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BOUNDS FOR CHARACTERISTIC FUNCTIONS AND LAPLACE TRANSFORMS OF PROBABILITY DISTRIBUTIONS

Используя обобщение неравенства типа Рао–Крамера, мы устанавливаем некоторые неравенства, связывающие характеристические функции и преобразования Лапласа решетчатых распределений. Мы также указываем, как эти результаты могут быть распространены на случай непрерывных плотностей, и иллюстрируем некоторые применения.

Ключевые слова и фразы: характеристическая функция, преобразование Лапласа, решетчатое распределение, информация Фишера, неравенство.

1. Introduction. The apparatus of characteristic functions is of fundamental importance in probability theory, it provides a powerful analytical tool in analyzing random phenomena. In this paper, we will focus on establishing several inequalities for comparing characteristic functions and Laplace transforms of probability distributions.

Let F be a probability distribution function, that is, $F: \mathbf{R} \rightarrow [0, 1]$ is nondecreasing, right continuous, with $\lim_{t \rightarrow -\infty} F(t) = 0$ and $\lim_{t \rightarrow \infty} F(t) = 1$. Let

$$\phi(t) = \int_{\mathbf{R}} e^{itx} dF(x), \quad t \in \mathbf{R},$$

be its characteristic function [2], [7] and put

$$\psi(r) = \int_{\mathbf{R}} e^{-rx} dF(x), \quad r \in \mathbf{R},$$

if the integral is finite. When F is supported on the positive (negative) real axis, $\psi(r)$ is always well defined for $r \geq 0$ ($r \leq 0$), and in this case, ψ is sometimes referred to as the generating function of F .

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In the terminology of harmonic analysis, $\phi(-t)$ is precisely the Fourier transform, and $\psi(r)$ the Laplace transform, of the distribution F . We also recall that the conventional Laplace transform of F is defined only when F is supported on $[0, \infty)$ and $r \geq 0$. But in this paper we will understand the Laplace transform in the above generalized sense.

In many situations, one usually needs to estimate upper bounds for $|\phi|$. For example, in the study of local limit theorems of lattice distributions supported on the set of integers, one encounters the quantity $\omega(\delta) = \max_{\delta \leq |t| \leq \pi} |\phi(t)|$, $0 \leq \delta \leq \pi$, and the integral

$$\int_{\delta \leq |t| \leq \pi} |\phi(t)|^n dt, \quad 0 \leq \delta \leq \pi,$$

and is required to estimate their upper bounds [1], [3], [4].

A mathematically obvious yet physically important inequality is $|\phi(t)| \leq 1$ for any $t \in \mathbf{R}$. A less trivial result is that if there exist constants ε and θ such that $|\phi(t)| \leq \varepsilon < 1$ whenever $|t| \geq \theta$, then [7, Corollary to Theorem 4.1.2]

$$|\phi(t)| \leq 1 - \frac{1 - \varepsilon^2}{8\theta^2} t^2 \quad \forall |t| < \theta.$$

Another important inequality concerning characteristic functions which has interesting implications for quantum dynamics is [2], [11]

$$(\operatorname{Re} \phi(t))^2 \leq \frac{1 + \operatorname{Re} \phi(2t)}{2} \quad \forall t \in \mathbf{R}. \quad (1)$$

There are also various lower bounds for characteristic functions which have important applications in estimating survival probabilities of quantum states and in quantifying the time-energy uncertainty relations [8]–[10], [12].

In this paper, we are interested in establishing an improvement of inequality (1) and in proving several other inequalities concerning characteristic functions and the Laplace transforms under the additional condition that the probability distribution F is a lattice distribution. The results will then be extended to continuous density cases. The basic idea is to follow the powerful argument typical in the Cramér–Rao type inequality, which in turn is simply some applications of the ubiquitous Schwarz inequality.

