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META-TIMES AND EXTENDED SUBORDINATION

Обсуждается задача определения субординатора однородного базиса Леви с помощью неотрицательного однородного базиса Леви. Приводится явная конструкция, обобщающая обычный одномерный случай. Эта конструкция задействует некоторые случайные замены метавремени.

Ключевые слова и фразы: базисы Леви, листы Леви, замена метавремени.

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1. Introduction. In recent years the fundamental concept of subordination of a Lévy process by a subordinator has been generalized in many directions; see, e.g., [3], [4], [8]. Related to this, Barndorff-Nielsen [2] considered the following problem: Given an $\mathbf{R}^d$ -valued homogeneous Lévy basis $\Lambda_X = \{\Lambda_X(A): A \in \mathcal{B}_b(\mathbf{R}^k)\}$, and an independent $\mathbf{R}_+$ -valued homogeneous Lévy basis $\Lambda_T = \{\Lambda_T(A): A \in \mathcal{B}_b(\mathbf{R}^k)\}$ how should one define subordination of Λ_X by Λ_T ?

Let us first consider the case $k = 1$. There are Lévy processes $X = \{X_t: t \geq 0\}$ and $T = \{T_t: t \geq 0\}$ associated with Λ_X and Λ_T in the sense that

$$X_t = \Lambda_X((0, t]) \quad \text{and} \quad T_t = \Lambda_T((0, t]) \quad (1.1)$$

for $t \geq 0$. Thus, we can simply define a subordinated process $Y = \{Y_t: t \geq 0\}$ in the usual way as $Y_t = X_{T_t}$. However, when $k \geq 2$ there is no immediate analogue. To see this, note that there are so-called Lévy sheets $X = \{X_t: t \in \mathbf{R}_+^k\}$ and $T = \{T_t: t \in \mathbf{R}_+^k\}$ associated with Λ_X and Λ_T , and these are defined as in (1.1), where $(0, t]$ now is an interval in $\mathbf{R}^k$. But T_t is one-dimensional while t is k -dimensional, thus excluding the possibility of defining Y_t as X_{T_t} when $k \geq 2$. Barndorff-Nielsen argued that one should not construct a subordinated process; rather, the appropriate concept is a subordinated random measure $M = \{M(A): A \in \mathcal{B}_b(\mathbf{R}^k)\}$ defined in such a way that conditional on Λ_T , $M(A_1), \dots, M(A_n)$ are independent for all disjoint $A_1, \dots, A_n$, and the distribution of $M(A)$ for $A \in \mathcal{B}_b(\mathbf{R}_+^k)$ is $\mu^{\Lambda_T(A)}$, where $\mu = \mathcal{L}(\Lambda_X((0, e]))$ and $e = (1, \dots, 1) \in \mathbf{R}_+^k$ is the vector of ones.

In the present paper we give an explicit construction in terms of Λ_X and Λ_T of Barndorff-Nielsen's subordinated measure M . For notational convenience, instead of considering M and Λ_X as Lévy bases on $\mathbf{R}^k$ we look at the restriction to $\mathbf{R}_+^k$; the general case follows trivially from this. Specifically, we argue that a natural definition of $M = \{M(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ is $M(A) = \Lambda_X(\phi_T^{-1}(A))$, where $\phi_T: \mathbf{R}_+^k \rightarrow \mathbf{R}^k$ is a (random) mapping satisfying that $\text{Leb}(\phi_T^{-1}(A)) = \Lambda_T(A)$ for $A \in \mathcal{B}_b(\mathbf{R}_+^k)$. We also use the notation $\mathbf{T} = \phi_T^{-1}$ for the inverse image of ϕ_T . In a sense one can think of $\mathbf{T}$ as a kind of *meta-time change*, replacing *time changes* in the case $k = 1$. We show that this definition generalizes the case $k = 1$ in a natural way and, in particular, that M is a homogeneous Lévy basis.

This construction gives emphasis to the viewpoint that in the multiparameter case $k \geq 2$ the right concept is a subordinated measure instead of subordinated process.

In Section 2 we recall the definitions of homogeneous Lévy sheets and bases and show that these are in one-to-one correspondence. To pave the way for the analysis of M we state in Section 3 a lemma showing that it is possible to define a mapping ϕ_T with the above properties. Section 4 is about meta-time changes, meaning that we consider the measure M for a fixed outcome of T . Although meta-times exist for any T they are in full generality somewhat involved to define explicitly. However, in practice the most important case is when the measure induced by T is the sum of a constant times Lebesgue measure and a discrete measure. In this case alternative useful representations of M are given. In the last section it is shown that M is a homogeneous Lévy basis.

2. Homogeneous Lévy sheets and Lévy bases. Let d and k be positive integers. For $x = (x^1, \dots, x^d)$ and $y = (y^1, \dots, y^d)$ in $\mathbf{R}^d$ let $\langle x, y \rangle$ denote their inner product and $|x|$ the corresponding norm. Let $D = \{x \in \mathbf{R}^d: |x| \leq 1\}$. Throughout the paper all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{L}(X)$ denote the law of a random vector X . For a set S and two families $\{X_t: t \in S\}$ and $\{Y_t: t \in S\}$ of random vectors with X_t and Y_t in $\mathbf{R}^d$ write $\{X_t: t \in S\} \stackrel{\mathcal{D}}{=} \{Y_t: t \in S\}$ if the finite dimensional marginals are the same. We say that $\{X_t: t \in S\}$ is a modification of $\{Y_t: t \in S\}$ if $X_t = Y_t$ a.s. for all $t \in S$. Let $\hat{\mu}$ denote the characteristic function of a distribution μ on $\mathbf{R}^d$, $\hat{\mu}(z) = \int_{\mathbf{R}^d} e^{i\langle z, x \rangle} \mu(dx)$ for $z \in \mathbf{R}^d$.

