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ON BOUNDS IN LIMIT THEOREMS FOR SOME U -STATISTICS

Предлагается оценка скорости сходимости для распределения U -статистики второго порядка, ассоциированной с оператором Гильберта–Шмидта. Эта оценка получена для метрики Колмогорова при некоторых дополнительных условиях.

Ключевые слова и фразы: U -статистика второго порядка, предельное распределение, метрика Колмогорова, оператор Гильберта–Шмидта, собственные значения.

1. Introduction and formulation of the result. This paper pertains to the U -statistics which are currently attracting a growing attention in view of various applications (see [5]). We obtain the estimation of the convergence rate for the distribution of a second order U -statistic.

We begin by introducing some notation. Let $X, X_1, \dots, X_n$ be independent identically distributed (i.i.d.) random variables taking values in a measurable space $(\Theta, \mathfrak{R})$. Let $h: \Theta^2 \rightarrow \mathbf{R}$ and $g: \Theta \rightarrow \mathbf{R}$ be real-valued measurable functions. Let h be symmetric, that is, $h(x, y) = h(y, x)$ for all $x, y \in \Theta$. Consider the U -statistic

$$T = T(X_1, \dots, X_n) = n^{-1} \sum_{1 \leq i < k \leq n} h(X_i, X_k) + n^{-1/2} \sum_{1 \leq i \leq n} g(X_i), \quad (1)$$

assuming that

$$\mathbf{E} g(X) = 0, \quad \mathbf{E} h(x, X) = 0, \quad \text{for all } x \in \Theta. \quad (2)$$

We write

$$\gamma_s = \mathbf{E} |g(X)|^s, \quad \beta_s = \mathbf{E} |h(X, X_1)|^s, \quad (3)$$

and assume that

$$\beta_2 > 0. \quad (4)$$

The condition $\beta_2 > 0$ ensures that the quadratic part of the statistic is not asymptotically negligible and therefore T is not asymptotically normal. To

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be more precise, the asymptotic distribution of T is non-Gaussian and is given by the distribution of the random variable

$$T_0 = \frac{1}{2} \sum_{i \geq 1} q_i (\eta_i^2 - 1). \quad (5)$$

Here η_i is a sequence of i.i.d. standard normal random variables and $q_1, q_2, \dots$ denote eigenvalues of the Hilbert–Schmidt operator, say Q , associated with the kernel h (see Section 2 for detailed definitions). Without loss of generality, we assume throughout that $|q_1| \geq |q_2| \geq \dots$.

Denote by $\mathcal{L}(Y)$ the distribution of a random variable Y . Let us introduce the notation

$$\Delta_n = \rho(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T_0)),$$

where ρ is a Kolmogorov (or uniform) metric.

Korolyuk and Borovskikh [3] proved that $\Delta_n = o(n^{-1/2})$ if $q_i \neq 0$ for all i . Bentkus and Götze [1] improved this estimation and found the optimal bound $\Delta_n = O(n^{-1})$, under the less restrictive condition $q_9 \neq 0$. From the method of Bentkus and Götze [1], it is clear that bounds to Δ_n are related to estimates of the convergence rate in the central limit theorem in Hilbert spaces. Senatov [7] constructed the lower bounds for the convergence rate on the balls in these spaces, of the order $n^{k/12} |q_1 \cdots q_k|^{1/2}$, for $k \leq 6$, if there is information that $q_k \neq 0$. In this article, the rate of convergence of a U -statistic to its limit distribution is investigated for the case, where the conditions on eigenvalues are less restrictive among all the possible ones (there is information about the first eigenvalue only). In such a case, the lower bound of Senatov is of the order $n^{-1/12}$.

Let

$$\beta = (\beta_{18/5})^{5/36} + (\gamma_{18/5})^{5/36} < \infty. \quad (6)$$

By c we denote positive absolute constants that may differ from line to line and from formula to formula. The following theorem is our result.

Theorem 1. *If $q_1 > 0$, then*

$$\Delta_n \leq \frac{c}{\sqrt{|q_1|}} \left(\frac{\beta}{n^{1/12}} + \frac{d}{n^{1/4}} \right), \quad (7)$$

where d is defined in (17) and depends on q_i and a_i , $i = 1, 2, \dots$, (a_i are some constants which we shall define in Section 2).

R e m a r k 1. The result of Senatov [7] allows us to suppose that the order of bound (7) is unimprovable.

In the one-dimensional case, that is, in the case where we have additional information that $q_2 = q_3 = \dots = 0$, the bound (7) $\Delta_n = O(n^{-1/12})$ of our theorem improves to $\Delta_n = O(n^{-1/4})$, and the rate $O(n^{-1/4})$ is the best possible, see [9].

2. Special representations of bivariate U -statistics. Bentkus and Götze [1] suggested the following representation of bivariate U -statistics. Consider the measurable space $(\Theta, \mathfrak{R}, \mu)$ with a measure $\mu = \mathcal{L}(X)$. Let $L^2 = L^2(\Theta, \mathfrak{R}, \mu)$ denote the real Hilbert space of square integrable real functions. The Hilbert–Schmidt operator is defined via

$$Qf(x) = \int_{\Theta} h(x, y)f(y) \mu(dy), \quad f \in L^2.$$

Let $\{e_j: j \geq 1\}$ denote an orthonormal complete system of eigenfunctions of Q ordered by decreasing absolute values of the corresponding eigenvalues $|q_1| \geq |q_2| \geq \dots$. Then

$$\mathbf{E} h^2(X, X_1) = \sum_{j \geq 1} q_j^2 < \infty, \quad h(x, y) = \sum_{j \geq 1} q_j e_j(x) e_j(y). \quad (8)$$

Consider the subspace $L^2(g, h) \subset L^2$ generated by g and eigenfunctions e_j corresponding to nonzero eigenvalues $q_j \neq 0$. Introducing, if necessary, a normalized eigenfunction, say e_0 , such that $Qe_0 = 0$, we may assume that $e_0, e_1, \dots$ is an orthonormal basis of $L^2(g, h)$. Thus, we can write

$$g(X) = \sum_{j \geq 0} a_j e_j(X) \quad \text{in } L^2, \quad \gamma_2 = \mathbf{E} g^2(X) = \sum_{j \geq 0} a_j^2, \quad (9)$$

with $a_j = \mathbf{E} g(X) e_j(X)$. It is easy to see that $\mathbf{E} e_j(X) = 0$, for all j . Therefore $(e_j(X))_{j \geq 0}$ is an orthonormal system of mean zero random variables.