To state some of the results and to gain their flavor, let $a \in \mathbf{R}$ and $h > 0$ be any two constants and $\mathbf{Z}$ the set of all integers, let $\mathbf{p} = \{p_{a+\nu h} : \nu \in \mathbf{Z}\}$ be a discrete probability distribution supported on the lattice $\mathbf{Z}_{a,h} = \{a + \nu h : \nu \in \mathbf{Z}\}$. Then its characteristic function is

$$\phi_{\mathbf{p}}(t) = \sum_{\nu \in \mathbf{Z}} e^{it(a+\nu h)} p_{a+\nu h}, \quad t \in \mathbf{R}. \quad (2)$$

Inspired by [6], let us define two discrete versions of the Fisher information as follows:

$$I_{-}(\mathbf{p}) = \sum_{\nu \in \mathbf{Z}} \frac{(p_{a+\nu h} - p_{a+(\nu-1)h})^2}{p_{a+(\nu-1)h}}, \quad I_{+}(\mathbf{p}) = \sum_{\nu \in \mathbf{Z}} \frac{(p_{a+\nu h} - p_{a+(\nu+1)h})^2}{p_{a+\nu h}},$$

provided the series converge. It should be noted that such kinds of quantities were considered earlier by Gnedenko [3] and by Gamkrelidze [5].

Proposition 1. *Let $\phi_{\mathbf{p}}(t)$ be the characteristic function of $\mathbf{p}$ defined by (2), then*

$$\frac{\sin^2(th/2)}{I_{-}(\mathbf{p})} \left(\operatorname{Re} \phi_{\mathbf{p}}(t) \sin \frac{th}{2} - \operatorname{Im} \phi_{\mathbf{p}}(t) \cos \frac{th}{2} \right)^2 \leq \frac{1 - 2(\operatorname{Re} \phi_{\mathbf{p}}(t))^2 + \operatorname{Re} \phi_{\mathbf{p}}(2t)}{8}, \quad (3)$$

$$\frac{\sin^2(th/2)}{I_{-}(\mathbf{p})} \left(\operatorname{Re} \phi_{\mathbf{p}}(t) \cos \frac{th}{2} + \operatorname{Im} \phi_{\mathbf{p}}(t) \sin \frac{th}{2} \right)^2 \leq \frac{1 - 2(\operatorname{Im} \phi_{\mathbf{p}}(t))^2 - \operatorname{Re} \phi_{\mathbf{p}}(2t)}{8}. \quad (4)$$

The above two inequalities still hold true when $I_{-}(\mathbf{p})$ is replaced by $I_{+}(\mathbf{p})$ and in the meantime h is replaced by $-h$.

Noting that the left-hand sides of both inequalities (3) and (4) are nonnegative, we have the following simple consequences:

$$(\operatorname{Re} \phi_{\mathbf{p}}(t))^2 \leq \frac{1 + \operatorname{Re} \phi_{\mathbf{p}}(2t)}{2}, \quad (\operatorname{Im} \phi_{\mathbf{p}}(t))^2 \leq \frac{1 - \operatorname{Re} \phi_{\mathbf{p}}(2t)}{2},$$

which splits the trivial inequality $|\phi_{\mathbf{p}}(t)|^2 = (\operatorname{Re} \phi_{\mathbf{p}}(t))^2 + (\operatorname{Im} \phi_{\mathbf{p}}(t))^2 \leq 1$ into two parts.

By adding inequalities (3) and (4) together, and after some simple algebraic manipulation, we have the following result:

$$|\phi_{\mathbf{p}}(t)|^2 \leq \frac{I_-(\mathbf{p})}{I_-(\mathbf{p}) + 4 \sin^2(th/2)} \quad \forall t \in \mathbf{R}. \quad (5)$$

The above inequality still holds when $I_-(\mathbf{p})$ is replaced by $I_+(\mathbf{p})$. From the above inequality, we readily get bounds of $\omega(\delta) = \max_{\delta \leq |t| \leq \pi} |\phi_{\mathbf{p}}(t)|$, which may be compared with inequalities (2.5) and (2.16) of [1].

Let us define the Laplace transform of $\mathbf{p}$ as

$$\psi_{\mathbf{p}}(r) = \sum_{\nu \in \mathbf{Z}} e^{-r(a+\nu h)} p_{a+\nu h}, \quad r \in \mathbf{R},$$

provided the integral is finite.

Proposition 2. *It holds that*

$$\left(1 + \frac{(1 - e^{rh})^2}{I_-(\mathbf{p})}\right) \psi_{\mathbf{p}}^2(r) \leq \psi_{\mathbf{p}}(2r). \quad (6)$$

The above inequality still holds when $I_-(\mathbf{p})$ is replaced by $I_+(\mathbf{p})$ and in the meantime h is replaced by $-h$.