Let $\text{ID}(\mathbf{R}^d)$ denote the class of d -dimensional infinitely divisible distributions. Recall that a distribution μ on $\mathbf{R}^d$ is in $\text{ID}(\mathbf{R}^d)$ if and only if $\hat{\mu}$ is given by $\hat{\mu}(z) = \exp[-z\Sigma z^\top/2 +$

$i\langle\gamma, z\rangle + \int_{\mathbf{R}^d} g(z, x) \nu(dx)$, $z \in \mathbf{R}^d$, where $g(z, x) = e^{i\langle z, x\rangle} - 1 - i\langle z, x\rangle 1_D(x)$, $^\top$ denotes the transpose, and (Σ, ν, γ) is the characteristic triplet of μ ; that is, Σ is a $d \times d$ nonnegative definite matrix, ν is a Lévy measure on $\mathbf{R}^d$ and $\gamma \in \mathbf{R}^d$. Denote the entries of Σ by Σ^{ij} and the coordinates of γ by γ^j for $i, j = 1, \dots, d$. For $t \geq 0$ and $\mu \in \text{ID}(\mathbf{R}^d)$, μ^t denotes the distribution in $\text{ID}(\mathbf{R}^d)$ with $\hat{\mu}^t = \hat{\mu}^t$.

For $a = (a^1, \dots, a^k) \in \mathbf{R}_+^k$ and $b = (b^1, \dots, b^k) \in \mathbf{R}_+^k$ write $a \leq b$ if $a^j \leq b^j$ for all j and $a < b$ if $a^j < b^j$ for all j , and define the half-open interval $]a, b]$ as $]a, b] = \{t \in \mathbf{R}_+^k : a < t \leq b\}$. Let $[a, b] = \{t \in \mathbf{R}_+^k : a \leq t \leq b\}$.

For $F = \{F_t : t \in \mathbf{R}_+^k\}$ with $F_t \in \mathbf{R}^d$ and $a \leq b$ define the increment of F over $]a, b]$, $\Delta_a^b F$, as

$$\Delta_a^b F = \sum_{\epsilon_1=0}^1 \dots \sum_{\epsilon_k=0}^1 (-1)^{k-(\epsilon_1+\dots+\epsilon_k)} F_{(c^1(\epsilon_1), \dots, c^k(\epsilon_k))},$$

where $c^j(1) = b^j$ and $c^j(0) = a^j$. For example, if $k = 1$, then $\Delta_a^b F = F_b - F_a$ and when $k = 2$ we have $\Delta_a^b F = F_{(b^1, b^2)} + F_{(a^1, a^2)} - F_{(a^1, b^2)} - F_{(b^1, a^2)}$. Let $\mathcal{A} = \{t \in \mathbf{R}_+^k : t^j = 0 \text{ for some } j\}$. For $\mathcal{R} = (R_1, \dots, R_k)$, where R_j is either $\leq$ or $>$, write $a\mathcal{R}b$ if $a^j R_j b^j$ for all j .

We say that $F = \{F_t : t \in \mathbf{R}_+^k\}$ is *lamp* if the following three conditions are satisfied: (i) for $t \in \mathbf{R}_+^k$ the limit $F(t, \mathcal{R}) = \lim_{u \rightarrow t, t\mathcal{R}u} F_u$ exists for each of the 2^k relations $\mathcal{R} = (R_1, \dots, R_k)$, where R_j is either $\leq$ or $>$ (here we let $F(t, \mathcal{R}) = F_t$ if there is no u with $t\mathcal{R}u$); (ii) $F_t = F(t, \mathcal{R})$ for $\mathcal{R} = (\leq, \dots, \leq)$; (iii) $F_t = 0$ for $t \in \mathcal{A}$. Here *lamp* stands for *limits along monotone paths*. This is the multiparameter analogue of being càdlàg. See [1] for references to the literature on lamp trajectories. When F is lamp and $t \in \mathbf{R}_+^k \setminus \mathcal{A}$ define $\Delta_t F = \lim_{n \rightarrow \infty} \Delta_{t_n}^t F$, where t_n is any sequence with $t_n \rightarrow t$ and $t_n < t$. If F is continuous at the point t , then $\Delta_t F = 0$ but the converse is not true, that is, we can have $\Delta_t F = 0$ without F being continuous at t .

Definition 2.1. Let $X = \{X_t : t \in \mathbf{R}_+^k\}$ be a family of random vectors in $\mathbf{R}^d$. We say that X has *independent increments* if $X_t = 0$ for all $t \in \mathcal{A}$ a.s. and $\Delta_{a_1}^{b_1} X, \dots, \Delta_{a_n}^{b_n} X$ are independent whenever $n \geq 2$ and $]a_1, b_1], \dots,]a_n, b_n]$ are disjoint; if in addition X is continuous in probability and $\Delta_{t+a}^{t+b} X \stackrel{\mathcal{D}}{=} \Delta_a^b X$ for all $a, b, t \in \mathbf{R}_+^k$ with $a \leq b$, then X is called an *$\mathbf{R}^d$ -valued homogeneous Lévy sheet in law on $\mathbf{R}_+^k$* , and if also almost all sample paths are lamp, then X is called an *$\mathbf{R}^d$ -valued homogeneous Lévy sheet on $\mathbf{R}_+^k$* .

A homogeneous Lévy sheet is a special case of the additive processes considered in [1, p. 5], and of the Lévy sheets considered in [5] (in the case $k = 2$). In fact, a process satisfying all the above conditions except the homogeneity condition $\Delta_{t+a}^{t+b} X \stackrel{\mathcal{D}}{=} \Delta_a^b X$ would be called a Lévy sheet by Dalang and Walsh [5]. It follows, e.g., from [1, Proposition 4.1] that any homogeneous Lévy sheet in law has a modification which is a homogeneous Lévy sheet. It is easily seen that if $X = \{X_t : t \in \mathbf{R}_+^k\}$ is a homogeneous Lévy sheet in law, then $X_t = \Delta_0^t X$ a.s. for all $t \in \mathbf{R}_+^k$; moreover $\mathcal{L}(\Delta_a^b X) \in \text{ID}(\mathbf{R}^d)$ for all $a, b \in \mathbf{R}_+^k$ with $a \leq b$ and there is a $\mu \in \text{ID}(\mathbf{R}^d)$ such that $\mathcal{L}(\Delta_a^b X) = \mu^{\text{Leb}([a, b])}$ for all such a and b , where Leb denotes Lebesgue measure on $\mathbf{R}^k$. We say that X is *associated with μ* or with the characteristic triplet of μ .

