_k in W as $\{H_k\}$ and let $\{\nu_k\}$ be the sequence of corresponding measures. Let $\rho_k^s = \sum_{n=1}^{r-1} \nu_k^{n*}/(n!)$, $\rho_k^a = \sum_{n=r}^{\infty} \nu_k^{n*}/(n!)$, and $\rho_k = \delta_0 + \rho_k^s + \rho_k^a$. Let $a_k = \rho_k^s(H_k)$, $b_k = \rho_k^a(H_k)$. Define μ_q^1 by

$$\mu_1 = e^{-\sum_{k=1}^q c_k} \rho_1 * \rho_2 * \dots * \rho_q * \mu_q^1.$$

Let Z be a Borel set of W with $\Lambda_W(Z) = 0$. We have

$$\begin{aligned} \rho_1 * \rho_2 * \dots * \rho_q * \mu_q^1(Z) &= \left(\prod_{i=1}^q *(\delta_0 + \rho_i^s + \rho_i^a) \right) * \mu_q^1(Z) \\ &= \left\{ \sum_{p=0}^q \sum_{1 \leq k_1 < \dots < k_p \leq q} \rho_{k_1}^a * \dots * \rho_{k_p}^a * \prod_{i \neq k_1, \dots, k_p} *(\delta_0 + \rho_i^s) \right\} * \mu_q^1(Z). \end{aligned}$$

Here $\prod *$ is a convolution product. Let us denote the right-hand side of the above equalities by (Σ) . A term such that $\sum_{i=1}^p H_{k_i} = W$ disappears since $\rho_{k_1}^a * \rho_{k_2}^a * \dots * \rho_{k_p}^a \ll \Lambda_W$ by Lemma 2.1. If $\dim(H := \sum_{i=1}^p H_{k_i}) \leq m-1$ for $p \geq m-\ell$, then there are integers $\{j_1, \dots, j_m\}$ in $\{k_1, \dots, k_p\}$ such that $H = \sum_{i=1}^m H_{j_i}$. Let $\{i'\}$ be a subsequence in $\{1, \dots, q\}$ such that $H_{i'} \subset H$. Sum of the terms in the linear subspace H is not greater than

$$b_{j_1} \dots b_{j_m} \sum b_{k_1} \dots b_{k_p} = b_{j_1} \dots b_{j_m} \prod_i (1 + b_i),$$

where the sum is over $\{k_1, \dots, k_p\} \subset \{i'\} \setminus \{j_1, \dots, j_m\}$ for $0 \leq p \leq q-m$ and the product is taken over $i \in \{i'\} \setminus \{j_1, \dots, j_m\}$. Since $b_i \leq (c_i)^r e^{c_i}$ and

$D_{m-1} < \infty$, $\sum_{i'} b_{i'} < D_{m-1} e^{(D_{m-1})^{1/r}} =: D < \infty$. Hence we have

$$(\Sigma) \leq (D+1) \left\{ \sum_{p=0}^m \sum_{1 \leq j_1 < \dots < j_p \leq q} \prod_{i=1}^p \frac{b_{j_i}}{1+a_{j_i}} \right\} \prod_{i=1}^q (1+a_i).$$

We can choose a sequence of m -dimensional linear subspaces $\{W_n\}$ so that $\sum_k \nu_k(H_k \cap W_n)^r \rightarrow \infty$ as $n \rightarrow \infty$. Define $\mu_1^{(n)}$, $\mu_2^{(n)}$, $a_i^{(n)}$, $b_i^{(n)}$, and $c_i^{(n)}$ as above for W_n . Since $a_i^{(n)} \leq e^{c_i^{(n)}}$, $b_i^{(n)} \leq (c_i^{(n)})^r e^{c_i^{(n)}}$, $c_i^{(n)} \leq (D_{m-1})^{1/r}$ and $\sum_{i=1}^\infty (c_i^{(n)})^r \rightarrow \infty$ as $n \rightarrow \infty$ by the definition of D_m ,

$$(D+1)e^{-\sum_{k=1}^\infty c_k^{(n)}} \left\{ \sum_{p=0}^m \sum_{1 \leq j_1 < \dots < j_p} \prod_{k=1}^p \frac{b_{j_k}^{(n)}}{1+a_{j_k}^{(n)}} \right\} \prod_{k=1}^\infty (1+a_k^{(n)}),$$

upper bound of the mass of the singular part of $\mu_1^{(n)}$, tends to 0 as $n \rightarrow \infty$ by Lemma 2.5. We have, by Lemma 2.2,

$$\mu(Z) = \mu_1^{(n)} * \mu_2^{(n)}(Z) \rightarrow 0$$

as $n \rightarrow \infty$.

