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ON HITTING TIMES OF THE WINDING PROCESSES OF
PLANAR BROWNIAN MOTION AND OF
ORNSTEIN–UHLENBECK PROCESSES,
VIA BOUGEROL’S IDENTITY

Устанавливаются некоторые тождества по распределению для плоских комплекснозначных процессов Орнштейна–Уленбека ($Z_t = X_t + iY_t$, $t \geq 0$), включая плоское броуновское движение, эквивалентные хорошо известному тождеству Бужероля для линейного броуновского движения (β_t , $t \geq 0$): для любого фиксированного $u > 0$

$$\text{sh } \beta_u \stackrel{\text{law}}{=} \hat{\beta} \int_0^u ds \exp(2\beta_s),$$

где $(\hat{\beta}_t, t \geq 0)$ — броуновское движение, независимое от β .

Эти тождества по распределению для двумерных процессов позволяют изучать распределение моментов достижения $T_c^\theta \equiv \inf\{t: \theta_t = c\}$ ($c > 0$), $T_{-d,c}^\theta \equiv \inf\{t: \theta_t \notin (-d, c)\}$ ($c, d > 0$) и, в частности, $T_{-c,c}^\theta \equiv \inf\{t: \theta_t \notin (-c, c)\}$ ($c > 0$) для непрерывного процесса $\theta_t = \text{Im}(\int_0^t Z_s^{-1} dZ_s)$, $t \geq 0$, — угла, заметаемого комплексным процессом Орнштейна–Уленбека.

Ключевые слова и фразы: плоское броуновское движение; процесс Орнштейна–Уленбека; угол, заметаемый радиус-вектором комплекснозначного процесса; тождество Бужероля; момент выхода из конуса.

1. Introduction. The conformal invariance of planar Brownian motion has deep consequences as to the structure of its trajectories (see, e.g., [20]). In particular, a number of articles have been devoted to the study of its continuous winding process $(\theta_t, t \geq 0)$: [32], [34], [15], [27], [29], [21], [5], [37], [28], [4]. In this paper, we take up again the study of the first hitting times:

$$T_{-d,c}^\theta \equiv \inf\{t: \theta_t \notin (-d, c)\} \quad (c, d > 0),$$

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this time in relation with Bougerol's well-known identity (see [8], [2], and [38]): for fixed $u > 0$,

$$\operatorname{sh} \beta_u \stackrel{\text{law}}{=} \hat{\beta}_{\int_0^u ds \exp(2\beta_s)},$$

where $(\hat{\beta}_t, t \geq 0)$ is a Brownian motion¹⁾ independent of β . In particular, it turns out that, for fixed $c > 0$,

$$\theta_{T_c^{\hat{\beta}}} \stackrel{\text{law}}{=} C_{a(c)}, \quad (\star)$$

where $\hat{\beta}$ is a Brownian motion independent of $(\theta_u, u \geq 0)$, $T_c^{\hat{\beta}} = \inf\{t: \hat{\beta}_t = c\}$, $(C_t, t \geq 0)$ is a standard Cauchy process and $a(c) = \arg \operatorname{sh} c \equiv \ln(c + \sqrt{1 + c^2})$, $c \in \mathbf{R}$.

The identity $(\star)$ yields yet another proof of the celebrated Spitzer theorem:

$$\frac{2}{\ln t} \theta_t \xrightarrow[t \rightarrow \infty]{\text{law}} C_1,$$

with the help of Williams' «pinching method» (see [34] and [27]).

Moreover, we study the distributions of $T_{-\infty, c}^{\theta}$ and $T_{-c, c}^{\theta}$. In particular, we give explicit formulae for the density function of $T_{-c, c}^{\theta}$ and for the first moment of $\ln T_{-c, c}^{\theta}$.

The last section of the paper is devoted to developing similar results when planar Brownian motion is replaced by a complex valued Ornstein–Uhlenbeck process. We note that Bertoin and Werner [5] already made discussions of windings for planar Brownian motion using arguments related to Ornstein–Uhlenbeck processes.

Firstly, we obtain some analogue of $(\star)$ when $T_c^{\hat{\beta}}$ is replaced by $T_c^{(\lambda)} = T_{-c, c}^{\theta^Z} = \inf\{t: |\theta_t^Z| = c\}$, the corresponding time for an Ornstein–Uhlenbeck process with parameter λ . Secondly, we identify the distribution of $T_c^{(\lambda)}$. More specifically, we derive the asymptotics of $\mathbf{E}[T_c^{(\lambda)}]$ for λ large and for λ small.

2. The Brownian motion case.

2.1. A reminder on planar Brownian motion. Let $(Z_t = X_t + iY_t, t \geq 0)$ denote a standard planar Brownian motion, starting from $x_0 + i0$, $x_0 > 0$, where $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two independent linear Brownian motions, starting, respectively, from x_0 and 0.

It is well known (see, e.g., [19]) that, since $x_0 \neq 0$, the process $(Z_t, t \geq 0)$ does not visit a.s. the point 0 but keeps winding around 0 infinitely often.

¹⁾ When we simply write: Brownian motion, we always mean real-valued Brownian motion starting from 0. For 2-dimensional Brownian motion, we add qualifier *planar* or *complex*.

In particular, the continuous winding process $\theta_t = \text{Im}(\int_0^t Z_s^{-1} dZ_s)$, $t \geq 0$, is well defined.

Furthermore, there is the skew-product representation:

$$\ln |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \Big|_{u=H_t=\int_0^t |Z_s|^{-2} ds}, \quad (1)$$

where $(\beta_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\ln x_0 + i0$. For a study of the Bessel clock H , see [35].

Rewriting (1) as

$$\ln |Z_t| = \beta_{H_t}, \theta_t = \gamma_{H_t}, \quad (2)$$

we easily obtain that the total σ -fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.

Fig. 1. Exit time from a single border cone for a planar BM. This figure presents the exit times T_c^θ (t_1 does not matter because the angle is negative).

Fig. 2. Exit time from a double border cone for a planar BM. This figure presents the exit times $T_{-d,c}^\theta$ for a planar BM starting from $x_0 + i0$.

A number of studies of the properties of the first hitting time (see Figure 2)

$$T_{-d,c}^\theta \equiv \inf\{t: \theta_t \notin (-d, c)\} \quad (c, d > 0),$$

have been developed, going back to [32].

In particular, it is well known ([32], [9], [31, Exercise 2.21, p. 196]) that

$$\mathbf{E}[(T_{-d,c}^\theta)^p] < \infty \quad \text{if and only if} \quad p < \frac{\pi}{2(c+d)}. \quad (3)$$

Moreover, Spitzer's asymptotic theorem (see, e.g., [32]) states that

$$\frac{2\theta_t}{\ln t} \xrightarrow[t \rightarrow \infty]{\text{law}} C_1 \stackrel{\text{law}}{=} \gamma_{T_1^\beta}, \quad (4)$$

where C_1 is a standard Cauchy variable.

2.2. On the Laplace transform of the distribution of the hitting time $T_c^\theta \equiv T_{-\infty,c}^\theta$. Now, we use the representation (2) to access the distribution of T_c^θ (see Figure 1). We define $T_c^\gamma \equiv \inf\{t: \gamma_t \notin (-\infty, c)\}$ the hitting time associated to the Brownian motion $(\gamma_t, t \geq 0)$. Note that from (2) it follows that $H_{T_c^\theta} = T_c^\gamma$, hence $T_c^\theta = H_u^{-1}|_{u=T_c^\gamma}$, where

$$H_u^{-1} \equiv \inf\{t: H_t > u\} = \int_0^u ds \exp(2\beta_s) \equiv A_u. \quad (5)$$

Thus, we have obtained

$$T_c^\theta = A_{T_c^\gamma}, \quad (6)$$

where $(A_u, u \geq 0)$ and T_c^γ are independent, since β and γ are independent. We can write $\beta_s = \ln x_0 + \beta_s^{(0)}$, with $(\beta_s^{(0)}, s \geq 0)$ a standard one-dimensional Brownian motion starting from 0. Then, we deduce from (6) that

$$T_c^\theta = x_0^2 \left(\int_0^{T_c^\gamma} ds \exp(2\beta_s^{(0)}) \right). \quad (7)$$

From now on, for simplicity, we shall take $x_0 = 1$, but this is really no restriction, as the dependency in x_0 , which is exhibited in (7), is very simple. We shall also make use of Bougerol's identity ([8], [2], and [38, p. 200]) which is very useful to study the distribution of A_u (see, e.g., [25], [26]). For any fixed $u > 0$

$$\text{sh } \beta_u \stackrel{\text{law}}{=} \hat{\beta}_{A_u} = \hat{\beta}_{\int_0^u ds \exp(2\beta_s)}, \quad (8)$$

where on the right-hand side, $(\hat{\beta}_t, t \geq 0)$ is a Brownian motion independent of $A_u \equiv \int_0^u ds \exp(2\beta_s)$.

Thus, from (8) and (6), and as is well known (see, e.g., [31]), the law of $\beta_{T_c^\theta}$ is the Cauchy law with parameter c , i.e., with density

$$h_c(y) = \frac{c}{\pi(c^2 + y^2)},$$

we deduce the following proposition.

