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ON THE DISTRIBUTION OF COMPLEX ROOTS OF RANDOM POLYNOMIALS WITH HEAVY-TAILED COEFFICIENTS

Рассмотрим случайный полином $G_n(z) = \xi_n z^n + \dots + \xi_1 z + \xi_0$ с независимыми и одинаково распределенными комплексными коэффициентами. Предположим, что распределение случайной величины $\ln(1 + \ln(1 + |\xi_0|))$ имеет медленно меняющийся хвост. Тогда распределение комплексных корней полинома G_n концентрируется по вероятности при $n \rightarrow \infty$ около двух концентрических окружностей и является равномерным при $n \rightarrow \infty$. Окружности имеют радиусы $|\xi_0/\xi_\tau|^{1/\tau}$ и $|\xi_\tau/\xi_n|^{1/(n-\tau)}$, где ξ_τ обозначает максимальный по модулю коэффициент.

Ключевые слова и фразы: корни случайного полинома, концентрация корней, коэффициенты с тяжелыми хвостами.

1. Introduction. Consider the sequence of random polynomials

$$G_n(z) = \xi_n z^n + \xi_{n-1} z^{n-1} + \dots + \xi_1 z + \xi_0,$$

where $\xi_0, \xi_1, \dots, \xi_n, \dots$ are i.i.d. real- or complex-valued random variables. We would like to investigate the behavior of the complex roots of G_n .

The first results in this questions are due to Hammersley [2]. He derived an explicit formula for the r -point correlation function ($1 \leq r \leq n$) of the roots of G_n when the coefficients have an arbitrary absolutely continuous joint distribution.

Shparo and Shur [9] showed that under quite general assumptions the roots of G_n concentrate near the unit circle as n tends to ∞ with asymptotically uniform distribution of the argument. More precisely, denote by $R_n(a, b)$ (respectively, $S_n(\alpha, \beta)$) the number of the roots of G_n contained in the ring $\{z \in \mathbf{C}: a \leq |z| \leq b\}$ (respectively, the sector $\{z \in \mathbf{C}: \alpha \leq \arg z \leq \beta\}$). For $\varepsilon > 0$, $m \in \mathbf{Z}_+$ consider the function

$$f(t) = \left[\underbrace{\ln^+ \ln^+ \dots \ln^+ t}_{m+1} \right]^{1+\varepsilon} \cdot \prod_{k=1}^m \underbrace{\ln^+ \ln^+ \dots \ln^+ t}_k,$$

where $\ln^+ s = \max(1, \ln s)$. If for some $\varepsilon > 0$, $m \in \mathbf{Z}^+$

$$\mathbf{E} f(|\xi_0|) < \infty,$$

then for any $\delta \in (0, 1)$ and any α, β such that $-\pi \leq \alpha < \beta \leq \pi$

$$\begin{aligned} \frac{1}{n} R_n(1-\delta, 1+\delta) &\xrightarrow{\mathbf{P}} 1, & n \rightarrow \infty, \\ \frac{1}{n} S_n(\alpha, \beta) &\xrightarrow{\mathbf{P}} \frac{\beta - \alpha}{2\pi}, & n \rightarrow \infty. \end{aligned}$$

Ibragimov and Zaporozhets [4] improved this result as follows. They showed that

$$\mathbf{P} \left\{ \frac{1}{n} R_n(1-\delta, 1+\delta) \xrightarrow{n \rightarrow \infty} 1 \right\} = 1$$

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holds for any $\delta \in (0, 1)$ if and only if

$$\mathbf{E} \ln(1 + |\xi_0|) < \infty.$$

They also proved that for any α, β such that $-\pi \leq \alpha < \beta \leq \pi$

$$\mathbf{P}\left\{\frac{1}{n}S_n(\alpha, \beta) \xrightarrow{n \rightarrow \infty} \frac{\beta - \alpha}{2\pi}\right\} = 1$$

holds for *any* distribution of ξ_0 .

