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WU YONGFENG*

CONVERGENCE PROPERTIES OF THE MAXIMAL PARTIAL SUMS FOR
ARRAYS OF ROWWISE NA RANDOM VARIABLES¹⁾

Пусть $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ — схема серий отрицательно ассоциированных случайных величин, $\{a_n, n \geq 1\}$ — последовательность положительных действительных чисел таких, что $a_n \uparrow \infty$. В работе изучаются свойства сходимости максимальных частичных сумм $a_n^{-1} \max_{1 \leq j \leq n} |\sum_{k=1}^j X_{nk}|$. Полученные результаты обобщают соответствующие теоремы в [10].

Ключевые слова и фразы: отрицательно ассоциированные случайные величины, полная сходимость моментов, L^q -сходимость.

1. Introduction and preliminaries. The concept of negatively associated (NA) random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [2].

Definition 1.1. A finite family of random variables $\{X_k, 1 \leq k \leq n\}$ is said to be negatively associated (abbreviated to NA) if for any disjoint subsets A and B of $\{1, 2, \dots, n\}$ and any real coordinate-wise nondecreasing functions f on $\mathbf{R}^A$ and g on $\mathbf{R}^B$,

$$\text{Cov}(f(X_i, i \in A), g(Y_j, j \in B)) \leq 0 \quad (1.1)$$

whenever the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

Obviously sequences of NA random variables are a family of very wide scope, which contain independent random variable sequences. NA has been applied to the reliability theory, multivariate statistical analysis and percolation theory, and attracted extensive attentions. Since the paper of Joag-Dev and Proschan appeared, the convergence properties of NA random sequences have been studied by Matula [3], Su et al. [4], Shao [5], Shao and Su [6], Liang and Su [7], Liang [8], Roussas [9], Gan and Chen [10].

Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0, b_n > 0, q > 0$. If

$$\sum_{n=1}^{\infty} a_n \mathbf{E} \{b_n^{-1} |Z_n| - \varepsilon\}_+^q < \infty \text{ for some or, equivalently, all } \varepsilon > 0, \quad (1.2)$$

then (1.2) was called the complete moment convergence by Chow [11]. Chow [11] investigated the complete moment convergence for independent random variables. Wang and Su [12], Rosalsky et al. [13] extended and generalized Chow's result to a Rademacher type p ($1 < p < 2$) Banach space, Wang and Zhao [14], Liang et al. [15] to the NA sequence, and Li and Zhang [16] to moving average processes. Recently, Jiang and Zhang [17] discussed the precise rates in the law of the iterated logarithm for complete moment convergence.

Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise NA random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Also, let $\{\Psi_n(t), n \geq 1\}$ be a sequence of nonnegative even functions satisfying

$$\frac{\Psi_n(|t|)}{|t|^q} \uparrow \quad \text{and} \quad \frac{\Psi_n(|t|)}{|t|^p} \downarrow \quad \text{as} \quad |t| \uparrow. \quad (1.3)$$

* Department of Mathematics and Computer Science, Tongling University, Tongling 244000, China; e-mail: wyfwyf@126.com

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Introduce the conditions

$$\mathbf{E} X_{nk} = 0, \quad 1 \leq k \leq n, \quad n \geq 1, \quad (1.4)$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\mathbf{E} \Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty, \quad (1.5)$$

$$\sum_{n=1}^{\infty} \left[\sum_{k=1}^n \frac{\mathbf{E} |X_{nk}|^r}{a_n^r} \right]^s < \infty, \quad (1.6)$$

where $0 < r \leq 2, s > 0$.

This paper is motivated by a study by Gan and Chen [10]. However, it is worthy to point out that the methods used in this paper are different from theirs. The main goal of this paper is to study the complete moment convergence and the L^q convergence for rowwise NA random arrays. Now we state the main results. The proofs will be detailed in next section.

Theorem 1.1. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise NA random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Also, let $\{\Psi_n(t), n \geq 1\}$ be a sequence of nonnegative even functions satisfying (1.3) for $1 \leq q < p \leq 2$. Then conditions (1.4) and (1.5) imply

$$\sum_{n=1}^{\infty} a_n^{-q} \mathbf{E} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| - \varepsilon a_n \right\}_+^q < \infty \quad \forall \varepsilon > 0. \quad (1.7)$$

Theorem 1.2. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise NA random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Also, let $\{\Psi_n(t), n \geq 1\}$ be a sequence of nonnegative even functions satisfying (1.3) for $p > 2$ and $1 \leq q < p$. Then conditions (1.4), (1.5), and (1.6) imply (1.7).