Proposition 2.1. *For fixed $c > 0$, there is the following identity in law:*

$$\text{sh } C_c \stackrel{\text{law}}{=} \hat{\beta}_{T_c^\theta}, \quad (9)$$

where, on the left hand side, $(C_c, c \geq 0)$ denotes a standard Cauchy process and on the right-hand side, $(\hat{\beta}_u, u \geq 0)$ is a one-dimensional Brownian motion, independent from T_c^θ .

We may now identify the densities of the variables found on both sides of (9), i.e.,

$$\begin{aligned} \text{on the left-hand side: } & \frac{1}{\sqrt{1+x^2}} h_c(\arg \text{sh } x) = \frac{1}{\sqrt{1+x^2}} h_c(a(x)), \\ \text{on the right-hand side: } & \mathbf{E} \left[\frac{1}{\sqrt{2\pi T_c^\theta}} \exp \left(-\frac{x^2}{2T_c^\theta} \right) \right], \end{aligned}$$

where $a(x) = \arg \text{sh } x$.

Thus, we have obtained the following proposition.

Proposition 2.2. *The distribution of T_c^θ may be characterized by*

$$\mathbf{E} \left[\frac{1}{\sqrt{2\pi T_c^\theta}} \exp \left(-\frac{x}{2T_c^\theta} \right) \right] = \frac{1}{\sqrt{1+x}} \frac{c}{\pi(c^2 + \ln^2(\sqrt{x} + \sqrt{1+x}))}, \quad (10)$$

for $x \geq 0$.

The proof of Proposition 2.2 follows from the identity $a(y) = \arg \text{sh } y \equiv \ln(y + \sqrt{1+y^2})$ and by making the change of variable $y^2 = x$. Let us now define the probability

$$\mathbf{Q}_c = \sqrt{\frac{\pi c^2}{2T_c^\theta}} \cdot \mathbf{P}.$$

The fact that $\mathbf{Q}_c$ is a probability follows from (10) by taking $x = 0$. Thus we obtain that $c \mathbf{E}[\sqrt{\pi/(2T_c^\theta)}] = 1$, and we can write

$$\mathbf{E}_{\mathbf{Q}_c} \left[\exp \left(-\frac{x}{2T_c^\theta} \right) \right] = \frac{1}{\sqrt{1+x}} \frac{1}{1 + c^{-2} \ln^2(\sqrt{x} + \sqrt{1+x})}, \quad (11)$$

for $\forall x \geq 0$, which yields the Laplace transform of $1/T_c^\theta$ under $\mathbf{Q}_c$.

Let us now take a look at what happens if we make c tend to ∞ . Denote by $T_1^\beta \equiv \inf\{t: \beta_t = 1\}$ the first hitting time of level 1 for a standard

Brownian motion β and by N a standard Gaussian variable $\mathcal{N}(0, 1)$. Then, from equation (11), we obtain

$$\lim_{c \rightarrow \infty} \mathbf{E}_{\mathbf{Q}_c}[e^{-x/(2T_c^\theta)}] = \mathbf{E}[e^{-xN^2/2}] = \mathbf{E}[e^{-x/(2T_1^\beta)}], \quad (12)$$

which means that $T_c^\theta \xrightarrow{\text{law}} T_1^\beta$ as $c \rightarrow \infty$. (At this point, one may wonder whether there is some kind of convergence in law involving $(\theta_u, u \geq 0)$, under $\mathbf{Q}_c$, as $c \rightarrow \infty$, but we shall not touch this point.)

Proposition 2.2 has the following consequence.

Corollary 2.1. *Let $\varphi(x)$ denote the Laplace transform (11), that is the Laplace transform of $1/(2T_c^\theta)$ under $\mathbf{Q}_c$. Then, the Laplace transform of $1/(2T_c^\theta)$ under $\mathbf{P}$ is*

$$\mathbf{E}\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \int_x^\infty \frac{dw}{\sqrt{w-x}} \varphi(w). \quad (13)$$

P r o o f. By Fubini's theorem, we deduce from (11) that

$$\begin{aligned} \mathbf{E}\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] &= \int_0^\infty \frac{dy}{\sqrt{y}} \mathbf{E}\left[\frac{1}{\sqrt{2\pi T_c^\theta}} \exp\left(-\frac{x+y}{2T_c^\theta}\right)\right] \\ &= \int_0^\infty \frac{dy}{\sqrt{y}} \varphi(x+y) \stackrel{y=xt}{=} \sqrt{x} \int_0^\infty \frac{dt}{\sqrt{t}} \varphi(x(1+t)) \\ &\stackrel{v=1+t}{=} \sqrt{x} \int_1^\infty \frac{dv}{\sqrt{v-1}} \varphi(xv) \stackrel{w=xv}{=} \int_x^\infty \frac{dw}{\sqrt{w-x}} \varphi(w), \end{aligned}$$

which is formula (13).

2.3. Some related identities in law. This subsection is strongly related to [14]. A slightly different look at the combination of Bougerol's identity (8) and the skew-product representation (1) leads to the following striking identities in law.

Proposition 2.3. *Let $(\delta_u, u \geq 0)$ be a one-dimensional Brownian motion independent of the planar Brownian motion $(Z_u, u \geq 0)$ starting from $1 + i0$. Then, for any $b \geq 0$, the following identities in law hold:*

$$(i) \quad H_{T_b^\delta} \stackrel{\text{law}}{=} T_{a(b)}^\beta, \quad (ii) \quad \theta_{T_b^\delta} \stackrel{\text{law}}{=} C_{a(b)}, \quad (iii) \quad \bar{\theta}_{T_b^\delta} \stackrel{\text{law}}{=} |C_{a(b)}|,$$

where C_A is a Cauchy variable with parameter A and $\bar{\theta}_u = \sup_{s \leq u} \theta_s$.

P r o o f. By the symmetry principle (see [3] for the original note and [16] for a detailed discussion), Bougerol's identity can be stated equivalently as

$$\text{sh } \bar{\beta}_u \stackrel{\text{law}}{=} \bar{\delta}_{A_u(\beta)}. \quad (14)$$

Consequently, the laws of the first hitting times of a fixed level b by the processes on each side of (14) are identical, that is,

$$T_{a(b)}^\beta \stackrel{\text{law}}{=} H_{T_b^\delta},$$

which is (i).

(ii) follows from (i), since

$$\theta_u \stackrel{\text{law}}{=} \gamma_{H_u},$$

where $(\gamma_s, s \geq 0)$ is a Brownian motion independent of $(H_u, u \geq 0)$ and $(C_u, u \geq 0)$ can be represented as $(\gamma_{T_u^\theta}, u \geq 0)$.

(iii) follows from (ii), again with the help of the symmetry principle.

R e m a r k. Proposition 2.2 can also be derived from (iii) in Proposition 2.3. Indeed, for $c > 0$, starting from the left-hand side of (iii), and letting $N \sim \mathcal{N}(0, 1)$ independent from T_c^θ :

$$\begin{aligned} \mathbf{P}(\bar{\theta}_{T_b^\delta} < c) &= \mathbf{P}(T_b^\delta < T_c^\theta) = \mathbf{P}(b < \bar{\delta}_{T_c^\theta}) \\ &= \mathbf{P}\left(b < \sqrt{T_c^\theta}|N|\right) = \mathbf{P}\left(\frac{b}{\sqrt{T_c^\theta}} < |N|\right) \\ &= \sqrt{\frac{2}{\pi}} \mathbf{E}\left[\int_{b/\sqrt{T_c^\theta}}^\infty dy e^{-y^2/2}\right], \end{aligned} \quad (15)$$

while, on the right-hand side of (iii):

$$\mathbf{P}(|C_{a(b)}| < c) = 2 \int_0^c \frac{a(b) dy}{\pi(a^2(b) + y^2)} \stackrel{y=a(b)h}{=} \frac{2}{\pi} \int_0^{c/a(b)} \frac{dh}{1+h^2}. \quad (16)$$

Taking derivatives in (15) and (16) with respect to b and changing the variables $b = \sqrt{x}$, we obtain Proposition 2.2.

2.4. Recovering Spitzer's theorem. The identity (ii) in Proposition 2.3 is reminiscent of Williams' remark (see [34], [27]), that:

$$H_{T_r^R} \stackrel{\text{law}}{=} T_{\ln r}^\delta, \quad (17)$$

where R starts from 1 and δ starts from 0 (in fact, this is a consequence of (2)). For a number of variants of (17), see [36], [24]. This was D. Williams' starting point for a noncomputational proof of Spitzer's result (4). We note that in (ii), T_b^δ is independent of the process $(\theta_u, u \geq 0)$ while in (17) T_r^R depends on $(\theta_u, u \geq 0)$. Actually, we can mimic Williams' «pinching method» to derive Spitzer's theorem (4) from (ii) in Proposition 2.3.