Definition 2.2. Let $\Lambda = \{\Lambda(A) : A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$, where $\mathcal{B}_b(\mathbf{R}_+^k)$ is the set of bounded Borel sets in $\mathbf{R}_+^k$, denote a family of random vectors in $\mathbf{R}^d$. We call Λ an *$\mathbf{R}^d$ -valued homogeneous Lévy basis on $\mathbf{R}_+^k$* if the following conditions are satisfied: (i) $\Lambda(A_1), \dots, \Lambda(A_n)$ are independent whenever $A_1, \dots, A_n \in \mathcal{B}_b(\mathbf{R}_+^k)$ are disjoint; (ii) $\Lambda(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Lambda(A_n)$ a.s. whenever $A_1, A_2, \dots \in \mathcal{B}_b(\mathbf{R}_+^k)$ are disjoint with $\bigcup_{n=1}^\infty A_n \in \mathcal{B}_b(\mathbf{R}_+^k)$ (here the series converges almost surely); (iii) for all $t \in \mathbf{R}_+^k$ and $A \in \mathcal{B}(\mathbf{R}_+^k)$ we have $\Lambda(A) \stackrel{\mathcal{D}}{=} \Lambda(t + A)$.

If Λ is a homogeneous Lévy basis, then $\mathcal{L}(\Lambda(A)) \in \text{ID}(\mathbf{R}^d)$ for all $A \in \mathcal{B}_b(\mathbf{R}_+^k)$. Moreover, there is a $\mu \in \text{ID}(\mathbf{R}^d)$ such that $\mathcal{L}(\Lambda(A)) = \mu^{\text{Leb}(A)}$ for all $A \in \mathcal{B}_b(\mathbf{R}_+^k)$.

We say that the *homogeneous Lévy basis* is associated with μ or its characteristic triplet. Finally, recall that [9] calls $\Lambda = \{\Lambda(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ an independently scattered ID random measure if it satisfies (i) and (ii) of Definition 2.2 and $\mathcal{L}(\Lambda(A)) \in \text{ID}(\mathbf{R}^d)$ for all $A \in \mathcal{B}_b(\mathbf{R}_+^k)$. For simplicity we refer to this as a Lévy basis.

The following shows that, not surprisingly, there is a one-to-one correspondence between homogeneous Lévy sheets (in law) and homogeneous Lévy bases.

Theorem 2.1. *Let $X = \{X_t: t \in \mathbf{R}_+^k\}$ be a homogeneous Lévy sheet associated with $\mu \in \text{ID}(\mathbf{R}^d)$ with characteristic triplet (Σ, ν, γ) . Let*

$$J(C) = \#\{(t, \Delta_t X): t \in \mathbf{R}_+^k \setminus \mathcal{A}, (t, \Delta_t X) \in C \text{ and } \Delta_t X \neq 0\} \quad \text{for } C \in \mathcal{B}(\mathbf{R}_+^k \times \mathbf{R}^d).$$

Then we have the following.

1) $J = \{J(C): C \in \mathcal{B}(\mathbf{R}_+^k \times \mathbf{R}^d)\}$ is a Poisson random measure with intensity measure $\text{Leb} \times \nu$.

2) Let $\nu^1(B) = \nu(B \cap D)$ and $\nu^2(B) = \nu(B \cap D^c)$ for $B \in \mathcal{B}(\mathbf{R}^d)$. Define

$$X_t^1 = \int_{[0,t] \times \mathbf{R}^d} y 1_D(y) (J - \text{Leb} \times \nu)(d(s, y)), \quad X_t^2 = \int_{[0,t] \times \mathbf{R}^d} y 1_{D^c}(y) J(d(s, y)).$$

We then have that $X_t = X_t^1 + X_t^2 + X_t^g + t\gamma$, where $\{X_t^g: t \in \mathbf{R}_+^k\}$, $\{X_t^1: t \in \mathbf{R}_+^k\}$ and $\{X_t^2: t \in \mathbf{R}_+^k\}$ are independent, $\{X_t^g: t \in \mathbf{R}_+^k\}$ is a homogeneous Lévy sheet associated with $(\Sigma, 0, 0)$ and $\{X_t^i: t \in \mathbf{R}_+^k\}$ is a homogeneous Lévy sheet associated with $(0, \nu^i, 0)$ for $i = 1, 2$.

3) There exists one and up to modification only one homogeneous Lévy basis $\Lambda = \{\Lambda(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ satisfying $\Lambda([0, t]) = X_t$ a.s. for $t \in \mathbf{R}_+^k$. In addition Λ is given by

$$\begin{aligned} \Lambda(A) &= \int_{A \times \mathbf{R}^d} y 1_D(y) (J - \text{Leb} \times \nu)(d(t, y)) + \int_{A \times \mathbf{R}^d} y 1_{D^c}(y) J(d(t, y)) \\ &\quad + \int_A dX_t^g + \gamma \text{Leb}(A) \quad \text{a.s.} \quad \text{for } A \in \mathcal{B}_b(\mathbf{R}_+^k). \end{aligned} \quad (2.1)$$

Theorem 2.1, 1)–2) are essentially contained in [1, Theorem 4.6]. The only difference is that J above is a Poisson random measure on $\mathbf{R}_+^k \times \mathbf{R}^d$ while Theorem 4.6 of [1] is formulated in terms of Poisson random measures on $\mathbf{R}^d$. The proofs are essentially the same and hence we omit the proof of Theorem 2.1, 1)–2). See also [5] in the case $k = 2$.

For $A \in \mathcal{B}_b(\mathbf{R}_+^k)$ define $\int_A dX_t^g = \int 1_A(t) dX_t^g$, where we recall that $\int f(t) dX_t^g$ (a random vector in $\mathbf{R}^d$) is definable by approximation by step functions in the usual way for all measurable $f: \mathbf{R}_+^k \rightarrow \mathbf{R}$ satisfying $\int (f(t))^2 dt < \infty$. Moreover, we have $\mathcal{L}(\int f(t) dX_t^g) = N_d(0, \Sigma(f))$, where $\Sigma^{ij}(f) = \Sigma^{ij} \int (f(t))^2 dt$. The result in Theorem 2.1, 3) is immediate from fundamental properties of integrals with respect to (compensated) Poisson random measures, cf., e.g., [6]. In the case $k = 2$, Theorem 2.1, 3) can also be found in [5, Theorem 2.6].