Also, we shall assume throughout that all random variables and vectors are independent in aggregate, if the contrary is not clear from the context.

Let $\mathbf{R}^\infty$ denote the space of all real sequences $x = (x_0, x_1, x_2, \dots)$, $x_j \in \mathbf{R}$. The Hilbert space $l_2 \subset \mathbf{R}^\infty$ consists of $x \in \mathbf{R}^\infty$ such that

$$|x|^2 \stackrel{\text{def}}{=} \langle x, x \rangle, \quad |x| < \infty, \quad \langle x, y \rangle = \sum_{j \geq 0} x_j y_j.$$

Consider a random vector

$$\mathbf{X} \stackrel{\text{def}}{=} (e_0(X), e_1(X), \dots),$$

which takes values in $\mathbf{R}^\infty$. Since $\{e_j(X)\}_{j \geq 0}$ is a system of mean zero uncorrelated random variables with variances 1, the random vector $\mathbf{X}$ has identity covariance and mean zero. Using (8) and (9) we derive

$$h(X, X_1) = \langle Q\mathbf{X}, \mathbf{X}_1 \rangle, \quad g(X) = \langle a, \mathbf{X} \rangle, \quad (10)$$

where we define $Qx = (0, q_1 x_1, q_2 x_2, \dots)$, for $x \in \mathbf{R}^\infty$, and $a = (a_j)_{j \geq 0} \in \mathbf{R}^\infty$. Equalities (10) allow us to assume throughout that the measurable space Θ

is $\mathbf{R}^\infty$, the random variable X is a random vector taking values in $\mathbf{R}^\infty$ with mean zero and identity covariance and that

$$h(X, X_1) = \langle QX, X_1 \rangle, \quad g(X) = \langle a, X \rangle.$$

In particular, without loss of generality we shall assume throughout that the kernels $h(x, y)$ and $g(x)$ are linear functions in each of their arguments.

Let $G, G_i, 1 \leq i \leq n$, be i.i.d. Gaussian random vectors $G_i = (G_{1,i}, G_{2,i}, \dots)$ with values in $\mathbf{R}^\infty$, where $G_{1,i}, G_{2,i}, \dots$ denote i.i.d. standard normal random variables. We assume throughout that

$$\mathbf{E} g(G) = 0, \quad \mathbf{E} h(x, G) = 0, \quad \mathbf{E} g^2(G) = \mathbf{E} g^2(X), \quad (11)$$

$$\mathbf{E} h(x, G)h(y, G) = \mathbf{E} h(x, X)h(y, X), \quad \mathbf{E} h(x, G)g(G) = \mathbf{E} h(x, X)g(X)$$

for all x, y in Θ . Note that the possibility of selecting Gaussian random variables so that (11) be valid is substantiated by Bentkus and Götze in [1].

It has been also shown in [1] that it is possible to represent the statistic $T(G_1, \dots, G_n)$ in the form

$$T(G_1, \dots, G_n) = n^{-1} \sum_{1 \leq i < k \leq n} \langle QG_i, G_k \rangle + n^{-1/2} \sum_{1 \leq i \leq n} \langle a, G_i \rangle. \quad (12)$$

3. Proofs.

3.1. Lemmas. To prove the theorem we need the following lemma, which is well known.

Lemma 1. *Let η be a standard Gaussian random variable and let $q_1 > 0$. Then the distribution function $H(x) = \mathbf{P}\{q_1\eta^2 < x\}$ satisfies the Lipschitz condition*

$$\sup_x |H(x + \varepsilon) - H(x)| \leq c \sqrt{\frac{\varepsilon}{q_1}}, \quad (13)$$

where $\varepsilon > 0$.

The following lemma is easy to prove as well.

Lemma 2. *Let η be a standard Gaussian random variable and $q_1 > 0$. Then the following inequality is true:*

$$\sup_x \left| \mathbf{P} \left(q_1\eta_1^2 - \frac{1}{n}q_1\eta_1^2 + \frac{2}{\sqrt{n}}a_1\eta_1 \leq x \right) - \mathbf{P}(q_1\eta_1^2 \leq x) \right| \leq \frac{c}{\sqrt{q_1n}} \left(\frac{|a_1|}{\sqrt{q_1}} + 1 \right). \quad (14)$$

3.2. Proof of the theorem. We have to prove our theorem, i.e., the bound (7). According to the triangle inequality, we have

$$\begin{aligned} \Delta_n &\leq \rho(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T(G_1, \dots, G_n))) \\ &\quad + \rho(\mathcal{L}(T(G_1, \dots, G_n)), \mathcal{L}(T_0)), \end{aligned} \quad (15)$$

where $G_1, \dots, G_n$ were defined in Section 2.

In view of (15), to prove (7) it suffices to establish that

$$\rho(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T(G_1, \dots, G_n))) \leq c|q_1|^{-1/2} \beta n^{-1/12} \quad (16)$$

and

$$\begin{aligned} \rho(\mathcal{L}(T(G_1, \dots, G_n)), \mathcal{L}(T_0)) &\leq \frac{c}{n^{1/4}} \left(\sum_{i \geq 1} \left| \frac{q_i}{q_1} \right| \right)^{1/2} + \frac{c}{n^{1/4}} \left(\sum_{i \geq 1} \left(\frac{a_i}{q_1} \right)^2 \right)^{1/4} \\ &= \frac{cd}{n^{1/4}}. \end{aligned} \quad (17)$$

R e m a r k 2. It is possible that the sum $\sum_{i \geq 1} |q_i|$ is divergent, i.e.,

$$\sum_{i \geq 1} |q_i| = \infty. \quad (18)$$

In this case $\Delta_n = O(n^{-1})$.

Indeed, if (18) holds, then $|q_9| > 0$, since we assume that $|q_1| \geq |q_2| \geq \dots \geq |q_9| \geq \dots \geq |q_n| \geq 0$. Then we make use of Theorem 1.1 from [1]. The theorem states that if $|q_9| > 0$, then $\Delta_n = O(n^{-1})$. That is, if the series mentioned is divergent, then estimate (7) can be replaced by the improved estimate from [1].