Step 2. Let ℓ be an integer with $1 \leq \ell \leq d$. If $\sum_k (\nu_{\ell,k}(\mathbf{R}^d))^r < \infty$, then set $W_\ell = \{0\}$. Assume that $\sum_k (\nu_{\ell,k}(\mathbf{R}^d))^r = \infty$. If there is a $(d-1)$ -dimensional linear subspace K with $\sum_k (\nu_{\ell,k}(K^c))^r < \infty$, then set $W = K$. Otherwise set $W_\ell = \mathbf{R}^d$. For the above W , $\sum_k (\nu_{\ell,k}(W))^r = \infty$ holds by Lemma 2.3 and $\sum_k (\nu_{\ell,k}(W^c))^r < \infty$ holds by the definition of W . If there is no $(d-2)$ -dimensional linear subspace K with $\sum_k (\nu_{\ell,k}(W \cap K^c))^r < \infty$, then set $W_\ell = W$. If there is a $(d-2)$ -dimensional linear subspace K such that $\sum_k (\nu_{\ell,k}(W \cap K^c))^r < \infty$. Then by Lemma 2.3, $\sum_k (\nu_{\ell,k}(W \cap K))^r = \infty$. Set $W = K$. Repeating this procedure, we have the following two cases.

1. There is a linear subspace W such that $\ell+1 \leq \dim W = m \leq d$, $\sum_k (\nu_{\ell,k}(W^c))^r < \infty$ and $\sum_k (\nu_{\ell,k}(W \cap K^c))^r = \infty$ for any $(m-1)$ -dimensional linear subspace K of W . Then set W_ℓ as such W with largest dimension.

2. There is no such W for $m \leq \ell+1$. In this case, there is an $H_{\ell,k}$ such that $\nu_{\ell,k}(H_{\ell,k}) = \infty$ and $\sum_{j \neq k} \nu_{\ell,j}(H_{\ell,j})^r < \infty$. Set $W_\ell = H_{\ell,k}$. If $W_\ell = \mathbf{R}^d$, then $\mu \ll \Lambda_{\mathbf{R}^d}$ by Step 1. If $\dim W_\ell < d$, then by the induction hypothesis, μ_ℓ corresponding to $\sum_k \nu_{\ell,k}$ is absolutely continuous on W_ℓ . Suppose that $\dim(\sum_{k=1}^d W_k) < d$. Then for $(d-1)$ -dimensional linear subspace K containing $\sum_{k=1}^d W_k$,

$$\sum_{\ell=1}^d \sum_k (\nu_{\ell,k}(K^c))^r \leq \sum_{\ell=1}^d \sum_k (\nu_{\ell,k}((W_\ell)^c))^r < \infty.$$

This contradicts the assumption 2. Therefore, $\dim(\sum_{k=1}^d W_k) = d$ and $\mu \ll \Lambda_{\mathbf{R}^d}$ by Lemma 2.1. Theorem 1.1 is proved.

P r o o f o f T h e o r e m 1.2. We prove the theorem also by induction in d . If $d = 1$, then the conclusion is obviously valid. Assume that the theorem is valid for dimension less than $d \geq 2$.

Step 1. We prove the following statement: Let $1 \leq \ell \leq d - 1$. If

$$\nu^{(\ell)}(K^c) = \infty \quad (5)$$

for any $(d - 1)$ -dimensional linear subspace K in $\mathbf{R}^d$, then $\mu \ll \Lambda_{\mathbf{R}^d}$.

We denote $H_k^{(\ell)}$ by H_k and $\nu^{(\ell)}$ by ν . Let

$$D_m := \sup\{\nu(W) : W \text{ is an } m\text{-dimensional linear subspace of } \mathbf{R}^d\}.$$

Note that $\nu(W) = \sum_{H_k \subset W} \nu(H_k \cap W)$. We divide into the following cases:

$$D_1 \leq \dots \leq D_{m-1} < \infty = D_m = \dots = D_d, \quad m = \ell, \ell + 1, \dots, d.$$