Proposition 2.4 (A new proof of the Spitzer theorem). *As $t \rightarrow \infty$, $\theta_{T_{\sqrt{t}}^\delta} - \theta_t$ converges in law, which implies that*

$$\frac{1}{\ln t} \left(\theta_{T_{\sqrt{t}}^\delta} - \theta_t \right) \xrightarrow[t \rightarrow \infty]{\mathbf{P}} 0, \quad (18)$$

and, in turn, implies Spitzer's theorem (see formula (4)):

$$\frac{2}{\ln t} \theta_t \xrightarrow[t \rightarrow \infty]{\text{law}} C_1.$$

P r o o f. From equation (ii) of Proposition 2.3 we note that

$$\frac{1}{\ln b} \theta_{T_b^\delta} \stackrel{\text{law}}{=} \frac{C_{a(b)}}{\ln b} \xrightarrow[b \rightarrow \infty]{\text{law}} C_1.$$

So, for $b = \sqrt{t}$ we have

$$\frac{2}{\ln t} \theta_{T_{\sqrt{t}}^\delta} \xrightarrow[b \rightarrow \infty]{\text{law}} C_1.$$

On the other hand, following Williams' «pinching method», we note that

$$\frac{1}{\ln t} (\theta_{T_{\sqrt{t}}^\delta} - \theta_t) \xrightarrow[t \rightarrow \infty]{\text{law}} 0;$$

since $Z_u = x_0 + Z_u^{(0)}$ and also, by making the change of variables $u = tv$ and using the scaling property, we obtain

$$\theta_{T_{\sqrt{t}}^\delta} - \theta_t \equiv \text{Im} \left(\int_t^{T_{\sqrt{t}}^\delta} \frac{dZ_u}{Z_u} \right) \xrightarrow[t \rightarrow \infty]{\text{law}} \text{Im} \left(\int_1^{T_1^\delta} \frac{dZ_v^{(0)}}{Z_v^{(0)}} \right).$$

Here, the limit variable is, in our opinion, of no other interest than its existence which implies (18), hence (4).

2.5. On the distributions of $T_c^\theta \equiv T_{-\infty, c}^\theta$ and $T_{-c, c}^\theta$.

Proposition 2.5. *The following asymptotic equivalence holds:*

$$(\ln t) \mathbf{P}(T_c^\theta > t) \xrightarrow[t \rightarrow \infty]{} \frac{4c}{\pi}. \quad (19)$$

As a consequence, for $\eta > 0$, $\mathbf{E}[(\ln T_c^\theta)_+^\eta] < \infty$ if and only if $\eta < 1$ (where $(\cdot)_+$ denotes the positive part).

P r o o f. a) We rely upon the asymptotic distribution of $H_t \equiv \int_0^t |Z_s|^{-2} ds$ which is given by [31]:

$$\frac{4H_t}{(\ln t)^2} \xrightarrow[t \rightarrow \infty]{\text{law}} T_1^\beta \equiv \inf\{t: \beta_t = 1\}, \quad (20)$$

or, equivalently,

$$\frac{\ln t}{2\sqrt{H_t}} \xrightarrow[t \rightarrow \infty]{\text{law}} |N|, \quad (21)$$

where N is a standard Gaussian variable $\mathcal{N}(0, 1)$.

We note that, by the representation (2) of θ_t , the result (20) is equivalent to Spitzer's theorem [32]:

$$\frac{2\theta_t}{\ln t} \xrightarrow[t \rightarrow \infty]{\text{law}} C_1 \stackrel{\text{law}}{=} \gamma_{T_1^\beta}, \quad (22)$$

where C_1 is a standard Cauchy variable.

b) We shall now use this, in order to deduce Proposition 2.5. We denote $S_t^\theta \equiv \sup_{s \leq t} \theta_s \equiv S_{H_t}^\gamma$ and we note that (from scaling)

$$\mathbf{P}(T_c^\theta \geq t) = \mathbf{P}(S_{H_t}^\gamma \leq c) = \mathbf{P}(\sqrt{H_t} S_1^\gamma \leq c), \quad (23)$$

since γ and H are independent. Thus, we have (since $S_1^\gamma \stackrel{\text{law}}{=} |N|$) and by making the change of variable $x = cy/\sqrt{H_t}$:

$$\begin{aligned} \mathbf{P}(T_c^\theta \geq t) &= \sqrt{\frac{2}{\pi}} \mathbf{E} \left[\int_0^{c/\sqrt{H_t}} dx e^{-x^2/2} \right] \\ &= \sqrt{\frac{2}{\pi}} c \mathbf{E} \left[\int_0^1 \frac{dy}{\sqrt{H_t}} \exp \left(-\frac{c^2 y^2}{2H_t} \right) \right]. \end{aligned} \quad (24)$$

We now deduce from (21) that

$$\frac{\ln t}{2} \mathbf{P}(T_c^\theta \geq t) \xrightarrow[t \rightarrow \infty]{} \sqrt{\frac{2}{\pi}} c \mathbf{E}[|N|] = \frac{2}{\pi} c. \quad (25)$$

which is precisely (19).

It is now elementary to deduce from (25) that, for $\eta > 0$,

$$\mathbf{E}[(\ln T_c^\theta)_+^\eta] < \infty \quad \Leftrightarrow \quad 0 < \eta < 1,$$

since (25) is equivalent to

$$u \mathbf{P}(\ln T_c^\theta > u) \xrightarrow[u \rightarrow \infty]{} \frac{4c}{\pi}. \quad (26)$$

Consequently, Fubini's theorem yields

$$\mathbf{E}[(\ln T_c^\theta)_+^\eta] = \int_0^\infty du \eta u^{\eta-1} \mathbf{P}(\ln T_c^\theta > u),$$

and by (26) this is finite if and only if

$$\int_0^\infty du u^{\eta-2} < \infty \quad \Leftrightarrow \quad \eta < 1.$$

So, $\mathbf{E}[(\ln T_c^\theta)_+^\eta] < \infty$ if and only if $0 < \eta < 1$. Proposition 2.5 is proved.

Now we give several examples of random times $T: C(\mathbf{R}_+, \mathbf{R}) \rightarrow \mathbf{R}_+$ which can be studied quite similarly to T_c^θ . For such times T , it will always

be true that: $H_{T(\theta)} = T(\gamma)$ is equivalent to $T(\theta) = A_{T(\gamma)}$, defined with respect to Z , issued from $x_0 \neq 0$. Using Bougerol's identity, we obtain

$$\text{sh } \beta_{T(\gamma)} \stackrel{\text{law}}{=} \hat{\beta}_{A_{T(\gamma)}} = \hat{\beta}_{T(\theta)}, \quad (27)$$

where $(\hat{\beta}_u, u \geq 0)$ is a one-dimensional Brownian motion independent of (β, γ) (or equivalently, of Z). Consequently, denoting by h_T the density of $\beta_{T(\gamma)}$, we deduce from (27) that

$$\mathbf{E} \left[\frac{1}{\sqrt{2\pi T(\theta)}} \exp \left(-\frac{x^2}{2T(\theta)} \right) \right] = \frac{1}{\sqrt{1+x^2}} h_T \left(\ln(x + \sqrt{1+x^2}) \right), \quad (28)$$

or equivalently, changing x in $\sqrt{x}$, we obtain

$$\mathbf{E} \left[\frac{1}{\sqrt{2\pi T(\theta)}} \exp \left(-\frac{x}{2T(\theta)} \right) \right] = \frac{1}{\sqrt{1+x}} h_T \left(\ln(\sqrt{x} + \sqrt{1+x}) \right). \quad (29)$$

In a number of cases, h_T is known explicitly. We give two examples.

(i) First, we study the exit time from a cone for the planar Brownian motion, and we have

$$T(\gamma) = T_{-d,c}^\gamma \Leftrightarrow T(\theta) = \int_0^{T_{-d,c}^\gamma} ds \exp(2\beta_s) = T_{-d,c}^\theta.$$

Thus

$$\mathbf{E} \left[\frac{1}{\sqrt{2\pi T_{-d,c}^\theta}} \exp \left(-\frac{x}{2T_{-d,c}^\theta} \right) \right] = \frac{1}{\sqrt{1+x}} h_{-d,c} \left(\ln(\sqrt{x} + \sqrt{1+x}) \right), \quad (30)$$

where $h_{-d,c}$ is the density of the variable $\beta_{T_{-d,c}^\gamma}$. The law of $\beta_{T_{-d,c}^\gamma}$ can be obtained from its characteristic function which is given by [31, p. 73]:

$$\mathbf{E} \left[\exp \left(i\lambda \beta_{T_{-d,c}^\gamma} \right) \right] = \mathbf{E} \left[\exp \left(-\frac{\lambda^2}{2} T_{-d,c}^\gamma \right) \right] = \frac{\text{ch}(\lambda(c-d)/2)}{\text{ch}(\lambda(c+d)/2)}.$$

In particular, for $c = d$, we recover the very classical formula:

$$\mathbf{E} \left[\exp \left(i\lambda \beta_{T_{-c,c}^\gamma} \right) \right] = \frac{1}{\text{ch}(\lambda c)}.$$

It is well known that (see [23], [6])

$$\begin{aligned} \mathbf{E} \left[\exp \left(i\lambda \beta_{T_{-c,c}^\gamma} \right) \right] &= \frac{1}{\text{ch}(\lambda c)} = \frac{1}{\text{ch}(\pi(\lambda c/\pi))} \\ &= \int_{-\infty}^{\infty} e^{i(\lambda c/\pi)x} \frac{1}{2\pi} \frac{1}{\text{ch}(x/2)} dx \\ &\stackrel{y=cx/\pi}{=} \int_{-\infty}^{\infty} e^{i\lambda y} \frac{1}{2\pi} \frac{\pi/c}{\text{ch}(y\pi/(2c))} dy \\ &= \int_{-\infty}^{\infty} e^{i\lambda y} \frac{1}{2c} \frac{1}{\text{ch}(y\pi/(2c))} dy. \end{aligned} \quad (31)$$