3.3. Proof of (16). The estimation of $\rho(T(X_1, \dots, X_n), T(G_1, \dots, G_n))$ is analogous to the same one in [10].

Representation of the statistic $T(G_1, \dots, G_n)$. It has been shown in [1] that it is possible to represent the statistic $T(G_1, \dots, G_n)$ in the form

$$T(G_1, \dots, G_n) = n^{-1} \sum_{1 \leq i < k \leq n} \langle QG_i, G_k \rangle + n^{-1/2} \sum_{1 \leq i \leq n} \langle a, G_i \rangle,$$

where $G_i = (G_{1,i}, G_{2,i}, \dots)$ and $G_{1,i}, G_{2,i}, \dots$ denote independent standard normal random variables.

We can rewrite this expression in the form

$$T(G_1, \dots, G_n) = \sum_{i \geq 1} \left(n^{-1} q_i \sum_{1 \leq j < k \leq n} G_{i,j} G_{i,k} + n^{-1/2} \sum_{1 \leq j \leq n} a_i G_{i,j} \right).$$

It is easy to see that

$$2n^{-1} q_i \sum_{1 \leq j < k \leq n} G_{i,j} G_{i,k} = (n-1) q_i \overline{G}_i^2 - q_i S_i^2,$$

where

$$\overline{G}_i = n^{-1} \sum_{1 \leq j \leq n} G_{i,j} \quad \text{and} \quad S_i^2 = n^{-1} \sum_{1 \leq j \leq n} G_{i,j}^2 - \overline{G}_i^2.$$

It is well known (see [2]) that the random variables $\overline{G}_i$ and S_i are independent. The random variables $\overline{G}_i$ have independent normal distributions and can be written in the form $\overline{G}_i = \eta_i/\sqrt{n}$, where η_i are independent standard normal random variables, so we can write

$$n^{-1}q_i \sum_{1 \leq j < k \leq n} G_{i,j}G_{i,k} = \frac{n-1}{n} q_i \eta_i^2 - q_i S_i^2.$$

Then

$$\begin{aligned} T(G_1, \dots, G_n) &= \sum_{i \geq 1} \left(\frac{n-1}{2n} q_i \eta_i^2 - \frac{q_i}{2} S_i^2 + a_i \eta_i \right) \\ &= \sum_{i \geq 1} \left(\frac{1}{2} q_i (\eta_i^2 - 1) - \frac{1}{2n} q_i \eta_i^2 - \frac{q_i}{2} (S_i^2 - 1) + a_i \eta_i \right). \end{aligned}$$

Using of Lemma 1. Using the independence of η_i , η_j , S_i , S_j , $i \neq j$, it is easy to see, that

$$\begin{aligned} &\sup_x \left| \mathbf{P}(T(G_1, \dots, G_n) \leq x + \varepsilon) - \mathbf{P}(T(G_1, \dots, G_n) \leq x) \right| \\ &= \sup_x \left| \mathbf{P} \left(q_1 \eta_1^2 - \frac{1}{n} q_1 \eta_1^2 + 2a_1 \eta_1 \leq x + 2\varepsilon \right) \right. \\ &\quad \left. - \mathbf{P} \left(q_1 \eta_1^2 - \frac{1}{n} q_1 \eta_1^2 + 2a_1 \eta_1 \leq x \right) \right| \\ &= \sup_x \left| \mathbf{P} \left(\frac{q_1(n-1)}{n} \left(\eta_1 + \frac{a_1 n}{q_1(n-1)} \right)^2 \leq x + 2\varepsilon \right) \right. \\ &\quad \left. - \mathbf{P} \left(\frac{q_1(n-1)}{n} \left(\eta_1 + \frac{a_1 n}{q_1(n-1)} \right)^2 \leq x \right) \right|. \end{aligned}$$

Then from Lemma 1 we get

$$\sup_x \left| \mathbf{P}(T(G_1, \dots, G_n) \leq x + \varepsilon) - \mathbf{P}(T(G_1, \dots, G_n) \leq x) \right| \leq \frac{c'}{\sqrt{|q_1|}} \sqrt{\varepsilon}. \quad (19)$$

Using of the infinite differentiable functions. By definition we have

$$\begin{aligned} &\rho(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T(G_1, \dots, G_n))) \\ &= \sup_x \left| \mathbf{P}(T(X_1, \dots, X_n) \leq x) - \mathbf{P}(T(G_1, \dots, G_n) \leq x) \right|. \end{aligned}$$

To estimate the difference

$$\left| \mathbf{P}(T(X_1, \dots, X_n) \leq x) - \mathbf{P}(T(G_1, \dots, G_n) \leq x) \right|$$

we will use infinite differentiable functions φ such that $|\varphi^{(k)}(u)| \leq c_1/\varepsilon^k$, $k = 1, 2, 3$, and $0 \leq \varphi(u) \leq 1$,

$$\varphi(u) = 1, \text{ for } u \leq x - \varepsilon, \quad \text{and} \quad \varphi(u) = 0, \text{ for } u \geq x, \quad (20)$$

or

$$\varphi(u) = 1, \text{ for } u \leq x, \quad \text{and} \quad \varphi(u) = 0, \text{ for } u \geq x + \varepsilon. \quad (21)$$

Denote

$$\Delta = \max_{\varphi} \left| \mathbf{E} \varphi(T(X_1, \dots, X_n)) - \mathbf{E} \varphi(T(G_1, \dots, G_n)) \right|,$$

where max is taken over all such functions φ . Let us prove that, for any $\varepsilon > 0$, we have

$$\rho(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T(G_1, \dots, G_n))) \leq \frac{c'}{\sqrt{|q_1|}} \sqrt{\varepsilon} + \Delta. \quad (22)$$

Here and in the sequel $c', c_1, c_2, \dots$ are some absolute constants.