We call the process $X^g = \{X_t^g: t \in \mathbf{R}_+^k\}$ above the *Gaussian part* of X and the measure J the *jump measure* of X . We also denote it by J_X . Finally, we call Λ above the *homogeneous Lévy basis induced by X* , also to be denoted by Λ_X .

Proposition 2.1. *Let $\Lambda = \{\Lambda(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ be a homogeneous Lévy basis. Let $\tilde{X}_t = \Lambda([0, t])$. We then have the following results: For $a \leq b$, $\Delta_a^b \tilde{X} = \Lambda([a, b])$ a.s. In particular $\tilde{X} = \{\tilde{X}_t: t \in \mathbf{R}_+^k\}$ is a homogeneous Lévy sheet in law. Let $X = \{X_t: t \in \mathbf{R}_+^k\}$ be a homogeneous Lévy sheet which is a modification of $\tilde{X}$. Then for $A \in \mathcal{B}_b(\mathbf{R}_+^k)$ we have $\Lambda(A) = \Lambda_X(A)$ a.s., where Λ_X is the Lévy basis generated by X .*

P r o o f. It is easily seen that $\Delta_a^b \tilde{X} = \Lambda([a, b])$ a.s. The uniqueness part of Theorem 2.1, 3) implies $\Lambda = \Lambda_X$.

R e m a r k 2.1. 1) Theorem 2.1, 3) shows that a homogeneous Lévy sheet generates a homogeneous Lévy basis by (2.1), and Proposition 2.1 shows conversely that any

homogeneous Lévy basis is generated in this way. We call (2.1) *the Lévy-Itô decomposition of Λ* and call J in that equation the jump measure of Λ . We refer to [5] (for the case $k = 2$) and [7] for the Lévy-Itô decomposition of nonhomogeneous Lévy bases.

2) Let Λ be an $\mathbf{R}^d$ -valued homogeneous Lévy basis on $\mathbf{R}_+^k$ associated with the characteristic triplet (Σ, ν, γ) . Assume that $\int_{\mathbf{R}^d} (1 \wedge |x|) \nu(dx) < \infty$. Then for $A \in \mathcal{B}_b(\mathbf{R}_+^k)$ the representation (2.1) simplifies to

$$\Lambda(A) = \int_{A \times \mathbf{R}^d} y J(d(t, y)) + \int_A dX_t^g + \gamma_0 \text{Leb}(A) \quad \text{a.s.}, \quad (2.2)$$

where the first integral is defined pointwise almost surely and $\gamma_0 = \gamma - \int_A y 1_D(y) \nu(dy)$. Here *pointwise almost surely* signifies that, for almost all ω , $\int_{A \times \mathbf{R}^d} y J(d(t, y))(\omega)$ is a usual Lebesgue integral. Thus if in addition $\nu(\mathbf{R}^d \setminus \mathbf{R}_+^d) = 0$, $\Sigma = 0$ and $\gamma_0 \in \mathbf{R}_+^d$, then we can extend Λ in such a way that $\Lambda(A)$ is defined for all A in $\mathcal{B}(\mathbf{R}_+^k)$ rather than $\mathcal{B}_b(\mathbf{R}_+^k)$; however, some of the coordinates of $\Lambda(A)$ may be equal to ∞ . In addition, almost surely all coordinates of Λ are nonnegative measures.

3. Meta-times. The purpose of this section is to state a result showing that any measure m on $\mathbf{R}_+^k$ which is finite on compacts is the image measure of Leb under some mapping ϕ . This result is essentially well known, at least when m is finite, so in the next lemma we simply state a version of it which suits our purposes well.

Lemma 3.1. *Let $m = \{m(A) : A \in \mathcal{B}(\mathbf{R}_+^k)\}$ be a nonnegative measure on $\mathbf{R}_+^k$ satisfying $m(\mathcal{A}) = 0$ and $m(A) < \infty$ for all $A \in \mathcal{B}_b(\mathbf{R}_+^k)$. Then there exists a measurable mapping $\phi : \mathbf{R}_+^k \rightarrow \mathbf{R}^k$ such that*

$$m(A) = \text{Leb}(\phi^{-1}(A)) \quad \text{for all } A \in \mathcal{B}(\mathbf{R}_+^k) \quad (3.1)$$

and $\phi^{-1}(A)$ is a bounded set for all $A \in \mathcal{B}_b(\mathbf{R}_+^k)$.

R e m a r k 3.1. We refer to the inverse image ϕ^{-1} as a *meta-time* associated with m and we often denote it by $\mathbf{T}$. By the above lemma and properties of inverse images we can regard $\mathbf{T}$ as a mapping $\mathbf{T} : \mathcal{B}_b(\mathbf{R}_+^k) \rightarrow \mathcal{B}_b(\mathbf{R}^k)$ satisfying: (i) $\mathbf{T}(A)$ and $\mathbf{T}(B)$ are disjoint whenever $A, B \in \mathcal{B}_b(\mathbf{R}_+^k)$ are disjoint; (ii) $\mathbf{T}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \mathbf{T}(A_n)$ whenever $A_1, A_2, \dots$ are in $\mathcal{B}_b(\mathbf{R}_+^k)$ and $\bigcup_{n=1}^{\infty} A_n$ is in $\mathcal{B}_b(\mathbf{R}_+^k)$; (iii) $m(A) = \text{Leb}(\mathbf{T}(A))$ for all $A \in \mathcal{B}_b(\mathbf{R}_+^k)$.

P r o o f. Let $u \in \mathbf{R}^k \setminus \mathbf{R}_+^k$ be arbitrary.

1) First assume that $m(\mathbf{R}_+^k) < \infty$. Take an interval $[a, b]$ in $\mathbf{R}_+^k$ with $\text{Leb}([a, b]) = m(\mathbf{R}_+^k)$. Then m is the image measure of Lebesgue measure on $[a, b]$ under some mapping $\psi : [a, b] \rightarrow \mathbf{R}_+^k$. That is, $m(A) = \text{Leb}(\psi^{-1}(A))$ for all $A \in \mathcal{B}(\mathbf{R}_+^k)$. Indeed, this is essentially the well-known result (cf., e.g., [10]) that any distribution on $\mathbf{R}_+^k$ can be generated from k independent and uniformly distributed random variables. Letting

$$\phi(t) = \begin{cases} \psi(t), & t \in [a, b], \\ u, & t \in \mathbf{R}_+^k \setminus [a, b], \end{cases}$$

one sees that ϕ has the required properties.