Theorem 1.3. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise NA random variables with (1.4) and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Also, let $\{\Psi_n(t), n \geq 1\}$ be a sequence of nonnegative even functions satisfying (1.3) for $1 \leq q < p$.

1) If $1 < p \leq 2$ and

$$\sum_{k=1}^n \frac{\mathbf{E} \Psi_k(X_{nk})}{\Psi_k(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (1.8)$$

then

$$\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| \xrightarrow{L^q} 0. \quad (1.9)$$

2) If $p > 2$, (1.8) holds and

$$a_n^{-r} \sum_{k=1}^n \mathbf{E} |X_{nk}|^r \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (1.10)$$

where $0 < r \leq 2$, then we have (1.9).

R e m a r k 1.1. Take $q = 1$ in (1.3), then the conditions of the above theorems are the same as those of Theorem 1, Theorem 2, and Theorem 4 in [11]. Compared with [11], we study the complete moment convergence and the L^1 convergence under the same conditions. Since the L^1 convergence implies the convergence in probability, Theorem 1.3 is an improvement of Theorem 4 in [11].

Taking $\Psi_n(t) = |t|^\beta$ for $n \geq 1$ and $1 \leq q < \beta$ in the above theorems, we can have the following corollaries.

Corollary 1.1. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise NA random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. If $1 \leq q < \beta \leq 2$, then conditions (1.4) and

$$\sum_{n=1}^{\infty} \sum_{k=1}^n a_n^{-\beta} \mathbf{E} |X_{nk}|^{\beta} < \infty \quad (1.11)$$

imply (1.7).

Corollary 1.2. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise NA random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. If $1 \leq q < \beta$ and $\beta > 2$, then conditions (1.4), (1.6), and (1.11) imply (1.7).

Corollary 1.3. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise NA random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$.

1) If $1 \leq q < \beta \leq 2$, then (1.4) and

$$a_n^{-\beta} \sum_{k=1}^n \mathbf{E} |X_{nk}|^{\beta} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (1.12)$$

imply (1.9).

2) If $1 \leq q < \beta$ and $\beta > 2$, then (1.4), (1.10), for $0 < r \leq 2$, and (1.12) imply (1.9).

In this paper, the symbol C always stands for a generic positive constant which may differ from one place to another.

2. Main results. We need the following lemmas.

Lemma 2.1 (see [2]). Nondecreasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated.

Lemma 2.2 (see [5]). Let $\{X_k, 1 \leq k \leq n\}$ be a sequence of negatively associated random variables with zero means and finite second moments. Let $S_j = \sum_{k=1}^j X_k$ and $B_n = \sum_{k=1}^n \mathbf{E} X_k^2$. Then for all $x > 0$ and $a > 0$

$$\begin{aligned} \mathbf{P} \left\{ \max_{1 \leq j \leq n} |S_j| \geq x \right\} &\leq 2\mathbf{P} \left\{ \max_{1 \leq j \leq n} |X_j| > a \right\} + 4 \exp \left\{ -\frac{x^2}{8B_n} \right\} \\ &\quad + 4 \left\{ \frac{B_n}{4(xa + B_n)} \right\}^{x/12a}. \end{aligned}$$

P r o o f o f T h e o r e m 1.1. Since

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n^{-q} \mathbf{E} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| - \varepsilon a_n \right\}_+^q \\ &= \sum_{n=1}^{\infty} a_n^{-q} \int_0^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| - \varepsilon a_n > t^{1/q} \right\} dt \\ &= \sum_{n=1}^{\infty} a_n^{-q} \left(\int_0^{a_n^q} \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| > \varepsilon a_n + t^{1/q} \right\} dt \right. \\ &\quad \left. + \int_{a_n^q}^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| > \varepsilon a_n + t^{1/q} \right\} dt \right) \\ &\leq \sum_{n=1}^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| > \varepsilon a_n \right\} \\ &\quad + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| > t^{1/q} \right\} dt \triangleq I_1 + I_2, \end{aligned}$$

so to prove (1.7), it is enough to prove that $I_1 < \infty$ and $I_2 < \infty$. Because (1.3) for $q \geq 1$ implies (1.10) in [11], by Theorem 1 in [11], we have $I_1 < \infty$. To prove (1.7), it suffices to prove $I_2 < \infty$. Let

