Math-Net.Ru

Общероссийский математический портал

A. B. Simas, G. M. Cordeiro, S. Nadarajah, Asymptotic tail properties of the distributions in the class of dispersion models, *Теория вероятн. и ее примен.*, 2011, том 56, выпуск 4, 818–826

DOI: <https://doi.org/10.4213/tvp4428>

Использование Общероссийского математического портала Math-Net.Ru подразумевает, что вы прочитали и согласны с пользовательским соглашением

<http://www.mathnet.ru/rus/agreement>

Параметры загрузки:

IP: 35.173.137.237

26 апреля 2023 г., 18:22:57

By Lemma 5, the event $A_n \cap B_n$ implies that the polynomial $G_n(z)/\xi_n$ satisfies the conditions of Lemma 3. Thus we have proved the theorem for the roots of G_n lying outside the unit circumference. To treat the rest of the roots consider the associated polynomial

$$G_n^*(z) = z^n G_n\left(\frac{1}{z}\right) = \sum_j \xi_j z^{n-j}$$

and note that z_0 is a root of G_n if and only if z^{-1} is a root of $G_n^*(z)$.

REFERENCES

1. *Darling D.* The influence of the maximum term in the addition of independent random variables. — Trans. Amer. Math. Soc., 1952, v. 73, p. 95–107.
2. *Hammersley J. M.* The zeros of a random polynomial. — Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, v. II. Berkeley: Univ. California Press, 1956, p. 89–111.
3. *Ibragimov I., Zeitouni O.* On roots of random polynomials. — Trans. Amer. Math. Soc., 1997, v. 349, p. 2427–2441.
4. *Ibragimov I., Zaporozhets D.* On distribution of zeros of random polynomials in complex plane, arXiv:1102.3517v1, 2011, Prokhorov Festschrift, Springer-Verlag (to appear).
5. *Kesten H., Maller R. A.* The effect of trimming on the strong law of large numbers. — Proc. London Math. Soc., 1995, v. 71, p. 441–480.
6. *Ostrowski A. M.* Solution of Equations in Euclidean and Banach Spaces. New York: Academic Press, 1973, 412 p.
7. *Rahman Q. I., Schmeisser G.* Analytic Theory of Polynomials. Oxford: Oxford University Press, 2002, 742 p.
8. *Shepp L. A., Vanderbei R. J.* The complex zeros of random polynomials. — Trans. Amer. Math. Soc., 1995, v. 347, № 11, p. 4365–4384.
9. *Шнапо Д. И., Шур М. Г.* О распределении корней случайных полиномов. — Вестн. Моск. ун-та, 1962, № 3, с. 40–43.
10. *Запорожец Д. Н.* Пример случайного полинома с необычным поведением корней. — Теория вероятн. и ее примен., 2005, т. 50, в. 3, с. 549–555.

Поступила в редакцию
10.VI.2011

© 2011 г.

SIMAS A. B.*, CORDEIRO G. M.**,
NADARAJAH S.***

ASYMPTOTIC TAIL PROPERTIES OF THE DISTRIBUTIONS IN THE CLASS OF DISPERSION MODELS

Класс дисперсионных моделей, введенный Б. Йоргенсеном, содержит многие известные распределения, в том числе нормальное, Стьюдента t ,

* Universidade Federal da Paraíba, Centro de Ciências Exatas e da Natureza—Campus I, Departamento de Matemática, Campus I, Cidade Universitária, 58059-900—João Pessoa, PB—Brasil; e-mail: alesimas@impa.br

** Departamento de Estatística e Informática, Universidade Federal Rural de Pernambuco, Rua Dom Manoel de Medeiros s/n, Dois Irmãos, 52171-900 Recife-PE, Brasil; e-mail: gausscordeiro@uol.com.br

*** School of Mathematics, University of Manchester, Manchester M13 9PL, UK; e-mail: saralees.nadarajah@manchester.ac.uk

гамма-, обратное гауссовское, гиперболическое, Мизеса. Мы изучаем асимптотическое (при малой дисперсии) поведение плотностей вероятности для дисперсионных моделей, допускающих равномерно сходящуюся седловую аппроксимацию.

Ключевые слова и фразы: дисперсионные модели, седловая аппроксимация, свойства хвостов.