2) If $m(\mathbf{R}_+^k) = \infty$ instead, then we can take a sequence A_n , $n = 1, 2, \dots$, of disjoint bounded Borel sets in $\mathbf{R}_+^k$ covering $\mathbf{R}_+^k$ and satisfying that for all $t \in \mathbf{R}_+^k$ the interval $[0, t]$ is contained in the finite union of some of the A_n 's. Define, for all $n \geq 1$, $m_n = m(\cdot \cap A_n)$. Since the m_n 's are finite measures, there exist a sequence of disjoint intervals $[a_1, b_1], [a_2, b_2], \dots$ in $\mathbf{R}_+^k$ and measurable mappings $\psi_n : [a_n, b_n] \rightarrow A_n$ such that $m_n(A) = \text{Leb}(\psi_n^{-1}(A))$ for all $A \in \mathcal{B}(A_n)$. Since $m = \sum_{n \geq 1} m_n$, we can define

$$\phi(t) = \begin{cases} \psi_n(t), & t \in [a_n, b_n] \text{ for some } n, \\ u, & t \in \mathbf{R}_+^k \setminus (\bigcup_{n=1}^{\infty} [a_n, b_n]). \end{cases}$$

Clearly, since for any $t \in \mathbf{R}_+^k$ the interval $[0, t]$ is contained in the union of a finite number of A_n 's it follows that $\phi^{-1}([0, t])$ is contained in the union of a finite number of intervals $[a_n, b_n]$.

E x a m p l e 3.1. Let m be as in the lemma above. In many cases of interest, the mapping ϕ in the lemma has a very simple expression, as the following shows.

1) Assume m is concentrated on a set $\mathcal{T} = \{t_n\}_{n=1}^\infty \subseteq \mathbf{R}_+^k \setminus \mathcal{A}$. Take a disjoint sequence $R_1, R_2, \dots$ of bounded Borel sets in $\mathbf{R}_+^k$ such that $\text{Leb}(R_n) = m(\{t_n\})$ for all n . Define $\phi(t) = t_n$ when $t \in R_n$ for some n and let $\phi(t) = u$ for $t \in \mathbf{R}_+^k \setminus (\bigcup_{n=1}^\infty R_n)$, where $u \in \mathbf{R}^k \setminus \mathbf{R}_+^k$ is arbitrary. The sets R_n can be chosen arbitrarily, showing in particular that ϕ is not at all uniquely determined.

2) If $m = \text{Leb}/c$ for some $c > 0$, then we can use $\phi(t) = ct$.

3) The case when $m = m_1 + m_2$, where $m_1 = \text{Leb}/c$ and m_2 is concentrated on $\{t_n\}_{n=1}^\infty \subseteq \mathbf{R}_+^k \setminus \mathcal{A}$, can be handled as follows. Let the sets R_n above be subsets of

$$\{s = (s^1, \dots, s^k) \in \mathbf{R}_+^k : 0 \leq s^j \leq 1 \text{ for all } j = 1, \dots, k\}.$$

Let $e = (1, \dots, 1) \in \mathbf{R}_+^k$ be the vector of ones. By defining ϕ as

$$\phi(t) = \begin{cases} t_n & \text{if } t \in R_n \text{ for some } n, \\ u & \text{if } t \in \{s = (s^1, \dots, s^k) \in \mathbf{R}_+^k : s^j \in [0, 1] \} \setminus (\bigcup_{n=1}^\infty R_n), \\ c(t - e) & \text{if } t \in \{s = (s^1, \dots, s^k) \in \mathbf{R}_+^k : s^j > 1\}, \end{cases}$$

equation (3.1) is easily verified.

4) Assume $k = 1$ and let $T_t = m([0, t])$ for all $t \geq 0$. Define $\phi: \mathbf{R}_+ \rightarrow \mathbf{R}$ as

$$\phi(y) = \inf\{t \geq 0 : T(t) \geq y\},$$

where $\inf \emptyset = u \in \mathbf{R} \setminus \mathbf{R}_+$. Then

$$[0, T_t] = \mathbf{T}([0, t]) \quad \text{for all } t \geq 0 \quad (3.2)$$

and hence $m(A) = \text{Leb}(\mathbf{T}(A))$ for all $A \in \mathcal{B}(\mathbf{R}_+)$.

4. Meta-time changes. In the one-dimensional case $k = 1$, one uses increasing functions to model a *time change* as in (4.3) below. The purpose of the present section is to show that in the case $k \geq 2$ certain *meta-time changes* give similar results. In fact, we show that the appropriate generalization of the process Y in (4.3) is the random measure M in (4.1), where in the latter equation $\mathbf{T}$ is a meta-time as defined in Section 3.

Let $X = \{X_t : t \in \mathbf{R}_+^k\}$ be an $\mathbf{R}^d$ -valued homogeneous Lévy sheet on $\mathbf{R}_+^k$ associated with $\mu \in \text{ID}(\mathbf{R}^d)$. Denote the corresponding homogeneous Lévy basis by Λ_X . Let $m = \{m(A) : A \in \mathcal{B}(\mathbf{R}_+^k)\}$ be a nonnegative measure on $\mathbf{R}_+^k$ satisfying $m(\mathcal{A}) = 0$ and $m(A) < \infty$ for all $A \in \mathcal{B}_b(\mathbf{R}_+^k)$. Set $T_t = m([0, t])$ for all $t \in \mathbf{R}_+^k$ and let $\phi: \mathbf{R}_+^k \rightarrow \mathbf{R}^k$ be given as in Lemma 3.1. Let $\mathbf{T} = \phi^{-1}$ be the corresponding meta-time associated with m .

Define $M = \{M(A) : A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ as

$$M(A) = \Lambda_X(\mathbf{T}(A)) \quad \text{for } A \in \mathcal{B}_b(\mathbf{R}_+^k). \quad (4.1)$$

Using the properties of $\mathbf{T}$ in Remark 3.1 one sees that M is a (nonhomogeneous) Lévy basis. Since in addition Λ_X is a homogeneous Lévy basis associated with μ , it follows that

$$\mathcal{L}(M(A)) = \mu^{m(A)} \quad \text{for } A \in \mathcal{B}_b(\mathbf{R}_+^k). \quad (4.2)$$

In particular, for $t \in \mathbf{R}_+^k$, $M((0, t])$ has characteristic triplet $T_t(\Sigma, \nu, \gamma)$, where (Σ, ν, γ) is the characteristic triplet of μ . We say that M is defined from Λ_X by meta-time change with $\mathbf{T}$.