$$\begin{aligned} Y_{nk} &= X_{nk}I(|X_{nk}| \leq t^{1/q}) + t^{1/q}I(X_{nk} > t^{1/q}) - t^{1/q}I(X_{nk} < -t^{1/q}), \\ Z_{nk} &= (X_{nk} - t^{1/q})I(X_{nk} > t^{1/q}) + (X_{nk} + t^{1/q})I(X_{nk} < -t^{1/q}), \\ Y'_{nk} &= X_{nk}I(|X_{nk}| \leq a_n) + t^{1/q}I(X_{nk} > t^{1/q}) - t^{1/q}I(X_{nk} < -t^{1/q}), \\ Y''_{nk} &= Y_{nk} - Y'_{nk} = X_{nk}I(a_n < |X_{nk}| \leq t^{1/q}). \end{aligned}$$

Clearly, $\{Y_{nk}, 1 \leq k \leq n, n \geq 1\}$ is an array of rowwise NA random variables by Lemma 2.1. Obviously,

$$\mathbf{P}\left\{\max_{1 \leq j \leq n} \left|\sum_{k=1}^j X_{nk}\right| > t^{1/q}\right\} \leq \sum_{k=1}^n \mathbf{P}\{|X_{nk}| > t^{1/q}\} + \mathbf{P}\left\{\max_{1 \leq j \leq n} \left|\sum_{k=1}^j Y_{nk}\right| > t^{1/q}\right\}.$$

Hence

$$\begin{aligned} I_2 &\leq \sum_{n=1}^{\infty} \sum_{k=1}^n a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P}\{|X_{nk}| > t^{1/q}\} dt \\ &\quad + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P}\left\{\max_{1 \leq j \leq n} \left|\sum_{k=1}^j Y_{nk}\right| > t^{1/q}\right\} dt \\ &\triangleq I_3 + I_4. \end{aligned}$$

For I_3 , we have

$$I_3 \leq \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\mathbf{E}|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} \leq \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\mathbf{E}\Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty.$$

Then we prove $I_4 < \infty$. By (1.3), (1.4), and (1.5), we have

$$\begin{aligned} \max_{t \geq a_n^q} \max_{1 \leq j \leq n} t^{-1/q} \left|\sum_{k=1}^j \mathbf{E} Y_{nk}\right| &= \max_{t \geq a_n^q} \max_{1 \leq j \leq n} t^{-1/q} \left|\sum_{k=1}^j \mathbf{E} Z_{nk}\right| \\ &\leq \max_{t \geq a_n^q} t^{-1/q} \sum_{k=1}^n \mathbf{E}|X_{nk}|I(|X_{nk}| > t^{1/q}) \\ &\leq \sum_{k=1}^n \frac{\mathbf{E}|X_{nk}|I(|X_{nk}| > a_n)}{a_n} \leq \sum_{k=1}^n \frac{\mathbf{E}\Psi_k(X_{nk})}{\Psi_k(a_n)} \rightarrow 0. \end{aligned} \quad (2.1)$$

Therefore, while n is sufficiently large,

$$\max_{1 \leq j \leq n} \left|\sum_{k=1}^j \mathbf{E} Y_{nk}\right| \leq \frac{t^{1/q}}{2}$$

holds uniformly for $t \geq a_n^q$. Then

$$\mathbf{P}\left\{\max_{1 \leq j \leq n} \left|\sum_{k=1}^j Y_{nk}\right| > t^{1/q}\right\} \leq \mathbf{P}\left\{\max_{1 \leq j \leq n} \left|\sum_{k=1}^j (Y_{nk} - \mathbf{E} Y_{nk})\right| > \frac{t^{1/q}}{2}\right\}. \quad (2.2)$$

Let $B_n = \sum_{k=1}^n \mathbf{E}(Y_{nk} - \mathbf{E} Y_{nk})^2$ and $\delta > q$. Take $x = t^{1/q}/2$, $a = t^{1/q}/24\delta$. By (2.2) and Lemma 2.2, we have

$$I_4 \leq \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P}\left\{\max_{1 \leq j \leq n} \left|\sum_{k=1}^j (Y_{nk} - \mathbf{E} Y_{nk})\right| > t^{1/q}/2\right\} dt$$

$$\begin{aligned}
&\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} |Y_{nj} - \mathbf{E} Y_{nj}| > \frac{t^{1/q}}{24} \delta \right\} dt \\
&\quad + C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\exp \left\{ -\frac{t^{2/q}}{32B_n} \right\} + \left\{ \frac{B_n}{B_n + t^{2/q}/48\delta} \right\}^{\delta} \right) dt \\
&\triangleq I_5 + I_6.
\end{aligned}$$