1. Assume the above case. Fix an m -dimensional linear subspace W of $\mathbf{R}^d$ satisfying $\nu(W) > 0$. Let $T = T_{W^\perp}$. Let $\sigma_1 = \nu|_W$ and $\sigma_2 = \nu|_{W^c}$ and let μ_1 and μ_2 be infinitely divisible distributions determined through (1) by σ_1 and σ_2 , respectively. Note that $\dim W^\perp = d - m \leq d - 1$. By the assumption (5), $(T\sigma_2)(K^c) = \nu^{(\ell)}(W^c \cap (T^{-1}(K))^c) = \infty$ for any $(d - m - 1)$ -dimensional linear subspace K of $W^\perp$ since $\dim T^{-1}(K) = d - 1$ and $W^c \supset (T^{-1}(K))^c$. Let $\{V_k\}$ be a sequence of all j -dimensional linear subspaces of $W^\perp$ such that $0 < (T\sigma_2)^{(j)}(V_k) \leq \infty$. Note that $V_k = T(H_p^{(\ell)})$ for at least one p . Suppose that V be an r -dimensional ($1 \leq j \leq r < d - m$) linear subspace of $W^\perp$ such that $(T\sigma_2)^{(j)}(V) > 0$. Let S be an orthogonal projection of $W^\perp$ to $V^\perp$. Let $\{U_q\}$ be the sequence of all i -dimensional linear subspaces of $V^\perp$ such that $(S((T\sigma_2)^{(j)}|_{V^c \cap W^\perp}))^{(i)}(U_q) > 0$. Note that $U_q = (ST)(H_p)$ for at least one p and $\{U_q\}$ is a subsequence of $\{ST(H_p)\}$. We have

$$\begin{aligned} (S((T\sigma_2)^{(j)}|_{V^c \cap W^\perp}))^{(i)}(\cdot) &= (T\sigma_2)^{(j)}\left(S^{-1}(\cdot) \cap \left(\bigcup_{k'} V_{k'}\right) \cap V^c \cap W^\perp\right) \\ &= \sigma_2(T^{-1}\left(S^{-1}(\cdot) \cap \left(\bigcup_{k'} V_{k'}\right) \cap V^c \cap W^\perp\right)) \\ &= \nu\left((ST)^{-1}(\cdot) \cap \left(\bigcup_{k'} T^{-1}(V_{k'})\right) \cap (V \oplus W)^c\right) \\ &= ((ST)(\nu|_{(V \oplus W)^c}))^{(i)}(\cdot). \end{aligned}$$

Here, $\{V_{k'}\}$ is a subsequence of $\{V_k\}$ such that $\dim S(V_{k'}) = i$ and $\{T^{-1}(V_{k'})\}$ is a subsequence of $\{H_k\}$. Hence $T\sigma_2$ satisfies the assumption by the assumption for the original ν regarding ST as T . By the induction hypothesis, $T\mu \ll \Lambda_{W^\perp}$.

2. Now, we consider μ_1 . Relabel H_k in W as $\{H_k\}$ and let $\nu_{1,k} = \sigma_1|_{H_k}$. Let $c_k := \nu_{1,k}(H_k)$, $\rho_k = \sum_{j=1}^{\infty} \nu_{1,k}^{j*}/j!$. Let $b_k = \rho_k(H_k) = e^{c_k} - 1$. Define $\bar{\mu}_q$ by

$$\mu_1 = e^{-\sum_{k=1}^q c_k} (\delta_0 + \rho_1) * (\delta_0 + \rho_2) * \cdots * (\delta_0 + \rho_q) * \bar{\mu}_q.$$

Let Z be a Borel set of W with $\Lambda_W(Z) = 0$. We have

$$\begin{aligned} & (\delta_0 + \rho_1) * (\delta_0 + \rho_2) * \cdots * (\delta_0 + \rho_q) * \bar{\mu}_q(Z) \\ &= \left\{ \delta_0 + \sum_{p=1}^q \sum_{1 \leq j_1 < \cdots < j_p \leq q} \rho_{j_1} * \cdots * \rho_{j_p} \right\} * \bar{\mu}_q(Z). \end{aligned}$$

For the term with $\dim(\sum_{k=1}^p H_{j_k}) \leq m-1$ ($p \leq m$), there is a subset $\{j_{k_1}, \dots, j_{k_m}\}$ of $\{j_1, \dots, j_p\}$ such that $\sum_{k=1}^p H_{j_k} = \sum_{i=1}^m H_{j_{k_i}}$. The sum of such terms with the same $\{j_{k_1}, \dots, j_{k_m}\}$ is not greater than

$$\begin{aligned} & (e^{c_{j_{k_1}}} - 1) \cdots (e^{c_{j_{k_m}}} - 1) \sum_{p=0}^{q-m} \sum \rho_{j_1}(H_{j_1}) \cdots \rho_{j_p}(H_{j_p}) \\ & \leq b_{j_{k_1}} \cdots b_{j_{k_m}} \prod_{k'} (\delta_0 + \rho_{k'})(H_k) \leq b_{j_{k_1}} \cdots b_{j_{k_m}} e^{D_{m-1}}. \end{aligned}$$