R e m a r k 4.1. Let $k = 1$ and let us show that in this case the above definition generalizes the usual concept of a time change in a natural way. For this purpose, define a process $Y = \{Y_t : t \geq 0\}$ by time changing X with T :

$$Y_t = X_{T_t}. \quad (4.3)$$

Hence, Y is a càdlàg process with independent increments and it is an additive process (i.e., also continuous in probability) if T is continuous. The distribution of an increment is given as

$$\mathcal{L}(Y_t - Y_s) = \mu^{T_t - T_s} \quad \text{for } 0 \leq s < t. \quad (4.4)$$

Let ϕ be given as in Example 3.1, 4). Using (3.2) rewrite Y_t in terms of the Lévy basis Λ_X as

$$Y_t = \Lambda_X([0, T_t]) = \Lambda_X(\mathbf{T}([0, t])) \quad \text{for } t \geq 0. \quad (4.5)$$

This shows that (4.1) provides a natural generalization of (4.3) and (4.5), since we simply replace $[0, t]$ by an arbitrary bounded Borel set; in return we get a measure M instead of a process Y . Similarly, (4.4) is generalized by (4.2).

R e m a r k 4.2. There are many alternative representations of M and in the following we consider some of them. Let $\{L_t: t \geq 0\}$ denote an $\mathbf{R}^d$ -valued Lévy process with $\mu = \mathcal{L}(L_1)$. Thus, in the language of [2], $\{L_t: t \geq 0\}$ is a Lévy seed associated with μ .

1) If $A_1, \dots, A_r$ are disjoint bounded Borel sets, then

$$(M(A_1), \dots, M(A_r)) \stackrel{\mathcal{D}}{=} (L_{m(A_1)}^{A_1}, \dots, L_{m(A_r)}^{A_r}),$$

where $\{L_t^{A_j}: t \geq 0\}$, for $j = 1, \dots, r$, are independent copies of $\{L_t: t \geq 0\}$. This follows since $\mathcal{L}(L_{m(A_j)}^{A_j}) = \mu^{m(A_j)}$. If $A_1 \subseteq A_2 \subseteq \dots \subseteq A_r$ instead, then

$$(M(A_1), \dots, M(A_r)) \stackrel{\mathcal{D}}{=} (L_{m(A_1)}, \dots, L_{m(A_r)}).$$

2) Consider the case where m is given as in Example 3.1 1); that is, m is concentrated on $\mathcal{T} = \{t_n\}_{n=1}^\infty \subseteq \mathbf{R}_+^k \setminus \mathcal{A}$. For $A \in \mathcal{B}_b(\mathbf{R}_+^k)$ we then have

$$\begin{aligned} M(A) &= \Lambda_X(\mathbf{T}(A \cap \mathcal{T}^c)) + \Lambda_X(\mathbf{T}(A \cap \mathcal{T})) \\ &= \Lambda_X(\mathbf{T}(A \cap \mathcal{T})) = \sum_{n: t_n \in A} \Lambda_X(\mathbf{T}(\{t_n\})) \quad \text{a.s.}, \end{aligned} \quad (4.6)$$

where the series converges almost surely and the first term on the right-hand side of (4.6) vanishes, since $\text{Leb}(\mathbf{T}(A \cap \mathcal{T}^c)) = 0$ by (3.1). Since, by the same equation, $\text{Leb}(\mathbf{T}(\{t_n\})) = m(\{t_n\})$, we have $\Lambda_X(\mathbf{T}(\{t_n\})) \stackrel{\mathcal{D}}{=} L_{m(\{t_n\})}^n$. Taking a sequence $\{L_t^n: t \geq 0\}$, $n = 1, 2, \dots$, of independent copies of $\{L_t: t \geq 0\}$ we thus have for all $r \geq 1$ and $A_1, \dots, A_r \in \mathcal{B}_b(\mathbf{R}_+^k)$ that

$$(M(A_1), \dots, M(A_r)) \stackrel{\mathcal{D}}{=} \left(\sum_{n: t_n \in A_1} L_{m(\{t_n\})}^n, \dots, \sum_{n: t_n \in A_r} L_{m(\{t_n\})}^n \right). \quad (4.7)$$

If $\mu = N_d(\gamma, I)$ (where I is the $d \times d$ identity matrix), then this simplifies as follows. Let $\varepsilon_1, \varepsilon_2, \dots$ denote a sequence of independent and identically distributed random vectors with law $N_d(0, I)$. Then (4.7) is equivalent to

$$\begin{aligned} &(M(A_1), \dots, M(A_r)) \\ &\stackrel{\mathcal{D}}{=} \left(\sum_{n: t_n \in A_1} \gamma m(\{t_n\}) + [m(\{t_n\})]^{1/2} \varepsilon_n, \dots, \sum_{n: t_n \in A_r} \gamma m(\{t_n\}) + [m(\{t_n\})]^{1/2} \varepsilon_n \right). \end{aligned}$$

3) Finally consider the case $m = m_1 + m_2$ as in Example 1, 3), where $m_1 = \text{Leb}/c$ and m_2 is concentrated on $\mathcal{T}$. Then $M = M_1 + M_2$, where $M_i = \{M_i(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ for $i = 1, 2$ are independent and given by $M_1(A) = \Lambda_X(\mathbf{T}(A \cap \mathcal{T}^c))$ and $M_2(A) = \Lambda_X(\mathbf{T}(A \cap \mathcal{T}))$ for $A \in \mathcal{B}_b(\mathbf{R}_+^k)$. The measure M_1 is a homogeneous Lévy basis associated with $\mu^{1/c}$ and M_2 can be represented as M in 2).