By argument similar to the proof of (2.1), we can get

$$\max_{t \geq a_n^q} \max_{1 \leq j \leq n} t^{-1/q} |\mathbf{E} Y_{nj}| \rightarrow 0.$$

Hence, we have

$$\begin{aligned}
I_5 &\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} |Y_{nj}| > \frac{t^{1/q}}{48} \delta \right\} dt \\
&\leq C \sum_{n=1}^{\infty} \sum_{k=1}^n a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P} \left\{ |Y_{nk}| > \frac{t^{1/q}}{48} \delta \right\} dt \\
&\leq C \sum_{n=1}^{\infty} \sum_{k=1}^n a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P} \left\{ |Y'_{nk}| > \frac{t^{1/q}}{96} \delta \right\} dt \\
&\quad + C \sum_{n=1}^{\infty} \sum_{k=1}^n a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P} \left\{ |Y''_{nk}| > \frac{t^{1/q}}{96} \delta \right\} dt \\
&\triangleq I_7 + I_8.
\end{aligned}$$

By the Markov inequality, taking into account that $p > q$, we have

$$\begin{aligned}
I_7 &\leq C \sum_{n=1}^{\infty} \sum_{k=1}^n a_n^{-q} \int_{a_n^q}^{\infty} t^{-p/q} \left(\mathbf{E} |X_{nk}|^p I(|X_{nk}| \leq a_n) + t^{p/q} \mathbf{P}(|X_{nk}| > t^{1/q}) \right) dt \\
&\leq C \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\mathbf{E} |X_{nk}|^p I(|X_{nk}| \leq a_n)}{a_n^p} + C \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\mathbf{E} |X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} \\
&\leq C \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\mathbf{E} \Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty.
\end{aligned}$$

By argument similar to the proof of $I_3 < \infty$, we can prove $I_8 < \infty$.

Finally, we prove $I_6 < \infty$. Since $\delta > q \geq 1$, we have

$$\exp \left\{ -\frac{t^{2/q}}{32B_n} \right\} \leq C \left(\frac{32B_n}{t^{2/q}} \right)^{\delta}.$$

Hence, by C_r -inequality and $\delta > 1$, we have

$$\begin{aligned}
I_6 &\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\frac{B_n}{t^{2/q}} \right)^{\delta} dt \leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{k=1}^n \frac{\mathbf{E} Y_{nk}^2}{t^{2/q}} \right)^{\delta} dt \\
&\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{k=1}^n \frac{\mathbf{E} X_{nk}^2}{t^{2/q}} I(|X_{nk}| \leq a_n) \right. \\
&\quad \left. + \sum_{k=1}^n \frac{\mathbf{E} X_{nk}^2}{t^{2/q}} I(a_n < |X_{nk}| \leq t^{1/q}) + \sum_{k=1}^n \mathbf{P}(|X_{nk}| > t^{1/q}) \right)^{\delta} dt \\
&\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{k=1}^n \frac{\mathbf{E} X_{nk}^2}{t^{2/q}} I(|X_{nk}| \leq a_n) \right)^{\delta} dt
\end{aligned}$$

$$\begin{aligned}
& + C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{k=1}^n \frac{\mathbf{E}|X_{nk}|}{t^{1/q}} I(a_n < |X_{nk}| \leq t^{1/q}) \right)^{\delta} dt \\
& + C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{k=1}^n \mathbf{P}(|X_{nk}| > t^{1/q}) \right)^{\delta} dt \\
& \triangleq I_9 + I_{10} + I_{11}.
\end{aligned}$$

From $q < p \leq 2$ and $\delta > 1$, we have

$$\begin{aligned}
I_9 &= C \sum_{n=1}^{\infty} a_n^{-q} \left(\sum_{k=1}^n \mathbf{E} X_{nk}^2 I(|X_{nk}| \leq a_n) \right)^{\delta} \int_{a_n^q}^{\infty} t^{-2\delta/q} dt \\
&\leq C \sum_{n=1}^{\infty} a_n^{-2\delta} \left(\sum_{k=1}^n \mathbf{E} X_{nk}^2 I(|X_{nk}| \leq a_n) \right)^{\delta} \\
&\leq C \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \frac{\mathbf{E}|X_{nk}|^p}{a_n^p} I(|X_{nk}| \leq a_n) \right)^{\delta} \\
&\leq C \left(\sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\mathbf{E} \Psi_k(X_{nk})}{\Psi_k(a_n)} \right)^{\delta} < \infty.
\end{aligned}$$