Shepp and Vanderbei [8] considered real-valued standard Gaussian coefficients and proved that

$$\frac{1}{n} \mathbf{E} R_n(e^{-\delta/n}, e^{\delta/n}) \longrightarrow \frac{1 + e^{-2\delta}}{1 - e^{-2\delta}} - \frac{1}{\delta}, \quad n \rightarrow \infty,$$

for any $\delta > 0$. Ibragimov and Zeitouni [3] extended this relation to the case of arbitrary i.i.d. coefficients from the domain of attraction of an α -stable law:

$$\frac{1}{n} \mathbf{E} R_n(e^{-\delta/n}, e^{\delta/n}) \longrightarrow \frac{1 + e^{-\alpha\delta}}{1 - e^{-\alpha\delta}} - \frac{2}{\alpha\delta}, \quad n \rightarrow \infty. \quad (1)$$

It is interesting to consider the limit case when $\alpha \rightarrow 0$. Then

$$\frac{1 + e^{-\alpha\delta}}{1 - e^{-\alpha\delta}} - \frac{2}{\alpha\delta} \longrightarrow 0$$

and a natural assumption for the distribution of coefficients would be a slowly varying tail. In this case (1) becomes

$$\frac{1}{n} \mathbf{E} R_n(e^{-\delta/n}, e^{\delta/n}) \longrightarrow 0, \quad n \rightarrow \infty.$$

This result (in a slightly stronger form) is proved in Theorem 1.

In contrast to the concentration near the unit circumference, there exist random polynomials with quite a different asymptotic behavior of complex roots. Zaporozhets [10] constructed a random polynomial with i.i.d. coefficients such that in average $n/2 + o(1)$ of the complex roots concentrate near the origin and the same number tends to infinity as $n \rightarrow \infty$ (moreover, the expected number of *real* roots of this polynomial is at most 9 for all n). Theorem 2 generalizes this result.

The paper is organized as follows. In Section 2 we formulate our results. In Section 3 we prove some auxiliary lemmas. The theorems are proved in Section 4.

By $\sum_j$ we always denote a summation taken over all j from $\{0, 1, \dots, n\}$. If conditions are stated for the summation, they are applied to this default range j from $\{0, 1, \dots, n\}$.

2. Results. For the sake of simplicity, we assume that $\mathbf{P}\{\xi_0 = 0\} = 0$. To treat the general case it is enough to study in the same way the behavior of the roots on the sets $\{\alpha_n = k, \beta_n = l\}$, where

$$\alpha_n = \max\{j = 0, \dots, n: \xi_j \neq 0\}, \quad \beta_n = \min\{j = 0, \dots, n: \xi_j \neq 0\}.$$

Theorem 1. *If the distribution of $|\xi_0|$ has a slowly varying tail, then for any $\delta > 0$*

$$\mathbf{P}\{R_n(e^{-\delta/n}, e^{\delta/n}) = 0\} \longrightarrow 1, \quad n \rightarrow \infty.$$

Consider the index $\tau = \tau_n \in \{0, \dots, n\}$ such that $|\xi_\tau| \geq |\xi_j|$ for $j = 0, \dots, n$. If it is not unique, we take the minimum one. Let $\omega_1, \dots, \omega_n$ be the complex roots of the system of equations

$$z^\tau + \frac{\xi_0}{\xi_\tau} = 0, \quad z^{n-\tau} + \frac{\xi_\tau}{\xi_n} = 0.$$

Theorem 2. *If the distribution of $\ln(1 + \ln(1 + |\xi_0|))$ has a slowly varying tail, then for any $\varepsilon \in (0, 1)$*

$$\mathbf{P}\{F_n(\varepsilon)\} \rightarrow 1, \quad n \rightarrow \infty,$$

where $F_n(\varepsilon)$ denotes the event that it is possible to enumerate the roots $z_1, \dots, z_n$ of G_n in such a way that

$$|z_k - w_k| < \frac{\varepsilon}{n} |w_k|$$

for $k = 1, \dots, n$.

3. Auxiliary lemmas. First we need to formulate and prove some auxiliary results. The following result is due to Pellet.

Lemma 1. *Let $g(z) = \sum_j a_j z^j$ be a polynomial of degree n . Suppose for some $k = 1, \dots, n-1$ the associated polynomial*

$$\tilde{g}(z) = \sum_{j \neq k} |a_j| z^j - |a_k| z^k$$

has exactly two positive roots R and r , $R > r$. Then g has exactly k roots inside the circle $\{z \in \mathbf{C}: |z| = r\}$ and $n - k$ roots outside the circle $\{z \in \mathbf{C}: |z| = R\}$.

P r o o f. See, e.g., [7].

The next lemma is due to Ostrowski.

Lemma 2. *Let B be a closed region in the complex plane, the boundary of which consists of a finite number of regular arcs; let the functions $f(z)$, $h(z)$ be regular on B . Assume that for all values of the real parameter t , running in the interval $a \leq t \leq b$, the function $f(z) + t \cdot h(z)$ is nonzero on the boundary of B . Then the number of the roots of $f(z) + t \cdot h(z)$ inside B is independent of t for $a \leq t \leq b$.*

P r o o f. See [6].