1. Introduction. The class of dispersion models was defined to extend the exponential family models in such a way that most of their good properties were preserved. As a byproduct of this definition, another class could also be defined, the so-called proper dispersion models. The dispersion model, defined in the convex support C with mean parameter μ in an open interval $\Omega \subset C$ and dispersion parameter $\sigma^2 > 0$, is a family of distributions whose probability density functions (pdfs) with respect to some σ -finite measure (usually the Lebesgue or the counting measures) can be written in the form

$$f(y; \mu, \sigma^2) = a(y; \sigma^2) \exp \left\{ -\frac{1}{2\sigma^2} d(y; \mu) \right\}, \quad y \in C, \quad (1)$$

where $a(\cdot; \cdot) \geq 0$ is a suitable function and $d = d(y; \mu)$ is a unit deviance on $C \times \Omega$, $\mu \in \Omega$. We denote (1) by the symbol $\text{DM}(\mu, \sigma^2)$. The distribution belongs to the class of proper dispersion models [7], [8] if its pdf can be written in form (1) with $a(y; \sigma^2)$ being decomposed as $a(y; \sigma^2) = d_1(\sigma^2)d_2(y)$, where $d_1(\cdot)$ and $d_2(\cdot)$ are suitable functions. Further, the exponential dispersion models (see [6], [8]) can be obtained by taking in (1) $d(y; \mu) = \theta y - b(\theta) + h(y)$, where $\mu = b'(\theta)$ and $h(y)$ is such that $\theta\mu + h(\mu) = b(\theta)$. The class of proper dispersion models covers important distributions which are not covered by the exponential dispersion models, such as the log-gamma, the Leipnik and the reciprocal inverse Gaussian distributions. The von Mises distribution, which also belongs to the class of proper dispersion models and does not belong to the exponential dispersion models, is particularly useful for the analysis of circular data; see [9] and Fisher [4]. See [8] for other important examples of proper dispersion models. Moreover, the dispersion models have two important general properties. First, the distribution of the statistic $D = d(Y; \mu)$ does not depend on μ when σ^2 is known, that is, D is a pivotal statistic for μ . Second, (1) is an exponential family model with canonical statistic D when σ^2 is known.

2. Main theorem. First, we introduce some definitions. Let $\Omega \subset C \subset \mathbf{R}$ be intervals with Ω open. A function $d: C \times \Omega \rightarrow \mathbf{R}$ is called a unit deviance if it satisfies the conditions

$$d(y; y) = 0 \quad \forall y \in \Omega, \quad d(y; \mu) > 0 \quad \forall y \neq \mu. \quad (2)$$

Unit deviances may often be constructed on the basis of log-likelihoods. In the following, we use the notation $\partial_\mu d$ and $\partial_{\mu y}^2 d$ to denote the partial derivatives of $d = d(y; \mu)$ with respect to μ and μ and y , respectively, and analogously for higher order derivatives. A unit deviance d is called regular if $d(y; \mu)$ is twice continuously differentiable with respect to (y, μ) on $\Omega \times \Omega$ and satisfies

$$\partial_{\mu\mu}^2 d(\mu; \mu) > 0 \quad \forall \mu \in \Omega.$$

The unit variance function $V: \Omega \rightarrow \mathbf{R}_+$ of a regular unit deviance is defined by

$$V(\mu) = \frac{2}{\partial_{\mu\mu}^2 d(\mu; \mu)}.$$

Moreover, it is easy to show that a regular unit deviance d satisfies

$$\partial_\mu d(\mu; \mu) = 0, \quad \partial_y d(\mu; \mu) = 0 \quad (3)$$

for all $\mu \in \Omega$. This holds since (2) implies that the function $d(y; \cdot)$ has a minimum at y and that the function $d(\cdot; \mu)$ has a minimum at μ . It can be shown by the chain rule

$$\partial_{yy}^2 d(\mu; \mu) = \partial_{\mu\mu}^2 d(\mu; \mu) = -\partial_{\mu y}^2 d(\mu; \mu) \quad \forall \mu \in \Omega,$$

which gives three equivalent expressions to compute the variance function.

The saddlepoint approximation for a dispersion model with regular unit deviance d is defined for $y \in \Omega$ by

$$f(y; \mu, \sigma^2) \sim \{2\pi\sigma^2 V(y)\}^{-1/2} \exp \left\{ -\frac{1}{2\sigma^2} d(y; \mu) \right\} \quad \text{as } \sigma^2 \rightarrow 0 \quad (4)$$

with the notation $a(x) \sim b(x)$ meaning that the ratio $a(x)/b(x)$ goes to one as $x \rightarrow 0$. The saddlepoint approximation (4) holds when σ^2 tends to zero for an extensive range of models. It is even exact for a few special models, such as the normal and simplex distributions (see [1]). The approximation (4) is equivalent to

$$\sigma a(y; \sigma^2) \rightarrow \{2\pi V(y)\}^{-1/2} \quad \text{as } \sigma^2 \rightarrow 0. \quad (5)$$

The saddlepoint approximation is said to be *uniformly convergent* if the convergence in equation (5) is uniform in y , and is said to converge *uniform on compact sets* if the convergence in (5) is uniform in y on compact subsets of Ω . Note that uniformly convergent saddlepoint approximation is a much stronger requirement than uniform convergence of the saddlepoint approximation on compact sets. Moreover, Jørgensen [8] proved that the saddlepoint approximation is uniform on compacts for both proper dispersion models and exponential dispersion models. The asymptotic behavior as $\sigma^2 \rightarrow 0$ was described in [6] as small dispersion asymptotics. Consider the following theorem whose proof can be found in [8, p. 30].