R e m a r k 4.3. From the Lévy-Itô decomposition (2.1) of Λ_X we have, a.s. for $A \in \mathcal{B}_b(\mathbf{R}_+^k)$,

$$M(A) = \int_{\mathbf{T}(A) \times \mathbf{R}^d} y 1_D(y) (J_X - \text{Leb} \times \nu)(d(t, y)) + \int_{\mathbf{T}(A) \times \mathbf{R}^d} y 1_{D^c}(y) J_X(d(t, y))$$

$$+ \int_{\mathbf{T}(A)} dX_t^g + \gamma \text{Leb}(\mathbf{T}(A)).$$

Applying the transformation rule on the first two integrals we get the following Lévy–Itô type representation of M :

$$\begin{aligned} M(A) &= \int_{A \times \mathbf{R}^d} y 1_D(y) (\tilde{J}_X - m \times \nu)(d(t, y)) + \int_{A \times \mathbf{R}^d} y 1_{D^c}(y) \tilde{J}_X(d(t, y)) \\ &\quad + \int_{\mathbf{T}(A)} dX_t^g + \gamma m(A) \quad \text{a.s. for } A \in \mathcal{B}_b(\mathbf{R}_+^k), \end{aligned}$$

where $\tilde{J}_X = \{\tilde{J}_X(C): C \in \mathcal{B}(\mathbf{R}_+^k \times \mathbf{R}^d)\}$ is the Poisson random measure given by $\tilde{J}_X(A \times B) = J_X(\mathbf{T}(A) \times B)$ for all $A \in \mathcal{B}_b(\mathbf{R}_+^k)$ and $B \in \mathcal{B}_b(\mathbf{R}^d)$.

5. Extended subordination. Let $X = \{X_t: t \in \mathbf{R}_+^k\}$, $\Lambda_X = \{\Lambda_X(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$, μ and (Σ, ν, γ) be given as in the previous section. That is, X is an $\mathbf{R}^d$ -valued homogeneous Lévy sheet on $\mathbf{R}_+^k$ associated with μ , which has characteristic triplet (Σ, ν, γ) , and Λ_X is the homogeneous Lévy basis induced by X . Let $T = \{T_t: t \in \mathbf{R}_+^k\}$ be an $\mathbf{R}_+$ -valued homogeneous Lévy sheet associated with a distribution $\lambda \in \text{ID}(\mathbf{R})$. Let λ have Lévy measure ρ and drift $\beta \in \mathbf{R}_+$; that is, $\rho(\mathbf{R}_-) = 0$, $\int_{\mathbf{R}_+} (1 \wedge x) \rho(dx) < \infty$ and

$$\hat{\lambda}(u) = \exp \left[i\beta u + \int_{\mathbf{R}_+} (e^{iux} - 1) \rho(dx) \right] \quad \text{for } u \in \mathbf{R}.$$

Let $\Lambda_T = \{\Lambda_T(A): A \in \mathcal{B}(\mathbf{R}_+^k)\}$ be the nonnegative homogeneous Lévy basis induced by $T = \{T_t: t \in \mathbf{R}_+^k\}$. By removing a null set if necessary it follows from Remark 2.12) that Λ_T has the pointwise representation

$$\begin{aligned} \Lambda_T(A)(\omega) &= \int_{A \times \mathbf{R}_+} y J_T(d(t, y))(\omega) + \beta \text{Leb}_1(A) \\ &= \sum_{t \in A} \Lambda_T(\{t\})(\omega) + \beta \text{Leb}_1(A) \quad \text{for } \omega \in \Omega \text{ and } A \in \mathcal{B}(\mathbf{R}_+^k), \end{aligned} \quad (5.1)$$

where Leb_1 denotes Lebesgue measure on $\mathbf{R}_+$ and the series converges for all $A \in \mathcal{B}_b(\mathbf{R}_+^k)$ and $\omega \in \Omega$. Let $\mathcal{F}^T = \sigma(\Lambda_T(A): A \in \mathcal{B}_b(\mathbf{R}_+^k))$ be the sigma-field generated by Λ_T .

Pointwise the measure $A \rightarrow \Lambda_T(A)(\omega)$ is the sum of a discrete measure and a constant times Lebesgue measure. By the construction in Example 3.1, 3) there is an $(\mathcal{F}^T \times \mathcal{B}(\mathbf{R}_+^k), \mathcal{B}(\mathbf{R}^k))$ -measurable mapping $\phi_T: \Omega \times \mathbf{R}_+^k \rightarrow \mathbf{R}^k$ such that for all $\omega \in \Omega$ and $A \in \mathcal{B}_b(\mathbf{R}_+^k)$ the set $\mathbf{T}(A)(\omega)$, given by $\mathbf{T}(A)(\omega) = \{x \in \mathbf{R}_+^k: \phi_T(\omega, x) \in A\}$, is bounded, and

$$\Lambda_T(A)(\omega) = \text{Leb}(\mathbf{T}(A)(\omega)). \quad (5.2)$$

That is, for each ω , $\mathbf{T}(\cdot)(\omega)$ is a meta-time associated with $\Lambda_T(\cdot)(\omega)$.

Define $M = \{M(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ as

$$M(A) = \Lambda_X(\mathbf{T}(A)) \quad \text{for } A \in \mathcal{B}_b(\mathbf{R}_+^k), \quad (5.3)$$

where as usual we suppress ω on both sides. We say that M appears by *extended subordination* of Λ_X by Λ_T or of X by T ; and we write $M = \Lambda_X \wedge \Lambda_T$ or $M = X \wedge T$.

In practice the meta-time $\mathbf{T}$ can be hard to work with directly. Therefore it is important to note that if we condition on T , then, by (5.1), the useful representations of M in Remark 4. apply. For example, if λ above is a Poisson or negative binomial distribution, then almost surely Λ_T is concentrated on a finite number of points on compacts. If λ is a gamma or an inverse Gaussian distribution, then almost surely Λ_T is concentrated on a dense subset of $\mathbf{R}_+^k$.

The following corresponds to the theorem in [2, Section 3.1].

Theorem 5.1. Assume $M = \Lambda_X \wedge \Lambda_T$ as above. Then $M = \{M(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ is a homogeneous Lévy basis associated with the measure $\mu^\# \in \text{ID}(\mathbf{R}^d)$ with characteristic triplet $(\Sigma^\#, \nu^\#, \gamma^\#)$, where

$$\begin{aligned}\Sigma^\# &= \beta \Sigma, \quad \nu^\#(B) = \beta \nu(B) + \int_0^\infty \mu^s(B) \rho(ds), \quad B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}), \\ \gamma^\# &= \beta \gamma + \int_0^\infty \int_{|x| \leq 1} x \mu^s(dx) \rho(ds).\end{aligned}$$

P r o o f. Conditionally on $\mathcal{F}^T$, and hence also unconditionally, M satisfies the σ -additivity condition in Definition 2.2 (ii).