Lemma 3. *Consider a monic polynomial of degree n with complex coefficient $g(z) = \sum_j a_j z^j$ such that $a_n = 1$, $a_0 \neq 0$. Fix some $k = 1, \dots, n-1$ and denote by $w_1, \dots, w_{n-k}$ the roots of the equation $z^{n-k} + a_k = 0$. Put*

$$A_k = \sum_{j \neq k} |a_j|.$$

If for some $\varepsilon > 0$

$$A_k \leq \left(1 - \frac{\varepsilon}{n}\right) \left(\frac{\varepsilon}{n + \varepsilon}\right)^{n-k} |a_k|^{1/(n-k)}, \quad (2)$$

then g has exactly $n - k$ roots $z_1, \dots, z_{n-k}$ outside the unit circumference and it is possible to enumerate these roots in such a way that

$$|z_j - w_j| \leq \frac{\varepsilon}{n} |w_j|$$

for $j = 1, \dots, n - k$.

P r o o f. We will prove a stronger version of Lemma 3. Namely, we will show that the statement holds for the family of polynomials

$$g_t(z) = z^n + a_k z^k + t \sum_{j \neq k, n} a_j z^j, \quad 0 \leq t \leq 1.$$

In particular,

$$g_0(z) = z^n + a_k z^k, \quad g_1(z) = g(z).$$

Let us use Lemma 1 to estimate absolute values of the roots of g_t . Consider the associated polynomial

$$\tilde{g}_t(z) = z^n - |a_k| z^k + t \sum_{j \neq k, n} |a_j| z^j.$$

We have $\tilde{g}_t(0), \tilde{g}_t(\infty) > 0$ and it follows from (2) that $\tilde{g}_t(1) < 0$. Also, by Descartes' rule of signs, $\tilde{g}_t$ has at most two positive roots. Therefore $\tilde{g}_t$ has exactly two positive roots r_t and R_t such that

$$0 < r_t < 1 < R_t. \quad (3)$$

Now let us show that

$$\left(1 - \frac{\varepsilon}{n}\right) |a_k|^{1/(n-k)} \leq R_t \leq |a_k|^{1/(n-k)}. \quad (4)$$

Since $\tilde{g}_t(R_t) = 0$, we have

$$R_t^{n-k} + t \sum_{j \neq k, n} |a_j| R_t^{j-k} = |a_k|, \quad (5)$$

which proves the right inequality in (4).

We prove the left inequality by contradiction. Suppose, on the contrary, that

$$R_t < \left(1 - \frac{\varepsilon}{n}\right) |a_k|^{1/(n-k)}.$$

Then

$$\begin{aligned} R_t^{n-k} + t \sum_{j \neq k, n} |a_j| R_t^{j-k} &< \left(1 - \frac{\varepsilon}{n}\right)^{n-k} |a_k| + A_k R_t^{n-k-1} \\ &\leq \left(1 - \frac{\varepsilon}{n}\right)^{n-k} |a_k| + A_k \left(1 - \frac{\varepsilon}{n}\right)^{n-k-1} |a_k|^{1-1/(n-k)} \\ &= \left(1 - \frac{\varepsilon}{n}\right)^{n-k} |a_k| + \frac{A_k}{|a_k|^{1/(n-k)}} \left(1 - \frac{\varepsilon}{n}\right)^{n-k-1} |a_k|. \end{aligned}$$

It follows from (2) that

$$\frac{A_k}{|a_k|^{1/(n-k)}} \leq \frac{\varepsilon}{n},$$

therefore,

$$\begin{aligned} R_t^{n-k} + t \sum_{j \neq k, n} |a_j| R_t^{j-k} &< \left(1 - \frac{\varepsilon}{n}\right)^{n-k} |a_k| + \frac{\varepsilon}{n} \left(1 - \frac{\varepsilon}{n}\right)^{n-k-1} |a_k| \\ &= \left(1 - \frac{\varepsilon}{n}\right)^{n-k-1} |a_k| \leq |a_k|, \end{aligned}$$

which contradicts with (5). Thus (4) is proved.

It follows from (3), (4), and Lemma 1 that k roots of g_t lie inside the circle $\{z \in \mathbf{C}: |z| = 1\}$ and the other $n - k$ roots lie outside the circle $\{z \in \mathbf{C}: |z| = (1 - \varepsilon/n) |a_k|^{1/(n-k)}\}$ for all $t \in [0, 1]$.