Hence, the density of $\beta_{T_{-c,c}^\gamma}$ is

$$h_{-c,c}(x) = \frac{1}{2c} \frac{1}{\operatorname{ch}(x\pi/(2c))} = \frac{1}{c} \frac{1}{e^{x\pi/(2c)} + e^{-x\pi/(2c)}},$$

and

$$h_{-c,c}(\ln(\sqrt{x} + \sqrt{1+x})) = \frac{1}{c} \frac{1}{(\sqrt{x} + \sqrt{1+x})^\zeta + (\sqrt{x} + \sqrt{1+x})^{-\zeta}},$$

where $\zeta = \pi/(2c)$. However using the identity

$$(\sqrt{x} + \sqrt{1+x})^{-\zeta} = (\sqrt{1+x} - \sqrt{x})^\zeta, \quad (32)$$

we obtain

$$h_{-c,c}(\ln(\sqrt{x} + \sqrt{1+x})) = \frac{1}{c} \frac{1}{(\sqrt{x} + \sqrt{1+x})^\zeta + (\sqrt{1+x} - \sqrt{x})^\zeta}. \quad (33)$$

So we deduce that (for $c = d$)

$$\begin{aligned} \mathbf{E} \left[\frac{1}{\sqrt{2\pi T_{-c,c}^\theta}} \exp \left(-\frac{x}{2T_{-c,c}^\theta} \right) \right] \\ = \frac{1}{c} \frac{1}{\sqrt{1+x}} \frac{1}{(\sqrt{x} + \sqrt{1+x})^\zeta + (\sqrt{1+x} - \sqrt{x})^\zeta}. \end{aligned} \quad (34)$$

(ii) As a second example of a random time T , let us consider the time introduced in [33], [10, Exercise 6.2, p. 178] (we use a slightly different notation). Let $(\beta_t, t \geq 0)$ be a real-valued Brownian motion and define, for $c > 0$:

$$\begin{aligned} T(\theta) &\equiv T_c^{\hat{\theta}} = \inf \left\{ t: \sup_{s \leq t} \theta_s - \inf_{s \leq t} \theta_s = c \right\}, \\ T(\gamma) &\equiv T_c^{\hat{\gamma}} = \inf \left\{ t: \sup_{s \leq t} \gamma_s - \inf_{s \leq t} \gamma_s = c \right\}. \end{aligned}$$

Thus, from the skew-product representation (1), $\theta_u \equiv \gamma_{H_u}$, by replacing $u = T_c^{\hat{\theta}}$, we obtain

$$H_{T_c^{\hat{\theta}}} = T_c^{\hat{\gamma}} \quad \Rightarrow \quad T_c^{\hat{\theta}} = \int_0^{T_c^{\hat{\gamma}}} ds \exp(2\beta_s) \equiv A_{T_c^{\hat{\gamma}}}.$$

Thus,

$$\mathbf{E} \left[\frac{1}{\sqrt{2\pi T_c^{\hat{\theta}}}} \exp \left(-\frac{x}{2T_c^{\hat{\theta}}} \right) \right] = \frac{1}{\sqrt{1+x}} h_c(\ln(\sqrt{x} + \sqrt{1+x})), \quad (35)$$

where h_c is the density of the variable $\beta_{T_c^\gamma}$. The law of $\beta_{T_c^\gamma}$ can be obtained from its characteristic function which is given by (see [6], [10])

$$\begin{aligned} \mathbf{E}[\exp(i\lambda\beta_{T_c^\gamma})] &= \mathbf{E}\left[\exp\left(-\frac{\lambda^2}{2}T_c^\gamma\right)\right] \\ &= \frac{1}{(\operatorname{ch}(\lambda c/2))^2} = \frac{1}{(\operatorname{ch}(\pi\lambda c/(2\pi)))^2} = \int_{-\infty}^{\infty} e^{i(\lambda c/(2\pi))x} \frac{1}{2\pi} \frac{x}{\operatorname{sh}(x/2)} dx \\ &\stackrel{y=cx/(2\pi)}{=} \int_{-\infty}^{\infty} e^{i\lambda y} \frac{1}{2\pi} \frac{2\pi y/c}{\operatorname{sh}(\pi y/c)} \frac{2\pi}{c} dy = \int_{-\infty}^{\infty} e^{i\lambda y} \frac{2\pi}{c^2} \frac{y}{\operatorname{sh}(\pi y/c)} dy. \end{aligned} \quad (36)$$

So, the density of $\beta_{T_c^\gamma}$ is

$$h_c(y) = \frac{2\pi y}{c^2} \frac{1}{\operatorname{sh}(\pi y/c)} = \frac{4\pi}{c^2} \frac{y}{e^{\pi y/c} - e^{-\pi y/c}},$$

and

$$h_c\left(\ln(\sqrt{x} + \sqrt{1+x})\right) = \frac{4\pi}{c^2} \frac{\ln(\sqrt{x} + \sqrt{1+x})}{(\sqrt{x} + \sqrt{1+x})^{\hat{\zeta}} - (\sqrt{x} + \sqrt{1+x})^{-\hat{\zeta}}},$$

where $\hat{\zeta} = \pi/c$. Thus

$$\begin{aligned} \mathbf{E}\left[\frac{1}{\sqrt{2\pi T_c^\theta}} \exp\left(-\frac{x}{2T_c^\theta}\right)\right] \\ = \frac{4\pi}{c^2} \frac{1}{\sqrt{1+x}} \frac{\ln(\sqrt{x} + \sqrt{1+x})}{(\sqrt{x} + \sqrt{1+x})^{\hat{\zeta}} - (\sqrt{1+x} - \sqrt{x})^{\hat{\zeta}}}. \end{aligned} \quad (37)$$

We note that this study can be related to [30]; and, more precisely, $\beta_{T_c^\gamma}$ and T_c^γ correspond to the variables C_2 and $\widehat{C}_2$, respectively (see, e.g., [30, Table 6, p. 312]).

Let us now return to the case of $T_{-c,c}^\theta$ (example (i)). More specifically, we shall obtain its density function $f(t)$.

Proposition 2.6. *The density function f of $T_{-c,c}^\theta$ is given by*

$$f(t) = \frac{1}{\sqrt{2}c} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu_k)}{\Gamma(2\nu_k)} \frac{1}{\sqrt{t}} e^{-1/(4t)} M_{1/2, \nu_k}\left(\frac{1}{2t}\right), \quad (38)$$

where $M_{a,b}(\cdot)$ is the Whittaker function with parameters a, b . Equivalently,

$$\begin{aligned} f(t) &= \frac{\sqrt{2}}{c} \sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{t}} e^{-1/(2t)} \left(\frac{1}{2t}\right)^{\nu_k+1/2} \nu_k \\ &\quad \times \sum_{n=0}^{\infty} \frac{\Gamma(\nu_k+n)}{\Gamma(2\nu_k+n+1)} \frac{1}{n!} \left(\frac{1}{2t}\right)^n, \end{aligned} \quad (39)$$

where $\nu_k = \pi(2k+1)/(4c)$.

P r o o f. The following calculation relies upon [11]. We denote

$$\varphi_{\zeta}(x) = (\sqrt{x} + \sqrt{1+x})^{\zeta} + (\sqrt{1+x} - \sqrt{x})^{\zeta}.$$

Noting that

$$\sqrt{1+x} = \operatorname{ch} \frac{y}{2} \iff y = 2 \arg \operatorname{ch}(\sqrt{1+x}), \quad (40)$$

we get

$$\varphi_{\zeta}(x) = \left(\operatorname{sh} \frac{y}{2} + \operatorname{ch} \frac{y}{2} \right)^{\zeta} + \left(\operatorname{ch} \frac{y}{2} - \operatorname{sh} \frac{y}{2} \right)^{\zeta} = 2 \operatorname{ch} \frac{y\zeta}{2}.$$

Thus, from (34), we have

$$II \equiv \mathbf{E} \left[\frac{1}{\sqrt{2\pi T_{-c,c}^{\theta}}} \exp \left(-\frac{x}{2T_{-c,c}^{\theta}} \right) \right] = \frac{1}{\psi} \frac{1}{\operatorname{ch}(y/2)} \frac{1}{\operatorname{ch}(\pi y/(2\psi))}, \quad (41)$$

where $\psi = 2c$. However, expanding $\operatorname{ch}(\pi y/(2\psi))$, we get

$$\frac{1}{\operatorname{ch}(\pi y/(2\psi))} = 2 \frac{e^{-\pi y/(2\psi)}}{1 + e^{-\pi y/\psi}} = 2 \sum_{k=0}^{\infty} (-e^{-\pi y/\psi})^k e^{-\pi y/(2\psi)},$$

and from (41), we deduce that

$$\begin{aligned} II &= \sum_{k=0}^{\infty} \frac{2}{\psi} \frac{(-1)^k}{\operatorname{ch}(y/2)} e^{-\pi(2k+1)y/(2\psi)} \\ &= \sum_{k=0}^{\infty} \frac{4(-1)^k}{\psi \sqrt{2} \sqrt{2 \operatorname{sh}(y/2) \operatorname{ch}(y/2)}} \sqrt{\frac{\operatorname{sh}(y/2)}{\operatorname{ch}(y/2)}} e^{-\nu_k y} \\ &= \sum_{k=0}^{\infty} \frac{4(-1)^k}{\psi \sqrt{2} \sqrt{2 \operatorname{sh}(y/2) \operatorname{ch}(y/2)}} \sqrt{\operatorname{th} \frac{y}{2}} e^{-\nu_k y}, \end{aligned}$$

where $\nu_k = \pi(2k+1)/(2\psi)$.