If f is a differentiable probability density on $\mathbf{R}$ with characteristic function $\phi_f(t) = \int_{\mathbf{R}} e^{itx} f(x) dx$, then as a typical application by passing h to zero of the subsequent inequalities established in the lattice case, we will derive in Section 3 the following upper bound for $\phi_f(t)$.

Proposition 3. *For any $\lambda > 0$, it holds that*

$$|\phi_f(t)|^2 \leq \pi \frac{1 - e^{-2\lambda|t|}}{\lambda t^2} \int_{\mathbf{R}} (f'(x))^2 (\lambda^2 + x^2) dx \quad \forall t \in \mathbf{R}. \quad (7)$$

Here the prime ' denotes derivative. Note that

$$\lim_{t \rightarrow 0} \frac{1 - e^{-2\lambda|t|}}{\lambda t^2} = \lim_{t \rightarrow 0} \frac{2}{|t|} = \infty$$

while it is trivial that $|\phi_f(t)|^2 \leq 1$, thus the above inequality is nontrivial only for $|t|$ sufficiently large, more precisely, for t such that

$$\int_{\mathbf{R}} (f'(x))^2 (\lambda^2 + x^2) dx \leq \frac{\lambda t^2}{\pi(1 - e^{-2\lambda|t|})}.$$

In order to prove Propositions 1–3, we will in fact establish some more general comparison results for two characteristic functions and the Laplace transforms in Section 2. The main results are inequalities (10)–(14). We will illustrate how all these results can be adapted to continuous probability density cases in Section 3. Finally, some applications and discussions will be presented in Section 4.

2. Bounds for characteristic functions and Laplace transforms. For our purpose, we first establish a generalization of the Cramér–Rao type inequality which is essentially a consequence of the Schwarz inequality. These inequalities may be of independent interest. Though the original Cramér–Rao type inequality arises from the estimation of parameters, the idea and method in its derivation are of their own value and are used here to derive our main result.

For two discrete probability distributions $\mathbf{p} = \{p_{a+\nu h}: \nu \in \mathbf{Z}\}$ and $\mathbf{q} = \{q_{a+\nu h}: \nu \in \mathbf{Z}\}$ supported on the lattice $\mathbf{Z}_{a,h} = \{a + \nu h: \nu \in \mathbf{Z}\}$ with $q_\nu > 0$ for any $\nu \in \mathbf{Z}$, let us define

$$I_-(\mathbf{p}|\mathbf{q}) = \sum_{\nu \in \mathbf{Z}} \frac{(p_{a+\nu h} - p_{a+(\nu-1)h})^2}{q_{a+(\nu-1)h}}, \quad I_+(\mathbf{p}|\mathbf{q}) = \sum_{\nu \in \mathbf{Z}} \frac{(p_{a+\nu h} - p_{a+(\nu+1)h})^2}{q_{a+\nu h}}$$

provided the series converge. The freedom of choosing $\mathbf{q}$ is useful in setting various upper bounds for the characteristic function of $\mathbf{p}$. Clearly,

$$I_-(\mathbf{p}|\mathbf{p}) = I_-(\mathbf{p}), \quad I_+(\mathbf{p}|\mathbf{p}) = I_+(\mathbf{p}).$$

For a function $g: \mathbf{Z}_{a,h} \rightarrow \mathbf{R}$, the backward difference operator D_- and the forward difference operator D_+ are defined as

$$D_-g(x) = g(x) - g(x-h), \quad D_+g(x) = g(x+h) - g(x), \quad x \in \mathbf{Z}_{a,h}.$$

Theorem 1. Let X and Y be two random variables taking values in the lattice $\mathbf{Z}_{a,h}$ such that $\mathbf{P}(X = a + \nu h) = p_{a+\nu h}$ and $\mathbf{P}(Y = a + \nu h) = q_{a+\nu h} > 0$ respectively. Let $g: \mathbf{Z}_{a,h} \rightarrow \mathbf{R}$ be a function defined on $\mathbf{Z}_{a,h}$ such that the variance $\mathbf{D}g(Y) < \infty$. Then

$$\mathbf{D}g(Y) \geq \frac{|\mathbf{E}(D_-g(X))|^2}{I_-(\mathbf{p}|\mathbf{q})}, \quad (8)$$

and

$$\mathbf{D}g(Y) \geq \frac{|\mathbf{E}(D_+g(X))|^2}{I_+(\mathbf{p}|\mathbf{q})}. \quad (9)$$

Here $\mathbf{E}$ denotes the expectation.