Since $\delta > q$, we have

$$\begin{aligned}
I_{10} &\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{k=1}^n \frac{\mathbf{E}|X_{nk}|}{t^{1/q}} I(|X_{nk}| > a_n) \right)^{\delta} dt \\
&\leq C \sum_{n=1}^{\infty} a_n^{-q} \left(\sum_{k=1}^n \mathbf{E}|X_{nk}| I(|X_{nk}| > a_n) \right)^{\delta} \int_{a_n^q}^{\infty} t^{-\delta/q} dt \\
&\leq C \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \frac{\mathbf{E}|X_{nk}|}{a_n} I(|X_{nk}| > a_n) \right)^{\delta} \\
&\leq C \left(\sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\mathbf{E} \Psi_k(X_{nk})}{\Psi_k(a_n)} \right)^{\delta} < \infty.
\end{aligned}$$

Finally, we prove $I_{11} < \infty$. From (1.3), we know $\Psi_n(|t|) \uparrow$ as $|t| \uparrow$. Hence, we have

$$\begin{aligned}
\max_{t \geq a_n^q} \sum_{k=1}^n \mathbf{P}\{|X_{nk}| > t^{1/q}\} &\leq \sum_{k=1}^n \mathbf{P}\{|X_{nk}| > a_n\} \\
&\leq \sum_{k=1}^n \frac{\mathbf{E} \Psi_k(|X_{nk}|)}{\Psi_k(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Therefore, while n is sufficiently large,

$$\sum_{k=1}^n \mathbf{P}\{|X_{nk}| > t^{1/q}\} < 1$$

holds uniformly for $t \geq a_n^q$. As $\delta > 1$ and by argument similar to the proof of $I_3 < \infty$, we can get

$$I_{11} \leq C \sum_{n=1}^{\infty} \sum_{k=1}^n a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P}\{|X_{nk}| > t^{1/q}\} dt \leq C \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{\mathbf{E} \Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty.$$

The proof is complete.

P r o o f o f T h e o r e m 1.2. Following the notation and the methods of the proof in Theorem 1.1, $I_1 < \infty$, $I_3 < \infty$, $I_5 < \infty$, $I_{10} < \infty$, and $I_{11} < \infty$ hold. So we only need to show $I_9 < \infty$. Let $\delta > \max\{q, s\}$. By (1.6) and C_r -inequality, we have

$$\begin{aligned} I_9 &\leq C \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \frac{\mathbf{E} X_{nk}^2 I(|X_{nk}| \leq a_n)}{a_n^2} \right)^{\delta} \\ &= C \sum_{n=1}^{\infty} \left[\left(\sum_{k=1}^n \frac{\mathbf{E} X_{nk}^2 I(|X_{nk}| \leq a_n)}{a_n^2} \right)^s \right]^{\delta/s} \\ &\leq C \left[\sum_{n=1}^{\infty} \left(\sum_{k=1}^n \frac{\mathbf{E} |X_{nk}|^r I(|X_{nk}| \leq a_n)}{a_n^r} \right)^s \right]^{\delta/s} < \infty. \end{aligned}$$

The proof is complete.

P r o o f o f T h e o r e m 1.3. As follows the notation of the proof in Theorem 1.1. To start with, we prove (1.9) for the case that $1 < p \leq 2$. For any $\varepsilon > 0$,

$$\begin{aligned} \mathbf{E} \left(a_n^{-1} \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| \right)^q &= a_n^{-q} \int_0^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| > t^{1/q} \right\} dt \\ &\leq \varepsilon^q + a_n^{-q} \int_{(a_n \varepsilon)^q}^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j X_{nk} \right| > t^{1/q} \right\} dt \\ &\leq \varepsilon^q + a_n^{-q} \int_{(a_n \varepsilon)^q}^{\infty} \sum_{k=1}^n \mathbf{P} \{ |X_{nk}| > t^{1/q} \} dt \\ &\quad + a_n^{-q} \int_{(a_n \varepsilon)^q}^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j Y_{nk} \right| > t^{1/q} \right\} dt \\ &\triangleq \varepsilon^q + I_{12} + I_{13}. \end{aligned}$$