Consider $\delta^* = \mathbf{P}\{T(X_1, \dots, X_n) \leq x\} - \mathbf{P}\{T(G_1, \dots, G_n) \leq x\}$. Let us prove (22) in the case $\delta^* \geq 0$. Let us take a function φ such that (21) holds. Then

$$\begin{aligned} \delta^* &= \mathbf{E} I\{T(X_1, \dots, X_n) \leq x\} - \mathbf{P}\{T(G_1, \dots, G_n) \leq x\} \\ &\leq \left| \mathbf{E} \varphi(T(X_1, \dots, X_n)) - \mathbf{E} \varphi(T(G_1, \dots, G_n)) \right| \\ &\quad + \left| \mathbf{E} \varphi(T(G_1, \dots, G_n)) - \mathbf{P}\{T(G_1, \dots, G_n) \leq x\} \right| \\ &\leq \Delta + \mathbf{P}\{x \leq T(G_1, \dots, G_n) \leq x + \varepsilon\}. \end{aligned}$$

Using (19), we get (22).

The proof is similar if $\delta^* < 0$, taking the function φ that satisfies (20) instead of (21).

Estimation (22) is true for all the functions h and g for which conditions of Section 1 are fulfilled.

Method of induction. We write

$$\bar{\beta} = \beta_3 + \beta_{18/5} + \gamma_3 + \gamma_{18/5} + 1.$$

Let us prove that

$$\rho(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T(G_1, \dots, G_n))) \leq c_2 |q_1|^{-1/2} \bar{\beta}^{1/6} n^{-1/12} \quad (23)$$

for some constant c_2 . We shall use induction on n .

Proof for $n = 2$. It is easy to see that bounds (23) hold for $n = 2$. Indeed, let $n = 2$. Since $\rho \leq 1$, it is enough to show that $|q_1|^{-1/2} \bar{\beta}^{1/6} \geq 1$. We have

$$|q_1|^{-1/2} \bar{\beta}^{1/6} \geq |q_1|^{-1/2} (\mathbf{E} |h|^3)^{1/6} = (q_1^{-2} (\mathbf{E} |h|^3)^{2/3})^{1/4}.$$

Using (8), we obtain

$$|q_1|^{-1/2} \bar{\beta}^{1/6} \geq (q_1^{-2} \mathbf{E} |h|^2)^{1/4} = (q_1^{-2} (q_1^2 + q_2^2 + \dots))^{1/4} \geq 1.$$

The statement is proved for $n = 2$.

Proof for arbitrary n . Assume that, for all $m \leq n - 1$, the inequality

$$\begin{aligned} \rho\left(\mathcal{L}(T(X_1, \dots, X_i, G_{i+1}, \dots, G_m)), \mathcal{L}(T(G_1, \dots, G_m))\right) \\ \leq c_2 |q_1|^{-1/2} \bar{\beta}^{1/6} m^{-1/12} \end{aligned} \quad (24)$$

is true for all $1 \leq i \leq m$ and for all the functions h and g in which conditions of Section 1 are fulfilled. Here $\bar{\beta}$ is the moment of functions h and g , defined in Section 1.

Let us prove that (24) holds for $m = n$. We will prove (24) for $i = n$. By the same way it is possible to prove it for all $1 \leq i \leq n - 1$.

To get the bounds (24) for n we will use the following estimation:

$$\begin{aligned} \Delta^*(\varphi) &= \left| \mathbf{E} \varphi(T(X_1, \dots, X_n)) - \mathbf{E} \varphi(T(G_1, \dots, G_n)) \right| \\ &\leq \left| \mathbf{E} \varphi(T(X_1, \dots, X_n)) - \mathbf{E} \varphi(T(X_1, \dots, X_{n-1}, G_n)) \right| \\ &\quad + \left| \mathbf{E} \varphi(T(X_1, \dots, X_{n-1}, G_n)) - \mathbf{E} \varphi(T(X_1, \dots, X_{n-2}, G_{n-1}, G_n)) \right| \\ &\quad + \dots + \left| \mathbf{E} \varphi(T(X_1, G_2, \dots, G_n)) - \mathbf{E} \varphi(T(G_1, \dots, G_n)) \right| \\ &= \Delta_{1,n}^* + \dots + \Delta_{n,n}^*. \end{aligned} \quad (25)$$

Let us estimate $\Delta_{1,n}^*$, because we can estimate $\Delta_{i,n}^*$ analogously. We have

$$\begin{aligned} T(X_1, \dots, X_n) \\ = \frac{1}{n} \left\{ h(X_1, X_2) + \sqrt{n}g(X_1) + \dots + h(X_1, X_{n-1}) + \sqrt{n}g(X_{n-1}) \right. \\ \left. + h(X_1, X_n) + \sqrt{n}g(X_n) + h(X_2, X_3) + \dots + h(X_2, X_n) \right. \\ \left. + \dots + h(X_{n-2}, X_{n-1}) + h(X_{n-2}, X_n) + h(X_{n-1}, X_n) \right\}. \end{aligned}$$

We denote by w the sum of all members that do not have the argument X_n :

$$\begin{aligned} w = \frac{1}{n} \left\{ h(X_1, X_2) + \sqrt{n}g(X_1) + \dots + h(X_1, X_{n-1}) + \sqrt{n}g(X_{n-1}) \right. \\ \left. + \dots + h(X_{n-2}, X_{n-1}) \right\} \end{aligned}$$

and l_n is the sum of all members with the argument X_n :

$$l_n = \frac{1}{n} \left\{ h(X_1, X_n) + \sqrt{n}g(X_n) + \dots + h(X_{n-2}, X_n) + h(X_{n-1}, X_n) \right\}.$$

Replacing X_n by G_n , we get

$$\begin{aligned} T(X_1, \dots, X_{n-1}, G_n) \\ = w + \frac{1}{n} \left\{ h(X_1, G_n) + \sqrt{n}g(G_n) + \dots + h(X_{n-2}, G_n) + h(X_{n-1}, G_n) \right\} \\ = w + l_n^*, \end{aligned}$$

where l_n^* is obtained from l_n replacing X_n by G_n . We expand into the Taylor series

$$\varphi(x+y) = \varphi(x) + \varphi'(x)y + \frac{1}{2}\varphi''(x)y^2 + \frac{1}{2}\mathbf{E}\varphi'''(x+\tau y)(1-\tau)^2y^3.$$