From (40), we have that $1+x = \operatorname{ch}^2(y/2)$ if and only if $x = \operatorname{sh}^2(y/2)$, thus

$$\left(\operatorname{th} \frac{y}{2} \right)^{1/2} = \sqrt{\frac{\operatorname{sh}(y/2)}{\operatorname{ch}(y/2)}} = \left(\frac{\sqrt{x}}{\sqrt{1+x}} \right)^{1/2} = \left(\frac{x}{1+x} \right)^{1/4}.$$

Moreover, we know that (see [1, equation 8.6.10] or [22])

$$i \sqrt{\frac{\pi}{2 \operatorname{sh} y}} e^{-\nu_k y} = Q_{\nu_k - 1/2}^{1/2}(\operatorname{ch} y),$$

where $\{Q_b^a(\cdot)\}$ is the family of Legendre functions and $\text{ch } y = 2x + 1$. So, we deduce

$$II = \sum_{k=0}^{\infty} \frac{4(-i)}{\psi\sqrt{\pi}} (-1)^k \left(\frac{x}{1+x}\right)^{1/4} Q_{\nu_k-1/2}^{1/2}(2x+1). \quad (42)$$

By using formula (7.621.9) in [17]

$$\int_0^{\infty} e^{-sw} M_{l,\nu_k}(w) \frac{dw}{w} = \frac{2\Gamma(1+2\nu_k) e^{-i\pi l}}{\Gamma(1/2+\nu_k+l)} \left(\frac{s-1/2}{s+1/2}\right)^{l/2} Q_{\nu_k-1/2}^l(2s), \quad (43)$$

with $l = 1/2$, $\nu_k = \pi(2k+1)/(2\psi)$, $s = x + 1/2$ and $M_{\cdot,\cdot}(\cdot)$ denoting the Whittaker function, which is defined as

$$M_{a,b}(w) = w^{b+1/2} e^{-w/2} \frac{\Gamma(2b+1)}{\Gamma(1/2+b-a)} \sum_{n=0}^{\infty} \frac{\Gamma(1/2+b-a+n)}{\Gamma(2b+1+n)} \frac{w^n}{n!},$$

we have

$$-2i \frac{\Gamma(1+2\nu_k)}{\Gamma(1+\nu_k)} \left(\frac{x}{1+x}\right)^{1/4} Q_{\nu_k-1/2}^{1/2}(2x+1) = \int_0^{\infty} e^{-sw} M_{1/2,\nu_k}(w) \frac{dw}{w}. \quad (44)$$

From (42) and by changing the variable $w = 1/(2t)$, we deduce

$$\begin{aligned} II &= \sum_{k=0}^{\infty} \frac{2}{\psi\sqrt{\pi}} (-1)^k \frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)} \int_0^{\infty} \frac{dw}{w} \exp\left(-w\left(x+\frac{1}{2}\right)\right) M_{1/2,\nu_k}(w) \\ &= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{dt}{t} \frac{2}{\psi\sqrt{\pi}} (-1)^k \frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)} \exp\left(-\frac{1}{4t} - \frac{x}{2t}\right) M_{1/2,\nu_k}\left(\frac{1}{2t}\right). \end{aligned} \quad (45)$$

By using the equations (41) and (45), we conclude that

$$\begin{aligned} &\mathbf{E}\left[\frac{1}{\sqrt{2\pi T_{-c,c}^{\theta}}} \exp\left(-\frac{x}{2T_{-c,c}^{\theta}}\right)\right] \\ &= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{dt}{t} \frac{2}{\psi\sqrt{\pi}} (-1)^k \frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)} \exp\left(-\frac{1}{4t} - \frac{x}{2t}\right) M_{1/2,\nu_k}\left(\frac{1}{2t}\right) \\ &= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{dt}{t} \frac{2}{\psi\sqrt{\pi}} (-1)^k \frac{\Gamma(\pi(4c)^{-1}(2k+1)+1)}{\Gamma(2\pi(4c)^{-1}(2k+1)+1)} \\ &\quad \times \exp\left(-\frac{\pi}{4t} - \frac{x}{2t}\right) M_{1/2,\pi(4c)^{-1}(2k+1)}\left(\frac{1}{2t}\right). \end{aligned} \quad (46)$$

Thus, the density function f of $T_{-c,c}^{\theta}$ is given by

$$f(t) = \frac{2\sqrt{2}}{\psi} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)} \frac{1}{\sqrt{t}} e^{-1/(4t)} M_{1/2,\nu_k}\left(\frac{1}{2t}\right)$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{c} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\pi(4a)^{-1}(2k+1)+1)}{\Gamma(\pi/(2a)(2k+1)+1)} \frac{1}{\sqrt{t}} e^{-1/(4t)} \\
&\quad \times M_{1/2, \pi(4a)^{-1}(2k+1)} \left(\frac{1}{2t} \right) \\
&= \frac{\sqrt{2}}{c} \sum_{k=0}^{\infty} (-1)^k \frac{\nu_k \Gamma(\nu_k)}{2\nu_k \Gamma(2\nu_k)} \frac{1}{\sqrt{t}} e^{-1/(4t)} M_{\frac{1}{2}, \nu_k} \left(\frac{1}{2t} \right), \quad (47)
\end{aligned}$$

where the Whittaker function $M_{1/2, \nu_k}(1/(2t))$ is

$$\begin{aligned}
M_{1/2, \pi(4c)^{-1}(2k+1)} \left(\frac{1}{2t} \right) &= \left(\frac{1}{2t} \right)^{\pi(4c)^{-1}(2k+1)+1/2} \\
&\times e^{-1/(4t)} \frac{\Gamma(\pi(2c)^{-1}(2k+1)+1)}{\Gamma(\pi(4c)^{-1}(2k+1))} \\
&\times \sum_{n=0}^{\infty} \frac{\Gamma(\pi(4c)^{-1}(2k+1)+n)}{\Gamma(\pi(2c)^{-1}(2k+1)+1+n)} \frac{1}{n!} \left(\frac{1}{2t} \right)^n \\
&= \left(\frac{1}{2t} \right)^{\nu_k+1/2} e^{-1/(4t)} \frac{\Gamma(2\nu_k+1)}{\Gamma(\nu_k)} \sum_{n=0}^{\infty} \frac{\Gamma(\nu_k+n)}{\Gamma(2\nu_k+1+n)} \frac{1}{n!} \left(\frac{1}{2t} \right)^n \\
&= \left(\frac{1}{2t} \right)^{\nu_k+1/2} e^{-1/(4t)} \cdot 2\nu_k \frac{\Gamma(2\nu_k)}{\Gamma(\nu_k)} \sum_{n=0}^{\infty} \frac{\Gamma(\nu_k+n)}{(2\nu_k+n)\Gamma(2\nu_k+n)} \frac{1}{n!} \left(\frac{1}{2t} \right)^n. \quad (48)
\end{aligned}$$

Thus, from (47) and (48), we deduce (39). Proposition 2.6 is proved.

Next, we present the graphs of different approximations $f_{K,N}(t)$ of $f(t)$, in (39), where $f_{K,N}$ denotes the sum in the series in (39) of the terms for $k \leq K$, and $n \leq N$.

Fig. 3. Graph of $f_{9,9}(t)$, with $c = 2\pi$.

Fig. 4. Graph of $f_{K,N}(t)$ for several values of K and N , with $c = 2\pi$.

R e m a r k s. (i) Figure 3 represents the approximation of the density function f with respect to the time t (for K and $N \leq 9$), with $c = 2\pi$, whereas Figure 4 represents the approximation of f with respect to the time t for several values of k and n , with $c = 2\pi$.

(ii) From Figure 4, we can remark that the approximation K and $N \leq 9$ is sufficiently good (comparing to the one for K and $N \leq 100$).

(iii) For the case K and $N \leq 9$ it seems that locally, in a small area around 0, $f(t) < 0$ which is not right. This is due to the first negative ($k = 1$) term of the sum and to the fact that we have omitted many terms. However, this is not a problem because it appears only locally. Similar irregularities have already been observed in previous articles [18, p. 275].

2.6. On the first moment of $\ln T_{-c,c}^\theta$. This subsection is related to a result in [11].

Proposition 2.7. *The first moment of $\ln T_{-c,c}^\theta$ has the following integral representation:*

$$\mathbf{E}[\ln T_{-c,c}^\theta] = 2 \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} \ln \operatorname{sh}(cz) + \ln 2 + c_E, \quad (49)$$

where $c_E = -\Gamma'(1)$ is the Euler–Mascheroni constant (also called Euler’s constant).