P r o o f. Suppose that $\mathbf{E}g(Y) = m$ then simple manipulation shows that

$$\begin{aligned} & \sum_{\nu \in \mathbf{Z}} (p_{a+\nu h} - p_{a+(\nu-1)h})(g(a + (\nu-1)h) - m) \\ &= \sum_{\nu \in \mathbf{Z}} p_{a+\nu h}g(a + (\nu-1)h) - m \sum_{\nu \in \mathbf{Z}} p_{a+\nu h} \\ & \quad - \sum_{\nu \in \mathbf{Z}} p_{a+(\nu-1)h}g(a + (\nu-1)h) + m \sum_{\nu \in \mathbf{Z}} p_{a+(\nu-1)h} \\ &= \sum_{\nu \in \mathbf{Z}} p_{a+\nu h}g(a + (\nu-1)h) - \sum_{\nu \in \mathbf{Z}} p_{a+(\nu-1)h}g(a + (\nu-1)h) \\ &= - \sum_{\nu \in \mathbf{Z}} p_{a+\nu h}(g(a + \nu h) - g(a + (\nu-1)h)) = -\mathbf{E}(D_-g(X)). \end{aligned}$$

Accordingly, by the Schwarz inequality,

$$\begin{aligned} \left| \mathbf{E}(D_-g(X)) \right|^2 &= \left| \sum_{\nu \in \mathbf{Z}} \frac{p_{a+\nu h} - p_{a+(\nu-1)h}}{\sqrt{q_{a+(\nu-1)h}}} \sqrt{q_{a+(\nu-1)h}}(g(a + (\nu-1)h) - m) \right|^2 \\ &\leq \sum_{\nu \in \mathbf{Z}} \frac{(p_{a+\nu h} - p_{a+(\nu-1)h})^2}{q_{a+(\nu-1)h}} \sum_{\nu \in \mathbf{Z}} q_{a+(\nu-1)h}(g(a + (\nu-1)h) - m)^2 \\ &= I_-(\mathbf{p}|\mathbf{q})\mathbf{D}g(Y), \end{aligned}$$

which is the desired inequality (8).

Inequality (9) follows similarly by noting that

$$\sum_{\nu \in \mathbf{Z}} (p_{a+\nu h} - p_{a+(\nu-1)h})(g(a+\nu h) - m) = -\mathbf{E}(D_+ g(X))$$

and

$$\begin{aligned} \left| \mathbf{E}(D_+ g(X)) \right|^2 &= \left| \sum_{\nu \in \mathbf{Z}} \frac{p_{a+\nu h} - p_{a+(\nu-1)h}}{\sqrt{q_{a+\nu h}}} \sqrt{q_{a+\nu h}} (g(a+\nu h) - m) \right|^2 \\ &\leq \sum_{\nu \in \mathbf{Z}} \frac{(p_{a+\nu h} - p_{a+(\nu-1)h})^2}{q_{a+\nu h}} \sum_{\nu \in \mathbf{Z}} q_{a+\nu h} (g(a+\nu h) - m)^2 \\ &= I_+(\mathbf{p} | \mathbf{p}) \mathbf{D}(g(Y)). \end{aligned}$$

Theorem 2. Let $\mathbf{p}$ and $\mathbf{q}$ be two lattice distributions with characteristic functions $\phi_{\mathbf{p}}$ and $\phi_{\mathbf{q}}$, respectively. Then

$$\frac{\sin^2(th/2)}{I_-(\mathbf{p} | \mathbf{q})} \left(\operatorname{Re} \phi_{\mathbf{p}}(t) \sin \frac{th}{2} - \operatorname{Im} \phi_{\mathbf{p}}(t) \cos \frac{th}{2} \right)^2 \leq \frac{1 - 2(\operatorname{Re} \phi_{\mathbf{q}}(t))^2 + \operatorname{Re} \phi_{\mathbf{q}}(2t)}{8}, \quad (10)$$

$$\frac{\sin^2(th/2)}{I_-(\mathbf{p} | \mathbf{q})} \left(\operatorname{Re} \phi_{\mathbf{p}}(t) \cos \frac{th}{2} + \operatorname{Im} \phi_{\mathbf{p}}(t) \sin \frac{th}{2} \right)^2 \leq \frac{1 - 2(\operatorname{Im} \phi_{\mathbf{q}}(t))^2 - \operatorname{Re} \phi_{\mathbf{q}}(2t)}{8}. \quad (11)$$

The above inequalities still hold when $I_-(\mathbf{p} | \mathbf{q})$ is replaced by $I_+(\mathbf{p} | \mathbf{q})$ and in the meantime h is replaced by $-h$.