Here, the sum $\sum_{p=0}^{q-m} \sum$ is taken for $j_1, \dots, j_p$ such that $\sum_{i=1}^p H_{j_i} \subset \sum_{i=1}^m H_{j_{k_i}}$ and the product $\prod_{k'}$ is taken for all k' such that $H_{k'} \subset \sum_{i=1}^m H_{j_{k_i}}$. Hence the sum of terms $\rho_{j_1} * \cdots * \rho_{j_p} * \bar{\mu}_q(Z)$ with $\dim(\sum_{i=1}^p H_{j_i}) < m$ is bounded by

$$\sum_{p=1}^m \sum_{1 \leq j_1 < \cdots < j_p \leq q} b_{j_1} \cdots b_{j_p} e^{D_{m-1}}. \quad (6)$$

Next we consider a term with $\dim(\sum_{i=1}^p H_{j_i}) = m$. Let $L[x_1, \dots, x_p]$ be the linear subspace spanned by $x_1, \dots, x_p$. Let

$$K_{p,r} = \left\{ (x_1, \dots, x_p) \in \prod_{i=1}^p H_{j_i} : \dim L[x_1, \dots, x_p] = r \right\}$$

and $K_p(i_1, \dots, i_r) = \{(x_1, \dots, x_p) \in K_{p,r} : x_{i_1}, \dots, x_{i_r} \text{ are linearly independent}\}$. We have

$$\begin{aligned} & \int_{K_p(i_1, \dots, i_r)} \prod_{k=1}^p \rho_k(dx_k) = \int_{K_p(i_1, \dots, i_r)} \prod_{k=1}^r \rho_{j_{i_k}}(dx_{i_k}) \prod_{k \neq i_1, \dots, i_r} \rho_{j_k}(dx_k) \\ &= \int_{K(H_{i_1}, \dots, H_{i_r})} \prod_{k=1}^r \rho_{j_{i_k}}(dx_{i_k}) \prod_{k \neq i_1, \dots, i_r} \int_{L[x_{i_1}, \dots, x_{i_r}]} \rho_{j_k}(dx_k). \end{aligned}$$

Assume that $r < m$. If there is $k \neq i_1, \dots, i_r$ such that $H_k \not\subseteq L[x_{i_1}, \dots, x_{i_r}]$, then $\prod_{k \neq i_1, \dots, i_r} \int_{L[x_{i_1}, \dots, x_{i_r}]} \rho_k(dx_k) = 0$ and hence

$$\int_{K_p(i_1, \dots, i_r)} \prod_{k=1}^p \rho_{j_k}(dx_k) \leq \int_{K(H_{i_1}, \dots, H_{i_r})} \prod_{k=1}^r \rho_{j_{i_k}}(dx_{i_k}) e^{D_{m-1}} \leq \prod_{j=1}^r b_{i_j} e^{D_{m-1}}.$$

Hence

$$\int_{K_{p,r}} \prod_{k=1}^p \rho_{j_k}(dx_k) \leq e^{D_{m-1}} \sum_{1 \leq i_1 < \dots < i_r \leq p} \prod_{j=1}^r b_{i_j}. \quad (7)$$

Let $r = m$. Then, we have

$$\begin{aligned} & \int_{K_p(i_1, \dots, i_m)} 1_Z \left(\sum_{k=1}^m x_{i_k} + x + y \right) \prod_{k=1}^m \rho_{j_{i_k}}(dx_{i_k}) \left(\prod_k \rho_{j_k}(dx_k) \right) \bar{\mu}_q(dy) \\ &= \int_{K_p(i_1, \dots, i_m)} 1_{Z-x-y} \left(\sum_{k=1}^m x_{i_k} \right) \prod_{k=1}^m \rho_{j_{i_k}}(dx_{i_k}) \left(\prod_k \rho_{j_k}(dx_k) \right) \bar{\mu}_q(dy) = 0 \end{aligned} \quad (8)$$

by the assumption with $W = \{0\}$ and $T = I$, where I is an identity operator on $\mathbf{R}^d$ and $x = \sum_{k \in \{1, \dots, p\} \setminus \{i_1, \dots, i_m\}} x_k$. Letting $q \rightarrow \infty$, we have by (6)–(8),

$$\mu_1(Z) \leq 2 \left(\sum_{j=1}^m \left(\sum_{k=1}^{\infty} b_k \right)^j e^{D_{m-1}} \right).$$

As in Step 1 in the proof of Theorem 1.1, we have that $\mu(Z) = 0$ for Borel set Z with $\Lambda_{\mathbf{R}^d}(Z) = 0$.