P r o o f. Let us return to equations (2) and (6). So, for $t = T_{-c,c}^\theta$, we have

$$\theta_{T_{-c,c}^\theta} = \gamma_{H_{T_{-c,c}^\theta}} \iff H_{T_{-c,c}^\theta} = T_{-c,c}^\gamma \iff T_{-c,c}^\theta = A_{T_{-c,c}^\gamma}. \quad (50)$$

Thus, for $\varepsilon > 0$

$$\mathbf{E}[(T_{-c,c}^\theta)^\varepsilon] = \mathbf{E}[(A_{T_{-c,c}^\gamma})^\varepsilon].$$

Consider $(\delta_u, u \geq 0)$ a Brownian motion independent of A_t . Then, Bougerol's identity and the scaling property yield ($\mathcal{G}_a$ denotes a gamma variable with parameter a , and $N^2 \stackrel{\text{law}}{=} 2\mathcal{G}_{1/2}$)

$$\begin{aligned} \mathbf{E}[(\text{sh } B_t)^{2\varepsilon}] &= \mathbf{E}[(\delta_{A_t})^{2\varepsilon}] = \mathbf{E}[A_t^\varepsilon (\delta_1)^{2\varepsilon}] = \mathbf{E}[A_t^\varepsilon] \mathbf{E}[(2\mathcal{G}_{1/2})^\varepsilon] \\ &= \mathbf{E}[A_t^\varepsilon] \cdot 2^\varepsilon \frac{\Gamma(1/2 + \varepsilon)}{\Gamma(1/2)}, \end{aligned}$$

because

$$\mathbf{E}[(\mathcal{G}_{1/2})^\varepsilon] = \int_0^\infty x^{\varepsilon+1/2-1} \frac{e^{-x}}{\Gamma(1/2)} dx = \frac{\Gamma(1/2 + \varepsilon)}{\Gamma(1/2)}.$$

Thus, for $t = T_{-c,c}^\gamma$, we have

$$\mathbf{E}[(\text{sh } B_{T_{-c,c}^\gamma})^{2\varepsilon}] = \mathbf{E}[A_{T_{-c,c}^\gamma}^\varepsilon] \cdot 2^\varepsilon \frac{\Gamma(1/2 + \varepsilon)}{\Gamma(1/2)}. \quad (51)$$

Recall that [23], [6]

$$\mathbf{E} \left[\exp(i\lambda B_{T_{-c,c}^\gamma}) \right] = \mathbf{E} \left[\exp \left(-\frac{\lambda^2}{2} T_{-c,c}^\gamma \right) \right] = \frac{1}{\text{ch}(\lambda c)},$$

and the density of $\beta_{T_{-c,c}^\gamma}$ is

$$h_{-c,c}(y) = \frac{1}{2c} \frac{1}{\text{ch}(y\pi/(2c))} = \frac{1}{c} \frac{1}{e^{y\pi/(2c)} + e^{-y\pi/(2c)}}.$$

Thus, on the left-hand side of (51), we have:

$$\begin{aligned} \mathbf{E}[(\text{sh } B_{T_{-c,c}^\gamma})^{2\varepsilon}] &= \int_{-\infty}^\infty \frac{dy}{2c} \frac{1}{\text{ch}(\pi y/(2c))} (\text{sh } y)^{2\varepsilon} \\ &= \int_0^\infty \frac{dy}{c} \frac{1}{\text{ch}(\pi y/(2c))} (\text{sh } y)^{2\varepsilon} \\ &= \int_0^\infty dz \frac{1}{\text{ch}(\pi z/2)} (\text{sh } cz)^{2\varepsilon}, \end{aligned}$$

where we have made the change of variable $z = y/c$. Hence, from (51), by writing

$$\mathbf{E}[A_{T_{-c,c}^\gamma}^\varepsilon] = \mathbf{E}[(T_{-c,c}^\theta)^\varepsilon] = \mathbf{E}[e^{\varepsilon \ln T_{-c,c}^\theta}],$$

we deduce

$$\frac{\Gamma(1/2 + \varepsilon)}{\Gamma(1/2)} \mathbf{E}[e^{\varepsilon \ln T_{-c,c}^\theta}] = \frac{1}{2^\varepsilon} \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} (\operatorname{sh}(cz))^{2\varepsilon},$$

and by removing 1 from both sides, we obtain

$$\frac{\Gamma(1/2 + \varepsilon)}{\Gamma(1/2)} \mathbf{E}[e^{\varepsilon \ln T_{-c,c}^\theta}] - 1 = \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} \left(\frac{(\operatorname{sh}(cz))^{2\varepsilon}}{2^\varepsilon} - 1 \right). \quad (52)$$

On the left-hand side, we apply the trivial identity $ab - 1 = a(b - 1) + a - 1$ with $a = \Gamma(1/2 + \varepsilon)/\Gamma(1/2)$ and $b = \mathbf{E}[e^{\varepsilon \ln T_{-c,c}^\theta}]$, we divide by ε and we take the limit as $\varepsilon \rightarrow 0$. Thus,

$$\frac{a(b - 1)}{\varepsilon} = \frac{\Gamma(1/2 + \varepsilon)}{\Gamma(1/2)} \frac{\mathbf{E}[e^{\varepsilon \ln T_{-c,c}^\theta}] - 1}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \mathbf{E}[\ln T_{-c,c}^\theta],$$

and

$$\begin{aligned} \frac{a - 1}{\varepsilon} &= \frac{1}{\varepsilon} \left(\frac{\Gamma(1/2 + \varepsilon)}{\Gamma(1/2)} - 1 \right) = \frac{1}{\Gamma(1/2)} \left(\frac{\Gamma(1/2 + \varepsilon) - \Gamma(1/2)}{\varepsilon} \right) \\ &\xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\sqrt{\pi}} \Gamma' \left(\frac{1}{2} \right) = \frac{1}{\sqrt{\pi}} (-\sqrt{\pi})(c_E + 2 \ln 2) = -(c_E + 2 \ln 2). \end{aligned}$$

On the right-hand side of (52), we have

$$\frac{1}{\varepsilon} \left[\left(\frac{(\operatorname{sh}(cz))^2}{2} \right)^\varepsilon - 1 \right] = \frac{1}{\varepsilon} \left[\exp \left(\varepsilon \ln \frac{(\operatorname{sh}(cz))^2}{2} \right) - 1 \right],$$

hence

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} \left(\frac{(\operatorname{sh}(cz))^{2\varepsilon}}{2^\varepsilon} - 1 \right) &\xrightarrow{\varepsilon \rightarrow 0} \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} \ln \frac{(\operatorname{sh}(cz))^2}{2} \\ &= -\ln 2 + 2 \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} \ln \operatorname{sh}(cz), \end{aligned}$$

which finishes the proof.

R e m a r k. a) We denote now

$$F(c) \equiv \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} \ln \operatorname{sh}(cz). \quad (53)$$

Thus,

$$F(c) - \ln c \equiv \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} \ln \frac{\operatorname{sh}(cz)}{c} \xrightarrow{c \rightarrow 0} \int_0^\infty \frac{dz \ln z}{\operatorname{ch}(\pi z/2)} \approx -0.7832. \quad (54)$$

b) More generally, we denote

$$F(c, \delta) \equiv \int_0^\infty \frac{dz}{\operatorname{ch}(\delta z)} \ln \operatorname{sh}(cz), \quad (55)$$

and, changing the variables: $z = \frac{\pi}{2\delta}u$, we obtain

$$F(c, \delta) = \frac{\pi}{2\delta} \int_0^\infty \frac{du}{\operatorname{ch}(\pi/2u)} \ln \operatorname{sh} \frac{c\pi u}{2\delta} = \frac{\pi}{2\delta} F \left(c \frac{\pi}{2\delta} \right). \quad (56)$$

3. The Ornstein–Uhlenbeck case.

3.1. An identity in law for Ornstein–Uhlenbeck processes, which is connected to Bougerol’s identity. Consider the complex-valued Ornstein–Uhlenbeck process

$$Z_t = z_0 + \widetilde{Z}_t - \lambda \int_0^t Z_s ds, \quad (57)$$

where $\widetilde{Z}_t$ is a complex-valued Brownian motion, $z_0 \in \mathbf{C}$ and $\lambda \geq 0$, and let $T_c^{(\lambda)} \equiv T_{-c,c}^{\theta^Z} \equiv \inf\{t \geq 0: |\theta_t^Z| = c\}$ (θ_t^Z is the continuous winding process associated to Z) denote the first hitting time of the symmetric conic boundary of angle c for Z . It is well known (see, e.g., [31]) that

$$Z_t = e^{-\lambda t} \left(z_0 + \int_0^t e^{\lambda s} d\widetilde{Z}_s \right) = e^{-\lambda t} \mathbf{B}_{\alpha_t}, \quad (58)$$

where, by Dambis–Dubins–Schwarz theorem, $(\mathbf{B}_t, t \geq 0)$ is a complex-valued Brownian motion starting from z_0 and

$$\alpha_t = \int_0^t e^{2\lambda s} ds = \frac{e^{2\lambda t} - 1}{2\lambda}.$$

We are interested in the study of the continuous winding process $\theta_t^Z = \text{Im}(\int_0^t Z_s^{-1} dZ_s)$, $t \geq 0$. By applying Itô’s formula to (58), we have

$$dZ_s = e^{-\lambda s}(-\lambda)\mathbf{B}_{\alpha_s} ds + e^{-\lambda s} d\mathbf{B}_{\alpha_s}.$$

We divide by Z_s and we obtain

$$\frac{dZ_s}{Z_s} = -\lambda ds + \frac{d\mathbf{B}_{\alpha_s}}{\mathbf{B}_{\alpha_s}},$$

hence

$$\text{Im}\left(\frac{dZ_s}{Z_s}\right) = \text{Im}\left(\frac{d\mathbf{B}_{\alpha_s}}{\mathbf{B}_{\alpha_s}}\right),$$

which means that

$$\theta_t^Z = \theta_{\alpha_t}^{\mathbf{B}}.$$

Thus, the following proposition holds.