Here τ is a random variable uniformly distributed in $[0, 1]$ and independent of all the other random variables. Let us apply the expansion to $x = w$ and $y = l_n$. Write

$$\begin{aligned}\Delta_{1,n}^* &= \left| \mathbf{E}\varphi(T(X_1, \dots, X_n)) - \mathbf{E}\varphi(T(X_1, \dots, X_{n-1}, G_n)) \right| \\ &= \left| \mathbf{E}\varphi(w) - \mathbf{E}\varphi(w) + \mathbf{E}\varphi'(w)l_n - \mathbf{E}\varphi'(w)l_n^* + \frac{1}{2}\mathbf{E}\varphi''(w)l_n^2 \right. \\ &\quad - \frac{1}{2}\mathbf{E}\varphi''(w)(l_n^*)^2 + \frac{1}{2}\mathbf{E}\varphi'''(w+l_n\tau)l_n^3(1-\tau)^2 \\ &\quad \left. - \frac{1}{2}\mathbf{E}\varphi'''(w+l_n^*\tau)(l_n^*)^3(1-\tau)^2 \right|.\end{aligned}$$

Let $\mathbf{E}_{X_1, \dots, X_{n-1}, \tau}$ be the expectation with respect to random variables $X_1, \dots, X_{n-1}, \tau$, and $\mathbf{E}_{X_n}$ the expectation with respect to random variables X_n , and so on. Write

$$\begin{aligned}\Delta_{1,n}^* &= \left| \mathbf{E}_{X_1, \dots, X_{n-1}, \tau} \varphi'(w) \mathbf{E}_{X_n} l_n - \mathbf{E}_{X_1, \dots, X_{n-1}, \tau} \varphi'(w) \mathbf{E}_{G_n} l_n^* \right. \\ &\quad + \frac{1}{2} \mathbf{E}_{X_1, \dots, X_{n-1}, \tau} \varphi''(w) \mathbf{E}_{X_n} l_n^2 - \frac{1}{2} \mathbf{E}_{X_1, \dots, X_{n-1}, \tau} \varphi''(w) \mathbf{E}_{G_n} (l_n^*)^2 \\ &\quad \left. + \frac{1}{2} \mathbf{E} \varphi'''(w+l_n\tau) l_n^3 (1-\tau)^2 - \frac{1}{2} \mathbf{E} \varphi'''(w+l_n^*\tau) (l_n^*)^3 (1-\tau)^2 \right|.\end{aligned}$$

Using (11), we obtain

$$\Delta_{1,n}^* = \left| \frac{1}{2} \mathbf{E} \varphi'''(w+l_n\tau) l_n^3 (1-\tau)^2 - \frac{1}{2} \mathbf{E} \varphi'''(w+l_n^*\tau) (l_n^*)^3 (1-\tau)^2 \right|. \quad (26)$$

Denote $|\mathbf{E} \varphi'''(w+l_n\tau) l_n^3 (1-\tau)^2|$ by r and let us estimate its value. Write

$$l_n = l'_n + l''_n$$

with

$$\begin{aligned}l'_n &= \frac{1}{n} \left(\sqrt{n}g(X_n) + \sum_{1 \leq k \leq [(n-1)/2]} h(X_k, X_n) \right), \\ l''_n &= \frac{1}{n} \sum_{[(n-1)/2] < k \leq n-1} h(X_k, X_n),\end{aligned}$$

where $[(n-1)/2]$ is the integer part of the number $(n-1)/2$. In a reciprocal manner, we shall define $(l_n^*)'$ and $(l_n^*)''$. Using the inequality $(a+b)^3 \leq 8(a^3 + b^3)$ for $a > 0, b > 0$, we find

$$\begin{aligned} r &\leq \frac{8c_1}{\varepsilon^3} (\mathbf{E} \mathbf{I}(x \leq w + \tau l_n' + \tau l_n'' \leq x + \varepsilon) |l_n'|^3 \\ &\quad + \mathbf{E} \mathbf{I}(x \leq w + \tau l_n' + \tau l_n'' \leq x + \varepsilon) |l_n''|^3) \\ &= \frac{8c_1}{\varepsilon^3} (r_1 + r_2). \end{aligned} \quad (27)$$

We estimate only r_1 , since the estimation of r_2 is similar. Now we condition $X_1, \dots, X_{[(n-1)/2]}, X_n, \tau$. Let $\mathbf{E}_* = \mathbf{E}_{X_{[(n-1)/2]+1}, \dots, X_{n-1}}$, then

$$\begin{aligned} r_1 &= \mathbf{E} |l_n'|^3 \mathbf{E}_* \mathbf{I}(x \leq w + \tau l_n' + \tau l_n'' \leq x + \varepsilon) \\ &\leq \mathbf{E} |l_n'|^3 \sup_x \mathbf{P}\{x \leq w + \tau l_n' \leq x + \varepsilon\} \quad (\text{by the independency assumption}) \\ &= \mathbf{E} |l_n'|^3 \sup_x \mathbf{P}\{x \leq T^* \leq x + \varepsilon\}, \end{aligned}$$

where $T^* = T^*(X_{[(n-1)/2]+1}, \dots, X_{n-1})$. We derive T^* from $w + \tau l_n''$, then we condition $X_1, \dots, X_{[(n-1)/2]}, X_n, \tau$. The corresponding function $h^* = h$ is the same and g^* has the following form:

$$g^*(X_j) = g(X_j) + n^{-1/2} \sum_{1 \leq k \leq [(n-1)/2]} h(X_k, X_j) + n^{-1/2} \tau h(X_n, X_j),$$

where $j = [(n-1)/2] + 1, \dots, n-1$.

Using the inequalities (19), (24), we get

$$\sup_x \mathbf{P}(x \leq T^* \leq x + \varepsilon) \leq \frac{c'}{\sqrt{|q_1|}} \sqrt{\varepsilon} + \frac{2c_2}{\sqrt{|q_1|}} \left(\frac{2(\bar{\beta}^*)^2}{n-1} \right)^{1/12}, \quad (28)$$

where $\bar{\beta}^* = \beta_3 + \beta_{18/5} + \gamma_3^* + \gamma_{18/5}^* + 1$, $\gamma_s^* = \mathbf{E}_* |g^*(X)|^s$.