Step 2. Apply Step 2 in the proof of Theorem 1.1 setting $r = 1$. We have $\mu \ll \Lambda_{\mathbf{R}^d}$. Theorem 1.2 is proved.

4. Absolute continuity under GPDACR. Let S be a Borel measurable subset of $(d-1)$ -dimensional unit sphere center at the origin. We call a map

$$\varphi(u, \xi): (0, \infty) \times S \rightarrow \mathbf{R}^d \setminus \{0\}$$

generalized polar coordinate on $\mathbf{R}^d$ of class C^1 with S if it satisfies the following three conditions:

- 1) φ is measurable with respect to the ξ , class C^1 with respect to the u , one to one, $\varphi(0+, \xi) = 0$, $\varphi(1, \xi) = \xi$, and $\partial \varphi(u, \xi) / \partial u \neq 0$.
- 2) For $\xi \in S$ and a subset A of $\mathbf{R}^d$, set

$$C_\varphi(\xi, A) = \left\{ u \in (0, \infty): \frac{\partial}{\partial u} \varphi(u, \xi) \in A \right\}.$$

For every linear subspace H and any $\xi \in S$, the set $C_\varphi(\xi, H)$ coincides with $(0, \infty)$ or is of Lebesgue measure 0.

3) For any $\xi_1, \dots, \xi_k \in S$ ($2 \leq k \leq d$), let $F_k(\xi_1, \dots, \xi_k)$ be a subset of $(0, \infty)^k$ consisting of $(u_1, \dots, u_k)$ such that $\{\varphi(u_1, \xi_1), \dots, \varphi(u_k, \xi_k)\}$ are linearly independent and $\{\partial\varphi(u_1, \xi_1)/\partial u_1, \dots, \partial\varphi(u_k, \xi_k)/\partial u_k\}$ are linearly dependent. Then the k -dimensional Lebesgue measure of the set $F_k(\xi_1, \dots, \xi_k)$ is zero.

Example 4.1 [2]. Let Q be a $d \times d$ matrix whose all eigenvalues have positive real parts. Then $\varphi(u, \xi) = u^Q \xi := e^{(\ln u)Q} \xi$ is a generalized polar coordinate on $\mathbf{R}^d$ with $S = S_Q = \{\xi \in \mathbf{R}^d: |\xi| = 1, |u^Q \xi| > 1 \text{ for every } u > 1\}$.

We say that a Lévy measure ν has a generalized polar decomposition with absolutely continuous radial part (GPDACR) if there is a generalized polar coordinate (S, φ) of class C^1 such that for $B \in \mathcal{B}(\mathbf{R}^d)$,

$$\nu(B) = \int_S \lambda(d\xi) \int_{(0, \infty)} 1_B(\varphi(u, \xi)) \ell(\xi, u) du,$$

where $\lambda(d\xi)$ is a probability measure on S and $\ell(\xi, u)$ is nonnegative and measurable in each variable satisfying $\int_{(0, \infty)} \tilde{\ell}(\xi, u) du < \infty$ for λ -a.a. ξ , where

$$\tilde{\ell}(\xi, u) = |\varphi(u, \xi)|^2 (1 + |\varphi(u, \xi)|^2)^{-1} \ell(\xi, u). \quad (9)$$

In [10], the following fact is proved.

Assume that $\nu(\mathbf{R}^d) = \infty$ and GPDACR for ν . Then the following holds.

1. If ν is not concentrated on a proper linear subspace of $\mathbf{R}^d$ and either $\nu(H) = 0$ or $\nu(H) = \infty$ for every linear subspace H , then $\mu \ll \Lambda_{\mathbf{R}^d}$.

2. If $\nu(H) = 0$ for every $(d-2)$ -dimensional linear subspace H of $\mathbf{R}^d$ and there is no $(d-1)$ -dimensional linear subspace H such that $\nu(\mathbf{R}^d \setminus H) < \infty$, then $\mu \ll \Lambda_{\mathbf{R}^d}$.

We extend the above results as the following Theorems 4.1 and 4.2.

Theorem 4.1. Assume GPDACR for ν and

$$\sum_{\ell=1}^{d-1} \sum_k (\nu^{(\ell)}(H_k^{(\ell)} \cap K^c))^{d-1} + \nu^{(d)}(\mathbf{R}^d) = \infty \quad (10)$$

for any $(d-1)$ -dimensional linear subspace K of $\mathbf{R}^d$. Then, $\mu \ll \Lambda_{\mathbf{R}^d}$.