Proposition 3.1. *Using the previously introduced notation, we have*

$$\theta_t^Z = \theta_{\alpha_t}^{\mathbf{B}}, \quad (59)$$

and

$$T_c^{(\lambda)} = \frac{1}{2\lambda} \ln \left(1 + 2\lambda T_{-c,c}^{\theta^{\mathbf{B}}} \right), \quad (60)$$

where $T_{-c,c}^{\theta^{\mathbf{B}}}$ is the exit time from a cone of angle c for the complex-valued Brownian motion $\mathbf{B}$.

P r o o f. We define

$$T_c^{(\lambda)} \equiv T_{-c,c}^{\theta^Z} \equiv \inf\{t \geq 0: |\theta_t^Z| = c\} = \inf\{t \geq 0: |\theta_{\alpha t}^B| = c\}. \quad (61)$$

Thus, we deduce that $\alpha_{T_c^{(\lambda)}} = T_{-c,c}^{\theta^B} \equiv T_{-c,c}^{\theta}$. However, $T_{-c,c}^{\theta}$ (the exit time from a cone for the Brownian motion) has already been studied in the previous section and we know the explicit formula of its density function (Proposition 2.6). Thus

$$T_c^{(\lambda)} = \alpha^{-1}(T_{-c,c}^{\theta^B}) = \alpha^{-1}(T_{-c,c}^{\theta}), \quad (62)$$

where $\alpha^{-1}(t) = (2\lambda)^{-1} \ln(1 + 2\lambda t)$. Consequently,

$$T_c^{(\lambda)} = \frac{1}{2\lambda} \ln(1 + 2\lambda T_{-c,c}^{\theta}),$$

and

$$\mathbf{E}[T_c^{(\lambda)}] = \frac{1}{2\lambda} \mathbf{E}[\ln(1 + 2\lambda T_{-c,c}^{\theta})], \quad (63)$$

which finishes the proof.

From now on, for simplicity, we shall take $z_0 = 1$ (but this is really no restriction, as the dependency in z_0 , which is exhibited in (7), is very simple). The following proposition may be considered as an extension of the identity in law (ii) in Proposition 2.3, which results from Bougerol's identity.

Proposition 3.2. *Consider two independent Ornstein–Uhlenbeck processes $(Z_t^\lambda, t \geq 0)$ and $(U_t^\lambda, t \geq 0)$, the first one is complex valued and the second one is real valued, both starting from a point different from 0 and call $T_b^{(\lambda)}(U^\lambda) = \inf\{t \geq 0: e^{\lambda t} U_t^\lambda = b\}$. Then an Ornstein–Uhlenbeck extension of identity in law (ii) in Proposition 2.3 is the following:*

$$\theta_{T_b^{(\lambda)}(U^\lambda)}^{Z^\lambda} \stackrel{\text{law}}{=} C_{a(b)}, \quad (64)$$

where $a(x) = \arg \operatorname{sh} x$.

P r o o f. Consider the second Ornstein–Uhlenbeck process $(U_t^\lambda, t \geq 0)$ independent of the first one. Then, taking equation (58) for U_t^λ , we have

$$e^{\lambda t} U_t^\lambda = \delta_{(e^{2\lambda t} - 1)/(2\lambda)}, \quad (65)$$

where $(\delta_t, t \geq 0)$ is a complex-valued Brownian motion starting from $z_0 = 1$. Thus

$$T_b^{(\lambda)}(U^\lambda) = \frac{1}{2\lambda} \ln(1 + 2\lambda T_b^\delta). \quad (66)$$

Equation (59) for $t = (2\lambda)^{-1} \ln(1 + 2\lambda T_b^\delta)$ (equivalently, $\alpha(t) = T_b^\delta$) becomes (we suppose that $z_0 = 1$)

$$\theta_{T_b^{(\lambda)}(U^\lambda)}^{Z^\lambda} = \theta_{(2\lambda)^{-1} \ln(1 + 2\lambda T_b^\delta)}^{Z^\lambda} = \theta_{u=T_b^\delta}^B \stackrel{\text{law}}{=} C_{a(b)}.$$

3.2. On the distribution of $T_{-c,c}^\theta$ for an Ornstein–Uhlenbeck process. Now we turn to the study of the density function of

$$T_c^{(\lambda)} \equiv T_{-c,c}^{\theta^Z} \equiv \inf\{t \geq 0: |\theta_t^Z| = c\},$$

and its first moment.

Proposition 3.3. *Asymptotically for λ large, for $z_0 = 1$, we have*

$$2\lambda \mathbf{E}[T_c^{(\lambda)}] - \ln(2\lambda) \xrightarrow{\lambda \rightarrow \infty} \mathbf{E}[\ln T_{-c,c}^\theta] \quad (67)$$

and

$$\mathbf{E}[\ln T_{-c,c}^\theta] = 2 \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} \ln \operatorname{sh}(cz) + \ln 2 + c_E, \quad (68)$$

where c_E is Euler's constant.

For $c < \pi/8$, we have the asymptotic equivalence

$$\frac{1}{\lambda} \left(\mathbf{E}[T_c^{(\lambda)}] - \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^2] \right) \xrightarrow{\lambda \rightarrow 0} -\frac{1}{3} \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4]. \quad (69)$$

Equivalently,

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathbf{E}[T_c^{(\lambda)}] = \lim_{\lambda \rightarrow 0} \left[\frac{1}{\lambda} (\mathbf{E}[T_c^{(\lambda)}] - \mathbf{E}[T_c^{(0)}]) \right] = -\frac{1}{3} \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4]. \quad (70)$$

Moreover,

$$\mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4] = \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} (\operatorname{sh}(cz))^4. \quad (71)$$

More precisely, for $c < \pi/8$,

$$\mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4] = \frac{1}{8} \left(\frac{1}{\cos(4c)} - 4 \frac{1}{\cos(2c)} + 3 \right), \quad (72)$$

and asymptotically:

$$\mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4] \underset{c \rightarrow 0}{\simeq} 5c^4. \quad (73)$$

P r o o f. *Case 1: λ large.* Let us return to equation (63). As $\lambda \rightarrow \infty$, we have

$$\begin{aligned} \mathbf{E}[T_c^{(\lambda)}] &= \frac{1}{2\lambda} \mathbf{E} \left[\ln(1 + 2\lambda T_{-c,c}^\theta) \right] = \frac{1}{2\lambda} \mathbf{E} \left[\ln \left(2\lambda \left(T_{-c,c}^\theta + \frac{1}{2\lambda} \right) \right) \right] \\ &= \frac{\ln(2\lambda)}{2\lambda} + \frac{1}{2\lambda} \mathbf{E} \left[\ln \left(T_{-c,c}^\theta + \frac{1}{2\lambda} \right) \right]. \end{aligned}$$

Thus,

$$2\lambda \mathbf{E}[T_c^{(\lambda)}] - \ln(2\lambda) \xrightarrow{\lambda \rightarrow \infty} \mathbf{E}[\ln T_{-c,c}^\theta],$$

which is precisely (67). Moreover, by the integral representation (49) for $\mathbf{E}[\ln T_{-c,c}^\theta]$, we deduce (68).