P r o o f. Let X and Y be two random variables taking values in the lattice $\mathbf{Z}_{a,h}$ such that $\mathbf{P}(X = a + \nu h) = p_{a+\nu h}$ and $\mathbf{P}(Y = a + \nu h) = q_{a+\nu h} > 0$, respectively, then $\phi_{\mathbf{q}}(t) = \mathbf{E} e^{itY} = \mathbf{E} \cos(tY) + i \mathbf{E} \sin(tY) = \operatorname{Re} \phi_{\mathbf{q}}(t) + i \operatorname{Im} \phi_{\mathbf{q}}(t)$, and

$$\mathbf{D} \cos(tY) = \mathbf{E} \cos^2(tY) - (\mathbf{E} \cos(tY))^2 = \frac{1 + \operatorname{Re} \phi_{\mathbf{q}}(2t)}{2} - (\operatorname{Re} \phi_{\mathbf{q}}(t))^2.$$

On the other hand, from the identity $\cos(tX) - \cos(t(X-h)) = (1 - \cos(th)) \cos(tX) - \sin(th) \sin(tX)$ we have

$$\begin{aligned} \mathbf{E}(D_- \cos(tX)) &= (1 - \cos(th)) \operatorname{Re} \phi_{\mathbf{p}}(t) - \sin(th) \operatorname{Im} \phi_{\mathbf{p}}(t) \\ &= 2 \sin \frac{th}{2} \left(\operatorname{Re} \phi_{\mathbf{p}}(t) \sin \frac{th}{2} - \operatorname{Im} \phi_{\mathbf{p}}(t) \cos \frac{th}{2} \right). \end{aligned}$$

Now the desired inequality (10) follows by applying Theorem 1 to $g(x) = \cos(tx)$.

Similarly, we have

$$\mathbf{D} \sin(tY) = \mathbf{E} \sin^2(tY) - (\mathbf{E} \sin(tY))^2 = \frac{1 - \operatorname{Re} \phi_{\mathbf{q}}(2t)}{2} - (\operatorname{Im} \phi_{\mathbf{q}}(t))^2,$$

and from the identity $\sin(tX) - \sin(t(X-h)) = (1 - \cos(th)) \sin(tX) + \sin(th) \cos(tX)$ we have

$$\begin{aligned} \mathbf{E}(D_- \sin(tX)) &= (1 - \cos(th)) \operatorname{Im} \phi_{\mathbf{p}}(t) + \sin(th) \operatorname{Re} \phi_{\mathbf{p}}(t) \\ &= 2 \sin \frac{th}{2} \left(\operatorname{Re} \phi_{\mathbf{p}}(t) \cos \frac{th}{2} + \operatorname{Im} \phi_{\mathbf{p}}(t) \sin \frac{th}{2} \right). \end{aligned}$$

Consequently, inequality (11) follows by applying Theorem 1 to $g(x) = \sin(tx)$.

Proposition 1 follows readily from Theorem 2 by taking $\mathbf{q} = \mathbf{p}$.

By adding inequalities (10) and (11), we have

$$\frac{\sin^2(th/2)}{I_-(\mathbf{p} | \mathbf{q})} |\phi_{\mathbf{p}}(t)|^2 \leq \frac{1 - |\phi_{\mathbf{q}}(t)|^2}{4}. \quad (12)$$

If we choose $\mathbf{q}$ such that $\phi_{\mathbf{q}}(t)$ can easily be evaluated, then we obtain readily an upper bound for $|\phi_{\mathbf{p}}(t)|$.

In order to establish Proposition 2, we prove a slightly more general result.

Theorem 3. *Let $\mathbf{p}$ and $\mathbf{q}$ be two lattice distributions with Laplace transforms $\psi_{\mathbf{p}}$ and $\psi_{\mathbf{q}}$, respectively. Then*

$$\frac{(1 - e^{rh})^2}{I_-(\mathbf{p} | \mathbf{q})} \psi_{\mathbf{p}}^2(r) \leq \psi_{\mathbf{q}}(2r) - \psi_{\mathbf{q}}^2(r). \quad (13)$$

The above inequality still hold when $I_-(\mathbf{p} | \mathbf{q})$ is replaced by $I_+(\mathbf{p} | \mathbf{q})$ and in the meantime h is replaced by $-h$.