Proof. If ν satisfies GPDACR condition, then for $1 \leq \ell \leq d-1$, $(\tilde{\nu}^{(\ell)}|_{H_k^{(\ell)}})^{\ell*} \ll \Lambda_{H_k^{(\ell)}}$ for each k by the proof of Theorem 1 in [9]. Then $(\tilde{\nu}^{(\ell)}|_{H_k^{(\ell)}})^{(d-1)*} \ll \Lambda_{H_k^{(\ell)}}$ also holds for each k . Under the condition (3), the assumption 2 of Theorem 1.1 with $r = d-1$ holds and we get the conclusion. Theorem 4.1 is proved.

Theorem 4.2. Assume GPDACR for ν . Moreover, assume that for any linear subspace H satisfying $0 < \nu(H \setminus \bigcup_k H_k) \leq \infty$, where $\{H_k\}$ is the

totality of proper linear subspaces of H , $T\varphi(\cdot, \cdot)$ with $T = T_{H^\perp}$ satisfies the condition 3 of generalized polar coordinate. Then $\mu \ll \Lambda_{\mathbf{R}^d}$ if and only if

$$\nu(K^c) = \infty$$

for any $(d-1)$ -dimensional linear subspace K of $\mathbf{R}^d$.

P r o o f. We show that the assumption in Theorem 1.2 is satisfied. Let $W = \{0\}$. Then $W^\perp = \mathbf{R}^d$. Let $V_j = H_j^{(\ell)}$ and $V := \sum_{j=1}^r V_{i_j}$ with $\dim V = r$. Let $[\mathbf{h}_1, \dots, \mathbf{h}_r]$ be an orthonormal basis of V . Since $\sum_{i=1}^r \varphi(u_i, \xi_i) \in V$ for $\xi_1, \dots, \xi_r \in K(i_1, \dots, i_r)$, we can write

$$\sum_{i=1}^r \varphi(u_i, \xi_i) = \sum_{j=1}^r c_j(\mathbf{u}) \mathbf{h}_j,$$

where $\mathbf{u} = (u_1, \dots, u_r)$. Define a linear map $h: \mathbf{R}^r \rightarrow V$ by $h(\mathbf{v}) = \sum_{j=1}^r v_j \mathbf{h}_j$ for $\mathbf{v} = (v_1, \dots, v_r)$, which is an isomorphism. Let $\mathbf{c}(\mathbf{u}) = (c_1(\mathbf{u}), \dots, c_r(\mathbf{u}))$. Then $\mathbf{c}$ is continuously differentiable. Let $E = E_r(\xi_1, \dots, \xi_r) = (0, \infty)^r \setminus F_r$, where $F_r = F_r(\xi_1, \dots, \xi_r)$ is the set defined in property 3 of generalized polar coordinate φ . Let $a_{ij}(\mathbf{u}) = \frac{\partial}{\partial u_i} c_j(\mathbf{u})$. Then the matrix $A(\mathbf{u}) = (a_{ij}(\mathbf{u}))$ is invertible if $\mathbf{u} \in E$. Then, we have

$$\begin{aligned} \int_{(0, \infty)^r} 1_Z \left(\sum_{i=1}^r \varphi(u_i, \xi_i) \right) \prod_{i=1}^r du_i &= \int_E 1_Z(h(\mathbf{c}(\mathbf{u}))) d\mathbf{u} \\ &= \int_{\mathbf{c}(E)} 1_{h^{-1}(Z)}(\mathbf{v}) |\det(A^{-1}(\mathbf{v}))| d\mathbf{v} = 0. \end{aligned}$$

Since

$$\begin{aligned} &\int_{K(V_{i_1}, \dots, V_{i_r})} 1_Z \left(\sum_{j=1}^r x_j \right) \prod_{j=1}^r \tilde{\nu}^{(\ell)}(dx_j) \\ &\leq \int_{\prod_{j=1}^r V_{i_j}} \prod_{j=1}^r \lambda^{(\ell)}(d\xi_j) \int_E 1_Z \left(\sum_{j=1}^r \varphi(u_j, \xi_j) \right) \prod_{j=1}^r \tilde{\ell}(u_j, \xi_j) du_j, \end{aligned}$$

we have

$$\int_{K(V_{i_1}, \dots, V_{i_r})} 1_Z \left(\sum_{j=1}^r x_j \right) \prod_{j=1}^r \tilde{\nu}^{(\ell)}(dx_j) = 0.$$

Here, $\tilde{\ell}$ is defined by (9).