From Theorem 20 in [6], we derive that

$$\begin{aligned} n^{18/5} \mathbf{E} |l_n'|^{18/5} &= \mathbf{E} \left| n^{9/5} g(X_n) + \sum_{1 \leq k \leq [(n-1)/2]} h(X_k, X_n) \right|^{18/5} \\ &\leq c_4 n^{9/5} (\gamma_{18/5} + \beta_{18/5}) \leq c_4 \bar{\beta} n^{9/5}. \end{aligned} \quad (29)$$

There are the following inequalities in [1]:

$$\mathbf{E} |g(G)|^s \leq c_s \mathbf{E} |g(X)|^s, \quad \mathbf{E} |h(x, G)|^s \leq c_s \mathbf{E} |h(x, X)|^s, \quad s \geq 2, \quad (30)$$

where c_s are some constants, depending on s . Let us take c_4 so large that inequality (29) be true for $(l_n^*)'$ as well. By analogy we get

$$\begin{aligned} n^3 \mathbf{E} |l_n'|^3 &= \mathbf{E} \left| n^{3/2} g(X_n) + \sum_{1 \leq k \leq [(n-1)/2]} h(X_k, X_n) \right|^3 \\ &\leq c_3 n^{3/2} (\gamma_3 + \beta_3). \end{aligned}$$

It is easy to see that

$$\mathbf{E} |l'_n|^3 \leq c_3 \bar{\beta} n^{-3/2}. \quad (31)$$

Let us estimate $\mathbf{E} (|l'_n|^3 (\beta^*)^{1/6})$. Using the Hölder inequality we get

$$\begin{aligned} \mathbf{E} (|l'_n|^3 (\beta^*)^{1/6}) &\leq (\mathbf{E} |l'_n|^{18/5})^{5/6} (\mathbf{E} \beta^*)^{1/6} \\ &= (\mathbf{E} |l'_n|^{18/5})^{5/6} \left(\mathbf{E} (\beta_3 + \beta_{18/5} + |g^*(X)|^3 + |g^*(X)|^{18/5} + 1) \right)^{1/6}. \end{aligned}$$

The following bounds are true:

$$\mathbf{E} |g^*(X)|^{18/5} \leq c_4 (\gamma_{18/5} + \beta_{18/5}), \quad \mathbf{E} |g^*(X)|^3 \leq c_3 (\gamma_3 + \beta_3).$$

Denoting $c_5 = \max(c_4 + 1, c_3 + 1)$, we get

$$\mathbf{E} (|l'_n|^3 (\beta^*)^{1/6}) \leq c_5^{1/6} \bar{\beta}^{1/6} (\mathbf{E} |l'_n|^{18/5})^{5/6} \leq c_5 \bar{\beta} n^{-3/2}. \quad (32)$$

Combining (28), (31), (32) and using $\bar{\beta} > 1$, we obtain

$$r_1 \leq \frac{\bar{\beta}}{\sqrt{|q_1|} n^{3/2}} \left(c' c_3 \sqrt{\varepsilon} + 3c_2 c_5 \bar{\beta}^{1/6} (n-1)^{-1/12} \right).$$

We can construct the same bounds for the second summand in (26). Finally we have

$$\Delta_{1,n}^* \leq 16 \frac{c_1 c_5 \bar{\beta}}{\sqrt{|q_1|} \varepsilon^3 n^{3/2}} \left(c' \sqrt{\varepsilon} + c_2 \bar{\beta}^{1/6} (n-1)^{-1/12} \right).$$

Let $\varepsilon = \delta n^{-1/6} \bar{\beta}^{1/3}$, then we can write

$$\Delta_{1,n}^* \leq 16c_1 \left(\frac{c' c_3 \sqrt{\delta}}{\delta^3} + \frac{6c_2 c_4}{\delta^3} \right) |q_1|^{-1/2} \bar{\beta}^{1/6} n^{-13/12}.$$

In view of (25) and (22), we obtain

$$\begin{aligned} &\rho \left(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T(G_1, \dots, G_n)) \right) \\ &\leq c' |q_1|^{-1/2} \sqrt{\varepsilon} + 16c_1 c_5 (c' \sqrt{\delta} + 3c_2) \delta^{-3} |q_1|^{-1/2} \bar{\beta}^{1/6} n^{-1/12} \\ &= \left(c' \sqrt{\delta} + 16c_1 c_5 (c' \sqrt{\delta} + 3c_2) \delta^{-3} \right) |q_1|^{-1/2} \bar{\beta}^{1/6} n^{-1/12}. \end{aligned}$$

Choosing δ and c_2 so that the relations $112c_1 c_3 \leq \delta^3$, $c_2 \geq 2c' \sqrt{\delta}$ hold, we get

$$\rho \left(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T(G_1, \dots, G_n)) \right) \leq c_2 |q_1|^{-1/2} \bar{\beta}^{1/6} n^{-1/12}.$$

Note that

$$\frac{\bar{\beta}^{1/6}}{|q_1|^{1/2}} \leq \frac{(\beta_3 + \gamma_3)^{1/6}}{|q_1|^{1/2}} + \frac{(\beta_{18/5} + \gamma_{18/5})^{1/6}}{|q_1|^{1/2}} + \frac{1}{|q_1|^{1/2}}.$$

Using of the invariance of metric ρ under the linear transformations. Let $\lambda > 0$ be some constant. Using the fact that the metric ρ is invariant under the linear transformations, we get

$$\begin{aligned} & \rho(\mathcal{L}(\lambda T(X_1, \dots, X_n)), \mathcal{L}(\lambda T(G_1, \dots, G_n))) \\ &= \rho(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T(G_1, \dots, G_n))) \\ &\leq \frac{c}{\sqrt{|q_1|} n^{1/12}} \left((\beta_3 + \gamma_3)^{1/6} + \lambda^{1/10} (\beta_{18/5} + \gamma_{18/5})^{1/6} + \frac{1}{\lambda^{1/2}} \right). \end{aligned} \quad (33)$$

This inequality is true for all λ . Let

$$\lambda^{1/10} (\beta_{18/5} + \gamma_{18/5})^{1/6} = \frac{1}{\lambda^{1/2}}.$$