Without loss of generality we may assume $0 < \varepsilon < 1$. By the Markov inequality, (1.3), and (1.8), we have

$$\begin{aligned} I_{12} &\leq \sum_{k=1}^n a_n^{-q} \int_{(a_n \varepsilon)^q}^{\infty} \mathbf{P} \{ |X_{nk}| I(a_n \varepsilon < |X_{nk}| \leq a_n) > t^{1/q} \} dt \\ &\quad + \sum_{k=1}^n a_n^{-q} \int_{(a_n \varepsilon)^q}^{\infty} \mathbf{P} \{ |X_{nk}| I(|X_{nk}| > a_n) > t^{1/q} \} dt \\ &\leq \sum_{k=1}^n a_n^{-q} \mathbf{E} |X_{nk}|^p I(|X_{nk}| \leq a_n) \int_{(a_n \varepsilon)^q}^{\infty} t^{-p/q} dt \\ &\quad + \sum_{k=1}^n a_n^{-q} \mathbf{E} |X_{nk}|^q I(|X_{nk}| > a_n) \\ &\leq C \varepsilon^{q-p} \sum_{k=1}^n \frac{\mathbf{E} |X_{nk}|^p}{a_n^p} I(|X_{nk}| \leq a_n) + \sum_{k=1}^n a_n^{-q} \mathbf{E} |X_{nk}|^q I(|X_{nk}| > a_n) \\ &\leq (C \varepsilon^{q-p} + 1) \sum_{k=1}^n \frac{\mathbf{E} \Psi_k(X_{nk})}{\Psi_k(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Then we prove $I_{13} \rightarrow 0$ as $n \rightarrow \infty$. From (1.3), (1.4), and (1.8), we have

$$\max_{t \geq (a_n \varepsilon)^q} \max_{1 \leq j \leq n} t^{-1/q} \left| \sum_{k=1}^j \mathbf{E} Y_{nk} \right|$$

$$\begin{aligned}
&= \max_{t \geq (a_n \varepsilon)^q} \max_{1 \leq j \leq n} t^{-1/q} \left| \sum_{k=1}^j \mathbf{E} Z_{nk} \right| \leq (a_n \varepsilon)^{-1} \sum_{k=1}^n \mathbf{E} |X_{nk}| I(|X_{nk}| > a_n \varepsilon) \\
&\leq \varepsilon^{-1} \sum_{k=1}^n \frac{\mathbf{E} |X_{nk}|}{a_n} I(|X_{nk}| > a_n) + \varepsilon^{-p} \sum_{k=1}^n \frac{\mathbf{E} |X_{nk}|^p}{a_n^p} I(a_n \varepsilon < |X_{nk}| \leq a_n) \\
&\leq (\varepsilon^{-1} + \varepsilon^{-p}) \sum_{k=1}^n \frac{\mathbf{E} \Psi_k(X_{nk})}{\Psi_k(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Therefore, while n is sufficiently large, for $t \geq (a_n \varepsilon)^q$, we have

$$\mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j Y_{nk} \right| > t^{1/q} \right\} \leq \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j (Y_{nk} - \mathbf{E} Y_{nk}) \right| > t^{1/q}/2 \right\}.$$

Let $B_n = \sum_{k=1}^n \mathbf{E} (Y_{nk} - \mathbf{E} Y_{nk})^2$ and $\gamma > q$. Take $x = t^{1/q}/2$, $a = t^{1/q}/24\gamma$. By Lemma 2.2, we have

$$\begin{aligned}
I_{13} &\leq a_n^{-q} \int_{(a_n \varepsilon)^q}^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} \left| \sum_{k=1}^j (Y_{nk} - \mathbf{E} Y_{nk}) \right| > \frac{t^{1/q}}{2} \right\} dt \\
&\leq C a_n^{-q} \int_{(a_n \varepsilon)^q}^{\infty} \mathbf{P} \left\{ \max_{1 \leq j \leq n} |Y_{nj} - \mathbf{E} Y_{nj}| > \frac{t^{1/q}}{24} \gamma \right\} dt \\
&\quad + C a_n^{-q} \int_{(a_n \varepsilon)^q}^{\infty} \left(\exp \left\{ -\frac{t^{2/q}}{32 B_n} \right\} + \left\{ \frac{B_n}{B_n + t^{2/q}/48\gamma} \right\}^{\gamma} \right) dt \\
&\cong I_{14} + I_{15}.
\end{aligned}$$

By argument similar to the proof of $I_5 < \infty$ and $I_6 < \infty$, we may prove $I_{14} \rightarrow 0$ and $I_{15} \rightarrow 0$.

The proof of (1.9) for the case $p > 2$ is similar to that of Theorem 1.2, so we omit this proof.

The proof is complete.

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