Let W be an m -dimensional linear subspace, where $\ell \leq m \leq d$. Let $\{V_k\}$ be a sequence of all j -dimensional linear subspaces of $W^\perp$ such that $0 < T\nu_\ell(V_k) \leq \infty$, where $T\nu_\ell$ is defined in Theorem 1.2. Suppose that $\dim(V := \sum_{i=1}^r V_{k_i}) = r$. Then in the above argument for $W = \{0\}$, $\Lambda_{\mathbf{R}^r}(F_r) = 0$ by the assumption. Hence, the assumption of Theorem 1.2 is satisfied in this case. Theorem 4.2 is proved.

In the above theorems, we do not assume the following divergence condition:

$$\int_{(0,\infty)} \ell(\xi, u) du = \infty \quad \text{for } \lambda\text{-a.a. } \xi, \quad (11)$$

which is crucial in the proof of Theorem 27.10 in Sato's book [4].

5. An application to OU type processes. Let Q be a $d \times d$ -matrix and let $Z(t)$ be a Lévy process on $\mathbf{R}^d$ with (1). Let $X(t)$ be a stochastic process defined by a stochastic differential equation

$$dX(t) = -QX(t) dt + dZ(t), \quad X(0) = 0.$$

Such a process is called a process of Ornstein–Uhlenbeck type in [5]. Let μ_t , $t > 0$, be the distribution of $X(t)$. It is known [5] that μ_t is infinitely divisible and its Lévy measure ν_t is represented as

$$\nu_t(B) = \int_{\mathbf{R}^d} \nu(dy) \int_0^t 1_B(e^{-sQ}y) ds. \quad (12)$$

If all eigenvalues of Q have positive real parts, then $\varphi(u, \xi) = u^Q \xi$ is a generalized polar coordinate of class C^1 with $S = S_Q$ and by the above representation, we have

$$\nu_t(B) = \int_{S_Q} \lambda(d\xi) \int_0^\infty 1_B(u^Q \xi) \frac{k_\xi(u) - k_\xi(e^t u)}{u} du,$$

where λ is a probability measure on S_Q , $k_\xi(u)$ is nonnegative, Borel measurable in ξ and nonincreasing in u and satisfies

$$\int_{(0,\infty)} \frac{|u^Q \xi|^2}{1 + |u^Q \xi|^2} \frac{k_\xi(u) - k_\xi(e^t u)}{u} du < \infty.$$

Note that the divergence condition (11) is not satisfied in general in this case. Hence, the result on absolute continuity in [4] cannot be applied to this case.

Theorem 5.1. *Assume that Q is invertible. Then $\mu_t \ll \Lambda_{\mathbf{R}^d}$ if and only if $\nu_t(K^c) = \infty$ for every $(d-1)$ -dimensional linear subspace K .*

P r o o f. We show that the assumption in Theorem 1.2 is satisfied.

For the proof, we use an argument similar to the proof of Theorem 4.2. However, it is not clear whether GPDACR is satisfied in this situation. Therefore, we use the representation (12) of ν_t instead of the use of generalized polar coordinate decomposition.

1. Let $W = \{0\}$. That is, $W^\perp = \mathbf{R}^d$ and $T = I$. Let $V_i = H_i^{(\ell)}$ and $V := \sum_{j=1}^r V_{i_j}$ with $\dim V = r \leq \ell$. For $H = H_i^{(\ell)}$, the totality H_0 of $y \in H$ such that $e^{tQ}y \in H$ for all $t > 0$ is a linear subspace of H . If H is not