Let z_0 be a root of g_t from the second group, i.e.,

$$|z_0| > \left(1 - \frac{\varepsilon}{n}\right) |a_k|^{1/(n-k)}. \quad (6)$$

We have

$$|z_0^n + a_k z_0^k| = t \left| \sum_{j \neq k, n} a_j z_0^j \right| \leq A_k z_0^{n-1},$$

which leads to

$$\prod_{j=1}^{n-k} |z_0 - w_j| \leq A_k |z_0|^{n-k-1}.$$

This implies that there exists an index l such that

$$|z_0 - w_l| \leq \left(\frac{A_k}{|z_0|} \right)^{1/(n-k)} |z_0|.$$

Combining this with (2) and (6) we obtain

$$\begin{aligned} |z_0 - w_l| &< \left(\frac{A_k}{(1 - \varepsilon/n)|a_k|^{1/(n-k)}} \right)^{1/(n-k)} |z_0| \\ &\leq \frac{\varepsilon}{n + \varepsilon} |z_0| \leq \frac{\varepsilon}{n + \varepsilon} |w_l| + \frac{\varepsilon}{n + \varepsilon} |z_0 - w_l|, \end{aligned}$$

which produces

$$|z_0 - w_l| < \frac{\varepsilon}{n} |w_l| = \frac{\varepsilon}{n} |a_k|^{1/(n-k)}.$$

It means that all roots of g_t from the second group belong to $\bigcup_{m=1}^{n-k} B_m$, where $B_m = \{z \in \mathbf{C}: |z - w_m| < \varepsilon |w_m|/n\}$. Since $\varepsilon/n < \sin[\pi/(n-k)]$, all $B_1, \dots, B_{n-k}$ are disjoint. Therefore g_t does not vanish on the boundary of B_m for all $t \in [0, 1]$, $m = 1, \dots, n-k$. To conclude the proof, it remains to show that every B_m contains exactly one root of g_t . Obviously, this is true for $t = 0$. Therefore, by Lemma 2, this is also true for all $t \in [0, 1]$.

Lemma 4. Let $\{\eta_j\}_{j=0}^\infty$ be nonnegative i.i.d. random variables. Put

$$S_n = \sum_j \eta_j, \quad M_n = \max\{\eta_j\}_{j=0}^n.$$

(a) The distribution of η_0 has a slowly varying tail if and only if

$$\frac{M_n}{S_n} \xrightarrow{\mathbf{P}} 1, \quad n \rightarrow \infty.$$

(b) The distribution of η_0 has an infinite mean if and only if

$$\frac{S_n - M_n}{n} \xrightarrow{\text{a.s.}} \infty, \quad n \rightarrow \infty.$$

P r o o f. For (a) see [1], for (b) see [5, Theorem 2.1].

Lemma 5. Suppose $a_0, a_1, \dots, a_n \geq 0$ and $\varepsilon > 0$. If for some $k = 1, \dots, n-1$

$$\prod_{j \neq k} (1 + a_j)^{2n^2} \leq 1 + a_k$$

and

$$a_k \geq 2(1 - \varepsilon)^{-4n^2/(4n-1)} \varepsilon^{-4n^3/(4n-1)} (n + \varepsilon)^{4n^3/(4n-1)}, \quad (7)$$

then

$$\sum_{j \neq k} a_j + 1 \leq \left(1 - \frac{\varepsilon}{n}\right) \left(\frac{\varepsilon}{n + \varepsilon}\right)^{n-k} a_k^{1/(n-k)}.$$

P r o o f. Since $1 + \sum_{j \neq k} a_j \leq \prod_{j \neq k} (1 + a_j)$, it suffices to show that

$$(2a_k)^{1/(2n)^2} \leq (1 - \varepsilon) \left(\frac{\varepsilon}{n + \varepsilon}\right)^n a_k^{1/n},$$

which is equivalent to (7).