The estimation

$$\begin{aligned} & \rho(\mathcal{L}(T(X_1, \dots, X_n)), \mathcal{L}(T(G_1, \dots, G_n))) \\ &\leq \frac{c}{\sqrt{|q_1|} n^{1/12}} \left((\beta_3 + \gamma_3)^{1/6} + (\beta_{18/5} + \gamma_{18/5})^{5/36} \right) \end{aligned}$$

is the best one. Obviously,

$$(\beta_3 + \gamma_3)^{1/6} \leq \beta_3^{1/6} + \gamma_3^{1/6}, \quad (\beta_{18/5} + \gamma_{18/5})^{5/36} \leq (\beta_{18/5})^{5/36} + (\gamma_{18/5})^{5/36}.$$

Using the Hölder inequality, we get

$$\beta_3 \leq (\beta_{18/5})^{5/6}, \quad \gamma_3 \leq (\gamma_{18/5})^{5/6}.$$

From this expression we derive

$$\rho(T(X_1, \dots, X_n), T(G_1, \dots, G_n)) \leq \frac{c}{\sqrt{|q_1|}} \frac{\beta}{n^{1/12}}.$$

3.4. Proof of (17). Now we estimate $\rho(\mathcal{L}(T(G_1, \dots, G_n)), \mathcal{L}(T_0))$. Using of the representation of $T(G_1, \dots, G_n)$. Obviously,

$$\begin{aligned} & \rho(\mathcal{L}(T(G_1, \dots, G_n)), \mathcal{L}(T_0)) \\ &= \sup_x \left| \mathbf{P} \left(\sum_{i \geq 1} \left(\frac{1}{2} q_i (\eta_i^2 - 1) - \frac{1}{2n} q_i \eta_i^2 - \frac{q_i}{2} (S_i^2 - 1) + \frac{1}{\sqrt{n}} a_i \eta_i \right) \leq x \right) \right. \\ &\quad \left. - \mathbf{P} \left(\frac{1}{2} \sum_{i \geq 1} q_i (\eta_i^2 - 1) \leq x \right) \right| \\ &\leq \sup_x \left| \mathbf{P} \left(\sum_{i \geq 1} \left(\frac{1}{2} q_i (\eta_i^2 - 1) - \frac{1}{2n} q_i \eta_i^2 - \frac{q_i}{2} (S_i^2 - 1) + \frac{1}{\sqrt{n}} a_i \eta_i \right) \leq x \right) \right| \end{aligned}$$

$$\begin{aligned}
& -\mathbf{P}\left(\sum_{i \geq 1} \left(\frac{1}{2} q_i(\eta_i^2 - 1) - \frac{1}{2n} q_i \eta_i^2 + \frac{1}{\sqrt{n}} a_i \eta_i\right) \leq x\right) \\
& + \sup_x \left| \mathbf{P}\left(\sum_{i \geq 1} \left(\frac{1}{2} q_i(\eta_i^2 - 1) - \frac{1}{2n} q_i \eta_i^2 + \frac{1}{\sqrt{n}} a_i \eta_i\right) \leq x\right) \right. \\
& \quad \left. - \mathbf{P}\left(\frac{1}{2} \sum_{1 \leq i} q_i(\eta_i^2 - 1) \leq x\right) \right| = v_1 + v_2.
\end{aligned} \tag{34}$$

Estimation of v_1 . Using the independence of $S_i, \eta_i, \eta_j, i \neq j$, we obtain

$$\begin{aligned}
v_1 &= \left| \mathbf{P}\left(\frac{1}{2} q_1(\eta_1^2 - 1) - \frac{1}{2n} q_1 \eta_1^2 + \frac{1}{\sqrt{n}} a_1 \eta_1 - \sum_{i \geq 1} \frac{q_i}{2} (S_i^2 - 1) \leq x\right) \right. \\
& \quad \left. - \mathbf{P}\left(\frac{1}{2} q_1(\eta_1^2 - 1) - \frac{1}{2n} q_1 \eta_1^2 + \frac{1}{\sqrt{n}} a_1 \eta_1 \leq x\right) \right|.
\end{aligned}$$

It is possible to rewrite this expression in the following form, since the uniform distance is invariant under the monotone transformations:

$$v_1 = \left| \mathbf{P}\left(\frac{1}{2} q_1 \eta_1^2 - \sum_{i \geq 1} \frac{q_i}{2} (S_i^2 - 1) \leq x\right) - \mathbf{P}\left(\frac{1}{2} q_1 \eta_1^2 \leq x\right) \right|.$$

Using the independence of $S_i, \eta_i, \eta_j, i \neq j$, and inequality (13), we get

$$v_1 \leq c \left(|q_1|^{-1} \sum_{i \geq 1} |q_i| \mathbf{E} |S_1^2 - 1| \right)^{1/2}.$$

It follows that

$$\begin{aligned}
(\mathbf{E} |S_1^2 - 1|)^{1/2} &= \left(\mathbf{E} \left| n^{-1} \sum_{1 \leq j \leq n} (G_{1,j}^2 - 1) - \bar{G}_1^2 \right| \right)^{1/2} \\
&\leq c \left(\mathbf{E} \left| n^{-1} \sum_{1 \leq j \leq n} (G_{1,j}^2 - 1) \right| \right)^{1/2} + (\mathbf{E} \bar{G}_1^2)^{1/2}
\end{aligned}$$

Let us estimate the summands. We have $(\mathbf{E} \bar{G}_1^2)^{1/2} = n^{-1/2} (\mathbf{E} |\eta_1^2|)^{1/2} = n^{-1/2}$. Also, it is easy to see that

$$\begin{aligned}
\left(\mathbf{E} \left| n^{-1} \sum_{1 \leq j \leq n} (G_{1,j}^2 - 1) \right| \right)^{1/2} &= n^{-1/4} \left(\mathbf{E} \left| \sum_{1 \leq j \leq n} \frac{G_{1,j}^2 - 1}{\sqrt{n}} \right| \right)^{1/2} \\
&\leq n^{-1/4} \left(\mathbf{E} \left(\sum_{1 \leq j \leq n} \frac{G_{1,j}^2 - 1}{\sqrt{n}} \right)^2 \right)^{1/4} \leq cn^{-1/4}.
\end{aligned}$$

Consequently,

$$v_1 \leq c \left(\sum_{i \geq 1} \left| \frac{q_i}{q_1} \right| \right)^{1/2} n^{-1/4}.$$