Q -invariant, then H_0 is a proper subspace of H . Then $\nu_t^{(\ell)}(H_0) = 0$. If $y \in H \setminus H_0$, then $e^{tQ}y \notin H$ for a.a. $t > 0$ and hence $\nu_t^{(\ell)}(H \setminus H_0) = 0$ by (12). Hence H is Q -invariant and thus V is Q -invariant. Let $[\mathbf{h}_1, \dots, \mathbf{h}_r]$ be an orthonormal basis of V . Since $\sum_{i=1}^r e^{-u_i Q} y_i \in V$ for $y_1, \dots, y_r \in V$, we can represent as $\sum_{i=1}^r e^{-u_i Q} y_i = \sum_{j=1}^r c_j(\mathbf{u}) \mathbf{h}_j$, where $\mathbf{u} = (u_1, \dots, u_r)$. Define a linear map $h: \mathbf{R}^r \rightarrow V$ by $h(\mathbf{v}) = \sum_{j=1}^r v_j \mathbf{h}_j$ for $\mathbf{v} = (v_1, \dots, v_r)$, which is an isomorphism. Let $\mathbf{c}(\mathbf{u}) = (c_1(\mathbf{u}), \dots, c_r(\mathbf{u}))$. Then $\sum_{i=1}^r e^{-u_i Q} y_i = h(\mathbf{c}(\mathbf{u}))$ and $\mathbf{c}$ is continuously differentiable. Let $a_{ij}(\mathbf{u}) = \frac{\partial}{\partial u_i} c_j(\mathbf{u})$. If $e^{-u_1 Q} y_1, \dots, e^{-u_r Q} y_r$ are linearly independent, then since Q is invertible, $Qe^{-u_1 Q} y_1, \dots, Qe^{-u_r Q} y_r$ are linearly independent and the matrix $A(\mathbf{u}) = (a_{ij}(\mathbf{u}))$ is invertible. Let $F(y_1, \dots, y_r)$ be the set of $(u_1, \dots, u_r) \in [0, t]^r$ such that $e^{-u_1 Q} y_1, \dots, e^{-u_r Q} y_r$ are linearly independent. Then, $F(y_1, \dots, y_r)$ is empty or an open set and we have

$$\begin{aligned} & \int_{F(y_1, \dots, y_r)} 1_Z \left(\sum_{i=1}^r e^{-u_i Q} y_i \right) \prod_{i=1}^r du_i \\ &= \int_{\mathbf{c}(F(y_1, \dots, y_r))} 1_{h^{-1}(Z)}(\mathbf{v}) |\det(A^{-1}(\mathbf{v}))| d\mathbf{v} = 0 \end{aligned}$$

for Borel set Z in V with $\Lambda_V(Z) = 0$. Here, $\mathbf{v} = \mathbf{c}(\mathbf{u})$. Since

$$\begin{aligned} & \int_{K(V_{i_1}, \dots, V_{i_r})} 1_Z \left(\sum_{j=1}^r x_j \right) \prod_{j=1}^r \tilde{\nu}_t^{(\ell)}(dx_j) \\ & \leq \int_{\prod_{j=1}^r V_{i_j}} \prod_{j=1}^r \tilde{\nu}_t^{(\ell)}(dy_j) \int_{F(y_1, \dots, y_r)} 1_Z \left(\sum_{j=1}^r e^{-u_j Q} y_j \right) \prod_{j=1}^r du_j, \end{aligned}$$

we have

$$\int_{K(V_{i_1}, \dots, V_{i_r})} 1_Z \left(\sum_{j=1}^r x_j \right) \prod_{j=1}^r \tilde{\nu}_t^{(\ell)}(dx_j) = 0$$

for Borel set Z in V with $\Lambda_V(Z) = 0$.

2. Let W be an m -dimensional linear subspace of $\mathbf{R}^d$ such that $\nu_t^{(\ell)}(W \cap (\bigcup_k H_k^{(\ell)})) > 0$ and let $T = T_{W^\perp}$. Then, since W is Q invariant and $TQ = TQT$, TX_t satisfies $d(TX_t) = TQ(TX_t) dt + d(TZ_t)$ and TX_t is $W^\perp$ -valued. The Lévy measure of the distribution of TX_t is $T\nu_t$ and $T\nu_t$ has the same property as property 1. Hence, by Theorem 1.2, we have that $\mu \ll \Lambda_{\mathbf{R}^d}$. Theorem 5.1 is proved.

Note that the condition $\nu_t(K^c) = \infty$ for all $t > 0$ is equivalent to the condition $\nu_t(K^c)$ for some $t > 0$.

Bodnarchuk and Kulik [1] show that if $\nu(K^c) = \infty$ for every $(d-1)$ -dimensional linear subspace K (they call it Yamazato condition), then $\mu_t \ll \Lambda_{\mathbf{R}^d}$ for all $t > 0$ (Theorem 3 in [1]). Our assumption is weaker and the proof is elementary. The following example shows this situation and therefore their condition is not a necessary condition for the absolute continuity.

Example 5.1. Let $d = 2$ and let

$$Q = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha > \beta > 0.$$

Assume that ν is supported on the line $\{(x, x): x \in \mathbf{R}\}$ and $\nu(\mathbf{R}^2) = \infty$. Then ν does not satisfy Yamazato condition but ν_t does for $t > 0$.

Simon [6] obtains a result quite similar to Theorem 5.1. However, the necessary and sufficient condition for the absolute continuity is stated in a different way.

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