Let $n \geq 1$, and let $A_1, \dots, A_n \in \mathcal{B}_b(\mathbf{R}_+^k)$ be disjoint. Conditionally on $\mathcal{F}^T$ we are in the setting of the previous section. That is, $M(A_1), \dots, M(A_n)$ are independent conditionally on $\mathcal{F}^T$ and $\mathcal{L}(M(A_j) | \mathcal{F}^T) = \mu^{\Lambda_T(A_j)}$. Therefore, for arbitrary $z_1, \dots, z_n \in \mathbf{R}^d$ we have

$$\mathbf{E} \left[\prod_{j=1}^n e^{i\langle z_j, M(A_j) \rangle} \middle| \mathcal{F}^T \right] = \prod_{j=1}^n \widehat{\mu}(z_j)^{\Lambda_T(A_j)}.$$

Since $\Lambda_T(A_1), \dots, \Lambda_T(A_n)$ are independent, it thus follows that

$$\mathbf{E} \left[\prod_{j=1}^n e^{i\langle z_j, M(A_j) \rangle} \right] = \prod_{j=1}^n \mathbf{E} [\widehat{\mu}(z_j)^{\Lambda_T(A_j)}],$$

showing that $M(A_1), \dots, M(A_n)$ are independent. Since, moreover, $\mathcal{L}(\Lambda_T(A)) = \mathcal{L}(\Lambda_T(t+A))$ for all $t \in \mathbf{R}_+^k$ and $A \in \mathcal{B}_b(\mathbf{R}_+^k)$, it follows that $\mathcal{L}(M(A)) = \mathcal{L}(M(t+A))$. Thus, M is a homogeneous Lévy basis.

Choose an arbitrary set $A \in \mathcal{B}_b(\mathbf{R}_+^k)$ with $\text{Leb}(A) = 1$. Then $\mu^\# = \mathcal{L}(M(A))$ and by the above we have for $z \in \mathbf{R}^d$ that $\widehat{\mu}^\#(z) = \mathbf{E} [\widehat{\mu}(z)^{\Lambda_T(A)}]$.

Let $\{L_t: t \geq 0\}$ be a Lévy process with $\mathcal{L}(L_1) = \mu$ and let $\{H_t: t \geq 0\}$ be a subordinator independent of L with $\mathcal{L}(H_1) = \lambda$. It is easily seen that $\mathbf{E} [e^{i\langle z, L_{H_1} \rangle}] = \mathbf{E} [\widehat{\mu}(z)^{H_1}] = \widehat{\mu}^\#(z)$. In other words $\mathcal{L}(L_{H_1}) = \mu^\#$, which means that $\mu^\#$ appears as the law of a subordinated process in the usual sense. It is therefore well known, e.g., from [11, Theorem 30.1], that the characteristic triplet of $\mu^\#$ is as indicated.

R e m a r k 5.1. Above we assumed that Λ_T is a nonnegative homogeneous Lévy basis; however, it is possible to define $M = \Lambda_X \wedge \Lambda_T$ in a much more general context. For example, assume that $\Lambda_T = \{\Lambda_T(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ is stationary in the sense that $\{\Lambda_T(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\} \stackrel{\mathcal{D}}{=} \{\Lambda_T(t+A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ for all $t \in \mathbf{R}_+^k$ and that for all ω $A \mapsto \Lambda_T(A)(\omega)$ is a nonnegative measure on $\mathbf{R}_+^k$. Assume Λ_T and Λ_X are independent and note that we do no longer assume that Λ_T is a Lévy basis. Using Lemma 3.1 define $\phi_T: \Omega \times \mathbf{R}_+^k \rightarrow \mathbf{R}^k$ such that we have (5.2) and let $M = \{M(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ be given as in (5.3). Then $M = \{M(A): A \in \mathcal{B}_b(\mathbf{R}_+^k)\}$ is a homogeneous random measure in the sense that for any disjoint sequence $A_1, A_2, \dots$ with $A = \bigcup_{n=1}^\infty A_n \in \mathcal{B}_b(\mathbf{R}_+^k)$ we have $M(A) = \sum_{n=1}^\infty M(A_n)$ a.s. Moreover, by a slight modification of the above proof it follows that M is stationary. In general M is no longer a Lévy basis. But conditionally on T it is.

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BOUNDS FOR CHARACTERISTIC FUNCTIONS AND LAPLACE TRANSFORMS OF PROBABILITY DISTRIBUTIONS

Используя обобщение неравенства типа Рао–Крамера, мы устанавливаем некоторые неравенства, связывающие характеристические функции и преобразования Лапласа решетчатых распределений. Мы также указываем, как эти результаты могут быть распространены на случай непрерывных плотностей, и иллюстрируем некоторые применения.

Ключевые слова и фразы: характеристическая функция, преобразование Лапласа, решетчатое распределение, информация Фишера, неравенство.

1. Introduction. The apparatus of characteristic functions is of fundamental importance in probability theory, it provides a powerful analytical tool in analyzing random phenomena. In this paper, we will focus on establishing several inequalities for comparing characteristic functions and Laplace transforms of probability distributions.

Let F be a probability distribution function, that is, $F: \mathbf{R} \rightarrow [0, 1]$ is nondecreasing, right continuous, with $\lim_{t \rightarrow -\infty} F(t) = 0$ and $\lim_{t \rightarrow \infty} F(t) = 1$. Let

$$\phi(t) = \int_{\mathbf{R}} e^{itx} dF(x), \quad t \in \mathbf{R},$$

be its characteristic function [2], [7] and put

$$\psi(r) = \int_{\mathbf{R}} e^{-rx} dF(x), \quad r \in \mathbf{R},$$

if the integral is finite. When F is supported on the positive (negative) real axis, $\psi(r)$ is always well defined for $r \geq 0$ ($r \leq 0$), and in this case, ψ is sometimes referred to as the generating function of F .

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