4. Proofs of theorems.

P r o o f o f T h e o r e m 1. By Lemma 4 (a), for any $\delta > 0$ we have $\mathbf{P}\{A_n\} \rightarrow 1$, $n \rightarrow \infty$, where

$$A_n = \left\{ |\xi_\tau| > e^\delta \sum_{j \neq \tau} |\xi_j| \right\}.$$

Consider the associated polynomial

$$\tilde{G}_n(z) = \sum_{j \neq \tau} |\xi_j| z^j - |\xi_\tau| z^\tau.$$

Suppose A_n occurs. If $1 \leq t \leq e^{\delta/n}$, then

$$|\xi_\tau t^\tau| > e^\delta \sum_{j \neq \tau} |\xi_j| \geq t^n \sum_{j \neq \tau} |\xi_j| \geq \left| \sum_{j \neq \tau} t^j \xi_j \right|.$$

If $e^{-\delta/n} \leq t \leq 1$, then

$$|\xi_\tau t^\tau| \geq e^{-\delta} |\xi_\tau| > \sum_{j \neq \tau} |\xi_j| \geq \left| \sum_{j \neq \tau} t^j \xi_j \right|.$$

Therefore $\tilde{G}_n$ does not have real roots in the interval $[e^{-\delta/n}, e^{\delta/n}]$. Further, $\tilde{G}_n(0) > 0$, $\tilde{G}_n(\infty) > 0$, and $\tilde{G}_n(1) < 0$. By Descartes' rule of signs, $\tilde{G}_n$ has at most two positive roots. Thus $\tilde{G}_n$ has exactly two positive roots r and R such that $0 < r < e^{-\delta/n} < e^{\delta/n} < R$.

By Lemma 1, G_n has exactly τ roots inside the circle $\{z \in \mathbf{C}: |z| = e^{-\delta/n}\}$ and $n - \tau$ roots outside the circle $\{z \in \mathbf{C}: |z| = e^{\delta/n}\}$. Therefore, A_n implies that $R_n(e^{-\delta/n}, e^{\delta/n}) = 0$ which concludes the proof.

P r o o f o f T h e o r e m 2. Consider the events

$$A_n = \left\{ \prod_{j \neq \tau} \left(1 + \frac{|\xi_j|}{|\xi_n|} \right)^{2n^2} \leq 1 + \frac{|\xi_\tau|}{|\xi_n|} \right\}$$

and $B_n = \{|\xi_\tau|/|\xi_n| \geq 2(1 - \varepsilon)^{-4n^2/(4n-1)} \varepsilon^{-4n^3/(4n-1)} (n + \varepsilon)^{4n^3/(4n-1)}\}$. Since the distribution of $\ln(1 + \ln(1 + |\xi_0|))$ has a slowly varying tail, by Lemma 4 (a)

$$\mathbf{P} \left\{ 4 \sum_{j \neq \tau} \ln(1 + \ln(1 + |\xi_j|)) \leq \ln(1 + \ln(1 + |\xi_\tau|)) \right\} \rightarrow 1, \quad n \rightarrow \infty,$$

which implies

$$\mathbf{P} \left\{ \left(\sum_{j \neq \tau} \ln(1 + |\xi_j|) \right)^4 \leq \ln(1 + |\xi_\tau|) \right\} \rightarrow 1, \quad n \rightarrow \infty. \quad (8)$$

Since $\mathbf{E} \ln(1 + |\xi_0|) = \infty$, by Lemma 4 (b) with probability one

$$\frac{1}{n} \sum_{j \neq \tau} \ln(1 + |\xi_j|) \rightarrow \infty, \quad n \rightarrow \infty,$$

which together with (8) produces

$$\mathbf{P} \left\{ n^3 \sum_{j \neq \tau} \ln(1 + |\xi_j|) \leq \ln(1 + |\xi_\tau|) \right\} \rightarrow 1, \quad n \rightarrow \infty,$$

and $\mathbf{P}\{\ln(1 + |\xi_\tau|) \geq n^4\} \rightarrow 1$, $n \rightarrow \infty$. Since for any $\delta > 0$ there exists $T > 0$ such that $\mathbf{P}\{T^{-1} < |\xi_n| < T\} > 1 - \delta$, the last two inequalities imply $\mathbf{P}\{A_n\}, \mathbf{P}\{B_n\} \rightarrow 1$, $n \rightarrow \infty$.

By Lemma 5, the event $A_n \cap B_n$ implies that the polynomial $G_n(z)/\xi_n$ satisfies the conditions of Lemma 3. Thus we have proved the theorem for the roots of G_n lying outside the unit circumference. To treat the rest of the roots consider the associated polynomial

$$G_n^*(z) = z^n G_n\left(\frac{1}{z}\right) = \sum_j \xi_j z^{n-j}$$

and note that z_0 is a root of G_n if and only if z^{-1} is a root of $G_n^*(z)$.

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ASYMPTOTIC TAIL PROPERTIES OF THE DISTRIBUTIONS IN THE CLASS OF DISPERSION MODELS

Класс дисперсионных моделей, введенный Б. Йоргенсеном, содержит многие известные распределения, в том числе нормальное, Стьюдента t ,

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