P r o o f. Let X and Y be the random variables defined in Theorem 1. For any fixed $r \geq 0$, take $g(x) = e^{-rx}$ in Theorem 1, then $\mathbf{D}e^{-rY} = \mathbf{E}e^{-2rY} - (\mathbf{E}e^{-rY})^2 = \psi_{\mathbf{q}}(2r) - \psi_{\mathbf{q}}^2(r)$. On the other hand, from $D_-e^{-rX} = e^{-rX} - e^{-r(X-h)} = (1 - e^{rh})e^{-rX}$, we have $|\mathbf{E}D_-e^{-rX}|^2 = (1 - e^{rh})^2\psi_{\mathbf{p}}^2(r)$. Now inequality (13) follows from Theorem 1 by taking $g(x) = e^{-rx}$.

Proposition 2 follows from the above theorem by taking $\mathbf{q} = \mathbf{p}$.

3. Continuous limit. In this section, we indicate heuristically how to obtain the continuous analogues of all the above inequalities by passing the lattice spacing parameter h to zero.

Let f and g be two continuous probability densities on $\mathbf{R}$ with characteristic functions

$$\phi_f(t) = \int_{\mathbf{R}} e^{itx} f(x) dx, \quad \phi_g(t) = \int_{\mathbf{R}} e^{itx} g(x) dx,$$

respectively. Let

$$I(f | g) = \int_{\mathbf{R}} \frac{(f'(x))^2}{g(x)} dx.$$

Clearly, $I(f | f)$ is precisely the Fisher information

$$I(f) = \int_{\mathbf{R}} \frac{(f'(x))^2}{f(x)} dx$$

of f .

If we define

$$p_{a+\nu h} = \int_{a+(\nu-1/2)h}^{a+(\nu+1/2)h} f(x) dx, \quad q_{a+\nu h} = \int_{a+(\nu-1/2)h}^{a+(\nu+1/2)h} g(x) dx, \quad \nu \in \mathbf{Z},$$

then under mild conditions, we have

$$\lim_{h \rightarrow 0} \frac{I_-(\mathbf{p} | \mathbf{q})}{h^2} = \lim_{h \rightarrow 0} \frac{I_+(\mathbf{p} | \mathbf{q})}{h^2} = I(f | g),$$

and

$$\lim_{h \rightarrow 0} \phi_{\mathbf{p}}(t) = \phi_f(t), \quad \lim_{h \rightarrow 0} \phi_{\mathbf{q}}(t) = \phi_g(t).$$

Accordingly, by taking the limit $h \rightarrow 0$ in Theorem 2 and noting that

$$\lim_{h \rightarrow 0} \frac{\sin^2(th/2)}{h^2} = \frac{t^2}{4},$$

we have

$$\begin{aligned} \frac{t^2}{I(f | g)} (\operatorname{Im} \phi_f(t))^2 &\leq \frac{1 - 2(\operatorname{Re} \phi_g(t))^2 + \operatorname{Re} \phi_g(2t)}{2}, \\ \frac{t^2}{I(f | g)} (\operatorname{Re} \phi_f(t))^2 &\leq \frac{1 - 2(\operatorname{Im} \phi_g(t))^2 - \operatorname{Re} \phi_g(2t)}{2}. \end{aligned}$$

Moreover, by adding the above two inequalities together, we have the following result.

Theorem 4. *It holds that*

$$\frac{t^2}{I(f|g)} |\phi_f(t)|^2 \leq 1 - |\phi_g(t)|^2 \quad \forall t \in \mathbf{R}. \quad (14)$$

Alternatively, we have a variational characterization of an upper bound for $\phi_f(t)$ as

$$|\phi_f(t)|^2 \leq \frac{1}{t^2} \inf_g \frac{1 - |\phi_g(t)|^2}{I(f|g)} \quad \forall t \in \mathbf{R}.$$

By varying g , we obtain a family of upper bounds for $\phi_f(t)$. In particular, if we take

$$g(x) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + x^2}, \quad x \in \mathbf{R},$$

to be the Cauchy density with parameter $\lambda > 0$, then it is well known that $\phi_g(t) = e^{-\lambda|t|}$, and as a direct consequence of Theorem 4, we obtain inequality (7) which proves Proposition 3.