Case 2: λ small. We shall now study the case $\lambda \rightarrow 0$. We have

$$T_c^{(\lambda)} = \frac{1}{2\lambda} \ln(1 + 2\lambda T_{-c,c}^\theta).$$

For $c < \pi/8$, from (3), (at least) the first two positive moments of $T_{-c,c}^\theta$ are finite: $\mathbf{E}[(T_{-c,c}^\theta)^p] < \infty$ ($p = 1, 2$). We make the elementary computation

$$\begin{aligned} \frac{1}{\lambda} \left(\frac{\ln(1 + 2\lambda x)}{2\lambda} - x \right) &= \frac{1}{\lambda} \left(\frac{1}{2\lambda} \int_1^{1+2\lambda x} \frac{dy}{y} - x \right) \\ &\stackrel{y=1+a}{=} \frac{1}{2\lambda^2} \int_0^{2\lambda x} \left(\frac{1}{1+a} - 1 \right) da \stackrel{a=2\lambda b}{=} -2 \int_0^x \frac{b db}{1 + 2\lambda b} \xrightarrow{\lambda \rightarrow 0} -x^2. \end{aligned}$$

Consequently, by replacing $x = T_{-c,c}^\theta$, we have

$$\frac{1}{\lambda} \left(\mathbf{E}[T_c^{(\lambda)}] - \mathbf{E}[T_{-c,c}^\theta] \right) = \mathbf{E} \left[-2 \int_0^{T_{-c,c}^\theta} \frac{b db}{1 + 2\lambda b} \right].$$

We may now use the dominated convergence theorem [7], since the (db) integral is majorized by $(T_{-c,c}^\theta)^2$, which is integrable. Thus

$$\frac{1}{\lambda} \left(\mathbf{E}[T_c^{(\lambda)}] - \mathbf{E}[T_{-c,c}^\theta] \right) \xrightarrow{\lambda \rightarrow 0} -\mathbf{E}[(T_{-c,c}^\theta)^2].$$

Following the proof of Proposition 2.7 and using Bougerol's identity and the scaling property we obtain

$$\mathbf{E}[(\operatorname{sh} B_t)^2] = \mathbf{E}[\delta_{A_t}^2] = \mathbf{E}[A_t(\delta_1)^2] = \mathbf{E}[A_t] \mathbf{E}[(\delta_1)^2] = \mathbf{E}[A_t].$$

Thus, for $t = T_{-c,c}^\gamma$, we have

$$\mathbf{E}[A_{T_{-c,c}^\gamma}] = \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^2].$$

Similarly,

$$\mathbf{E}[(\operatorname{sh} B_t)^4] = \mathbf{E}[(\delta_{A_t})^4] = \mathbf{E}[(A_t)^2(\delta_1)^4] = \mathbf{E}[(A_t)^2] \mathbf{E}[(\delta_1)^4] = 3\mathbf{E}[(A_t)^2].$$

Thus, for $t = T_{-c,c}^\gamma$, we have

$$\mathbf{E}[(A_{T_{-c,c}^\gamma})^2] = \frac{1}{3} \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4].$$

So, because $A_{T_{-c,c}^\gamma} = T_{-c,c}^\theta$, we deduce (69). In order to prove (70), it suffices to remark that

$$\mathbf{E}[T_c^{(0)}] = \mathbf{E}[T_{-c,c}^\theta] = \mathbf{E}[A_{T_{-c,c}^\gamma}] = \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^2].$$

On the one hand, by using the density of $B_{T_{-c,c}^\gamma}$:

$$\begin{aligned} \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4] &= \int_{-\infty}^{\infty} \frac{dy}{2c} \frac{1}{\operatorname{ch}(\pi y/(2c))} (\operatorname{sh} y)^4 \\ &= \int_0^{\infty} \frac{dy}{c} \frac{1}{\operatorname{ch}(\pi y/(2c))} (\operatorname{sh} y)^4 \\ &\stackrel{z=y/c}{=} \int_0^{\infty} dz \frac{1}{\operatorname{ch}(\pi z/2)} (\operatorname{sh}(cz))^4, \end{aligned}$$

which is finite if and only if $c < \pi/8$. In order to prove this, it suffices to use the standard expressions: $\operatorname{sh} x = (e^x - e^{-x})/2$ and $\operatorname{ch} x = (e^x + e^{-x})/2$. On the other hand (note that $T \equiv T_{-c,c}^\gamma$), we remark that $-B_T \stackrel{\text{law}}{=} B_T$ and [31, Exercise 3.10], $\mathbf{E}[e^{kB_T}] = \mathbf{E}[e^{k^2 T/2}] = 1/\cos(kc)$, for $0 \leq k < \pi(2c)^{-1}$, thus

$$\begin{aligned} \mathbf{E}[(\operatorname{sh} B_T)^4] &= \frac{1}{2^4} \mathbf{E}[(e^{B_T} - e^{-B_T})^4] \\ &= \frac{1}{2^4} \mathbf{E}[e^{4B_T} - 4e^{3B_T - B_T} + 6e^{2B_T - 2B_T} - 4e^{B_T - 3B_T} + e^{-4B_T}] \\ &= \frac{1}{2^4} (2\mathbf{E}[e^{4B_T}] - 8\mathbf{E}[e^{2B_T}] + 6) \\ &= \frac{1}{2^3} \left(\frac{1}{\cos(4c)} - 4 \frac{1}{\cos(2c)} + 3 \right), \end{aligned}$$

which is precisely (72) and this is finite if and only if $c < \pi/8$. Moreover, asymptotically for $c \rightarrow 0$, by using the scaling property, we have

$$\begin{aligned} \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4] &= \mathbf{E}[(\operatorname{sh}(cB_{T_{-1,1}^\gamma}))^4] \underset{c \rightarrow 0}{\simeq} c^4 \mathbf{E}[(B_{T_{-1,1}^\gamma})^4] \\ &= c^4 \cdot 3\mathbf{E}[(T_{-1,1}^\gamma)^2] = c^4 \cdot 3 \frac{5}{3} = 5c^4, \end{aligned}$$

since $\mathbf{E}[(T_{-1,1}^\gamma)^2] = 5/3$ (see [30]; by using the notation of this paper, Table 3: $\mathbf{E}[X_t^2] = t(2 + 3t)/3$ for $X_t = C_1$ and $t = 1$). This asymptotics can also be obtained from (72) by developing $\cos(4c)$ and $\cos(2c)$ into series up to the second-order term and keeping the terms of the order c^4 .

R e m a r k. If we slightly modify the above study for the Ornstein–Uhlenbeck process by inserting a diffusion coefficient D :

$$Z_t = z_0 + \sqrt{2D} \widetilde{Z}_t - \lambda \int_0^t Z_s ds,$$

we obtain

$$Z_t = e^{-\lambda t} \left(z_0 + \sqrt{2D} \int_0^t e^{\lambda s} d\widetilde{Z}_s \right) = e^{-\lambda t} \mathbf{B}_{\alpha_t}, \quad (74)$$

where in the second equation we used the Dambis–Dubins–Schwarz theorem with

$$\alpha_t = 2D \int_0^t e^{2\lambda s} ds = D \frac{e^{2\lambda t} - 1}{\lambda} \Rightarrow \alpha_t^{-1} = \frac{1}{2\lambda} \ln \left(1 + \frac{\lambda}{D} t \right).$$

Thus,

$$2\lambda \mathbf{E}[T_c^{(\lambda)}] - \ln \frac{\lambda}{D} \xrightarrow{\lambda \rightarrow \infty} \mathbf{E}[\ln T_{-c,c}^\theta], \quad (75)$$

because

$$\begin{aligned} \mathbf{E}[T_c^{(\lambda)}] &= \frac{1}{2\lambda} \mathbf{E} \left[\ln \left(1 + \frac{\lambda}{D} T_{-c,c}^\theta \right) \right] = \frac{1}{2\lambda} \mathbf{E} \left[\ln \left(\frac{\lambda}{D} \left(T_{-c,c}^\theta + \frac{D}{\lambda} \right) \right) \right] \\ &= \frac{\ln(\lambda/D)}{2\lambda} + \frac{1}{2\lambda} \mathbf{E} \left[\ln \left(T_{-c,c}^\theta + \frac{D}{\lambda} \right) \right]. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbf{E}[\ln T_{-c,c}^\theta] &= 2 \ln z_0 + \mathbf{E}[\ln T_{-c,c}^{\theta(1)}] \\ &= 2 \ln z_0 + \int_0^\infty \frac{dz}{\operatorname{ch}(\pi z/2)} \ln \operatorname{sh}(cz) + \ln 2 + c_{\mathbf{E}}, \end{aligned} \quad (76)$$

where $T_{-c,c}^{\theta(1)}$ denotes the first hitting time of the symmetric conic boundary of angle c for a Brownian motion Z starting from 1.

For λ small, we replace $2T_{-c,c}^\theta$ by $(z_0^2/D)T_{-c,c}^\theta$ in the proof of Proposition 3.3 (λ small case) and we have

$$T_c^{(\lambda)} = \frac{1}{2\lambda} \ln \left(1 + \lambda \frac{z_0^2}{D} T_{-c,c}^\theta \right).$$

By repeating the previous calculation, we make the elementary computation:

$$\frac{1}{\lambda} \left(\frac{\ln(1 + (z_0^2/D)x)}{2\lambda} - \frac{z_0^2}{D} x \right) = -\frac{1}{2} \int_0^x \frac{(z_0^2/D)^2 b db}{1 + \lambda z_0^2/D b} \xrightarrow{\lambda \rightarrow 0} -\left(\frac{z_0^2}{2D} \right)^2 x^2.$$

We replace $x = T_{-c,c}^\theta$, and by the dominated convergence theorem [7], for $c < \pi/8$, we obtain

$$\begin{aligned} \frac{1}{\lambda} \left(\mathbf{E}[T_c^{(\lambda)}] - \frac{z_0^2}{2D} \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^2] \right) &\xrightarrow{\lambda \rightarrow 0} -\frac{1}{3} \left(\frac{z_0^2}{2D} \right)^2 \mathbf{E}[(T_{-c,c}^\theta)^2] \\ &= -\frac{1}{3} \left(\frac{z_0^2}{2D} \right)^2 \mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4], \end{aligned}$$

where $\mathbf{E}[(\operatorname{sh} B_{T_{-c,c}^\gamma})^4]$ is given by (71), (72) and asymptotically, as $c \rightarrow 0$ by (73).

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