Estimation of v_2 . Let us estimate the second summand in (34):

$$v_2 = \sup_x \left| \mathbf{P} \left(\sum_{i \geq 1} \left(q_i(\eta_i^2 - 1) - \frac{1}{n} q_i \eta_i^2 + \frac{2}{\sqrt{n}} a_i \eta_i \right) \leq x \right) - \mathbf{P}(q_i(\eta_i^2 - 1) \leq x) \right|.$$

Denote

$$\theta = -\frac{1}{n} q_1 \eta_1^2 + \frac{2}{\sqrt{n}} a_1 \eta_1, \quad Z = \sum_{i \geq 2} q_i(\eta_i^2 - 1), \quad Y = \sum_{i \geq 2} \left(-\frac{1}{n} q_i \eta_i^2 + \frac{2}{\sqrt{n}} a_i \eta_i \right).$$

Let $F_1(u) = \mathbf{P}(q_1(\eta_1^2 - 1) + \theta \leq u)$, $F(u) = \mathbf{P}(q_1(\eta_1^2 - 1) \leq u)$, then using the expression for the convolution, we get

$$v_2 = \sup_x |\mathbf{E} F_1(x - Z - Y) - \mathbf{E} F(x - Z)|.$$

Using the triangle inequality we find that

$$v_2 \leq \sup_x |\mathbf{E} F_1(x - Z - Y) - \mathbf{E} F_1(x - Z)| + \sup_x |\mathbf{E} F_1(x - Z) - \mathbf{E} F(x - Z)|. \quad (35)$$

We have

$$\sup_x |\mathbf{E} F_1(x - Z - Y) - \mathbf{E} F_1(x - Z)| \leq c|q_1|^{-1/2} \mathbf{E} \sqrt{Y} = c|q_1|^{-1/2} \mathbf{E} (Y^2)^{1/4}.$$

It is easy to see that $\mathbf{E} (Y^2)^{1/4} \leq cn^{-1/4} (\sum_{i \geq 2} a_i^2)^{1/4} + cn^{-1/2} (\sum_{i \geq 2} q_i^2)^{1/4}$. To estimate the second summand in (35) we apply Lemma 2 which yields

$$\sup_x |\mathbf{E} F_1(x - Z) - \mathbf{E} F(x - Z)| \leq \frac{c}{\sqrt{|q_1|n}} \left(\frac{|a_1|}{\sqrt{|q_1|}} + 1 \right).$$

Finally we have

$$v_2 \leq \frac{c}{\sqrt{|q_1|n}} \left(\frac{|a_1|}{\sqrt{|q_1|}} + 1 \right) + cn^{-1/4} \left(\sum_{i \geq 2} \left(\frac{a_i}{q_1} \right)^2 \right)^{1/4} + cn^{-1/2} \left(\sum_{i \geq 2} \left(\frac{q_i}{q_1} \right)^2 \right)^{1/4}.$$

Consequently,

$$\begin{aligned} \rho(T(G_1, \dots, G_n), T_0) &\leq \frac{c}{n^{1/4}} \left(\sum_{i \geq 1} \left| \frac{q_i}{q_1} \right| \right)^{1/2} + \frac{c}{n^{1/4}} \left(\sum_{i \geq 2} \left(\frac{a_i}{q_1} \right)^2 \right)^{1/4} \\ &\quad + \frac{c}{\sqrt{|q_1|n}} \left(\frac{|a_1|}{\sqrt{|q_1|}} + 1 \right). \end{aligned}$$

Using the invariance of metric ρ under the linear transformations. By analogy with (33), we get

$$\rho(T(G_1, \dots, G_n), T_0) \leq \frac{c}{n^{1/4}} \left(\sum_{i \geq 1} \left| \frac{q_i}{q_1} \right| \right)^{1/2} + \frac{c}{n^{1/4}} \left(\sum_{i \geq 2} \left(\frac{a_i}{q_1} \right)^2 \right)^{1/4} + \frac{c|a_1|}{|q_1|\sqrt{n}}.$$

If $c|a_1|/(|q_1|\sqrt{n}) \geq 1$, then

$$\rho(T(G_1, \dots, G_n), T_0) \leq \frac{c}{n^{1/4}} \left(\sum_{i \geq 1} \left| \frac{q_i}{q_1} \right| \right)^{1/2} + \frac{c}{n^{1/4}} \left(\sum_{i \geq 1} \left(\frac{a_i}{q_1} \right)^2 \right)^{1/4},$$

because $\rho \leq 1$. If $c|a_1|/(|q_1|\sqrt{n}) < 1$, then $\sqrt{c|a_1|/(|q_1|\sqrt{n})} \geq c|a_1|/(|q_1|\sqrt{n})$ and the statement (17) is true in both cases.

The theorem is proved.

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REFERENCES

1. Bentkus V., Götze F. Optimal bounds in non-Gaussian limit theorems for U -statistics. — Ann. Probab., 1999, v. 27, № 1, p. 454–521.
2. Крамер Г. Математические методы статистики. М.: Мир, 1975, 648 с.
3. Королюк В. С., Боровский Ю. В. Скорость сходимости для вырожденных функционалов Мизеса. — Теория вероятн. и ее примен., 1988, т. 33, в. 1, с. 136–146.
4. Нагаев С. А., Чеботарев В. И. Об оценках скорости сходимости в центральной предельной теореме для случайных векторов со значениями в пространстве l_2 . — Математический анализ и смежные вопросы математики. Новосибирск: Наука, 1978, с. 153–182.
5. Никитин Я. Ю. Асимптотическая эффективность непараметрических критериев. М.: Наука, 1995, 238 с.
6. Петров В. В. Предельные теоремы для сумм независимых случайных величин. М.: Наука, 1987, 317 с.
7. Senatov B. V. Качественные эффекты в оценках скорости сходимости в центральной предельной теореме в многомерных пространствах. — Тр. МИАН, 1997, т. 215, с. 3–239.
8. Yanushkevichiene O. On the rate of convergence of second-degree random polynomials. — J. Math. Sci. (New York), 1998, v. 92, № 3, p. 3955–3959.
9. Yanushkevichiene O. Optimal rates of convergence of second degree polynomials in several metrics. — J. Math. Sci. (New York), 2006, v. 138, № 1, p. 5472–5479.
10. Yanushkevichiene O. Asymptotic rate of convergence in the degenerate U -statistics of second order. — Banach Center Publ., 2010, v. 90, p. 275–284.

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