By taking the limit $h \rightarrow 0$ in Theorem 3 and noting that $\lim_{h \rightarrow 0} \psi_{\mathbf{p}}(r) = \psi_f(r) \equiv \int_{\mathbf{R}} e^{-rx} f(x) dx$ and a similar limit connecting $\mathbf{q}$ and g , we have

$$\frac{r^2}{I(f|g)} \psi_f^2(r) \leq \psi_g(2r) - \psi_g^2(r).$$

The corresponding continuous analogues of inequalities (3)–(6) follow similarly.

4. Discussion. We have established several inequalities which set upper bounds for characteristic functions and Laplace transforms of lattice distributions. These inequalities refine some well-known inequalities and involve some generalizations of the notion of Fisher information. When the lattice spacing parameter h tends to zero, we obtain some continuous analogues of the established inequalities. We also note that the behaviors of characteristic functions for probability densities and those for lattice distributions are qualitatively different. For example, according to the Riemann–Lebesgue lemma, for any probability density f , $\phi_f(t)$ tends to zero when t tends to infinite, but for a lattice distribution $\mathbf{p}$, $\phi_{\mathbf{p}}(t)$ may oscillate and return to 1 infinitely often when t tends to infinite. From our results, we have a more quantitative version of the above phenomenon: By inequality (5), we have for lattice distribution $\mathbf{p}$,

$$|\phi_{\mathbf{p}}(t)|^2 \leq \frac{I_-(\mathbf{p})}{I_-(\mathbf{p}) + 4 \sin^2(th/2)},$$

while for continuous density f ,

$$|\phi_f(t)|^2 \leq \frac{I(f)}{I(f) + t^2},$$

which follows readily from the lattice case by letting $h \rightarrow 0$ or follows from inequality (14) by putting $g = f$. Clearly, the above two upper bounds are fundamentally different.

As a simple application of inequality (5) relevant in local central limit theorems [1], we have

$$\omega(\delta) \equiv \max_{\delta \leq |t| \leq \pi} |\phi_{\mathbf{p}}(t)| \leq \sqrt{\frac{I_-(\mathbf{p})}{I_-(\mathbf{p}) + 4 \sin^2(\delta/2)}} \quad \forall 0 \leq \delta \leq \pi,$$

when $\mathbf{p}$ is a lattice probability distribution supported on the integer lattice $\mathbf{Z}$.

We have only treated the case when $p_{a+\nu h} > 0$ for all $\nu \in \mathbf{Z}$ for simplicity. If the probability $\mathbf{p}$ is supported on a half lattice, say $\mathbf{Z}_{a,h}^+ = \{a + \nu h: \nu \in \mathbf{Z}, \nu \geq 0\}$, then all the results proved in this paper involving $I_+(\mathbf{p})$ can be adapted readily to this case by putting $p_{a-h} = 0$ and modifying $I_+(\mathbf{p})$ as

$$I_+(\mathbf{p}) = \sum_{\nu \in \mathbf{Z}, \nu \geq 0} \frac{(p_{a+\nu h} - p_{a+(\nu-1)h})^2}{p_{a+\nu h}}.$$

Alternatively, if the probability $\mathbf{p}$ is supported on $\mathbf{Z}_{a,h}^- = \{a + \nu h: n \in \mathbf{Z}, \nu \leq 0\}$, then all the results involving $I_-(\mathbf{p})$ can be adapted readily to this case by putting $p_{a+h} = 0$ and modifying $I_-(\mathbf{p})$ as

$$I_-(\mathbf{p}) = \sum_{\nu \in \mathbf{Z}, \nu \leq -1} \frac{(p_{a+\nu h} - p_{a+(\nu-1)h})^2}{p_{a+(\nu-1)h}}.$$

In the more general case, we may always assume $p_{a+\nu h} > 0$ for $\nu \in \bigcup_j E_j$ with $E_j = \{n_j, n_j + 1, \dots, n_j + l_j\}$. Here $n_j \in \mathbf{Z}$ and l_j are positive integers. By considering each integer interval E_j separately with some straightforward modification concerning the endpoints of each interval, we may derive similar results, but the expressions are considerably more complicated and we will not pursue them here.

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