Theorem 1. *Let $Y \sim \text{DM}(\mu_0 + \sigma\mu, \sigma^2)$ be a dispersion model with uniformly convergent saddlepoint approximation. Then,*

$$\frac{Y - \mu_0}{\sigma} \xrightarrow{d} N(\mu, V(\mu_0)) \quad \text{as } \sigma^2 \rightarrow 0,$$

where $\xrightarrow{d}$ denotes convergence in distribution.

Theorem 1 holds for both dispersion and exponential dispersion models. It generalizes the well-known result that the pdf of the Student's t_n random variable with n degrees of freedom converges to the standard normal pdf, i.e., $t_n \xrightarrow{d} N(0, 1)$ as $n \rightarrow \infty$. We can verify this fact by noting that $t_n/\sqrt{n} + \mu$ is a $\text{DM}(\mu, (n+1)^{-1})$ with unit deviance $d(y; \mu) = \ln\{1 + (y - \mu)^2\}$. So, from the definition of the variance function, it follows that $V(\mu) = 1$ and using Theorem 1 we obtain

$$\frac{\sqrt{n+1}}{\sqrt{n}} t_n \rightarrow N(0, 1).$$

Further, in the proof of Theorem 1 given in [8, p. 30], it is shown that the pdf of $(Y - \mu_0)/\sigma$, where Y is a $\text{DM}(\mu_0 + \sigma\mu, \sigma^2)$, converges to the normal pdf when $\sigma^2 \rightarrow 0$, i.e.,

$$f(x; \mu, \sigma^2) \sim \phi(x; \mu, V(\mu_0))$$

for all $x \in \mathbf{R}$, where $\phi(x; \mu, V(\mu_0))$ denotes the normal pdf with mean μ and variance $V(\mu_0)$. We now give our main result.

Theorem 2. *Let $Y \sim \text{DM}(\mu_0 + \sigma\mu; \sigma^2)$ be a dispersion model with uniformly convergent saddlepoint approximation and let $Z = (Y - \mu_0)/\sigma$. We assume that the unit deviance d is of class C^k , that is, is continuously k times differentiable, k a positive integer, and $\partial_{y \dots y}^k d(\mu_0; \mu_0) \neq 0$, where $\partial_{y \dots y}^k$ denotes k derivatives with respect to y . Suppose further that $\partial_{y \dots y}^i d(\mu_0 + \sigma\mu; \mu_0 + \sigma\mu) = 0$ for σ sufficiently small and for $2 < i < k$. So, if $x_\sigma \in (0, \infty)$ for $\sigma \in (0, \infty)$ is such that $\lim_{\sigma^2 \rightarrow 0} (x_\sigma - \mu)^k \sigma^{k-2} = \beta \in [0, \infty]$, then the pdf $f(x; \mu, \sigma^2)$ of Z satisfies*

$$\lim_{\sigma^2 \rightarrow 0} \frac{f(x_\sigma; \mu, \sigma^2)}{\phi(x_\sigma; \mu, V(\mu_0))} = \exp \left\{ \frac{-\beta}{2k!} \partial_{y \dots y}^k d(\mu_0; \mu_0) \right\}.$$

R e m a r k 1. The conclusions of Theorem 2 also hold if (i) the saddlepoint approximation is uniform on compact sets, and if (ii) there exists a compact set $C \subset \Omega \cap (0, \infty)$ such that $x_\sigma \in C$ as $\sigma^2 \rightarrow 0$. As previously remarked, condition (i) holds for exponential dispersion models and proper dispersion models. Condition (ii) holds if, for instance, $\mu \in \text{int } \Omega \cap (0, \infty)$ and $0 < \beta < \infty$, since this implies that x_σ will be in a neighborhood of μ and that μ is away from the boundary of Ω and away from 0.

This theorem shows that data with small dispersions can be analyzed by means of the normal distribution, using the traditional method of transformations, where a transformation of the data is sought to linearize the model and stabilize the variance. This theorem can also be applied to such problems as the false discovery rate in multiple testing problems with large numbers of hypotheses and extremely small critical values for the smallest ordered p values; see [2].

3. Proof of the theorem. The proof is given for $k = 4$ since other cases are analogous. Let $f_Y(y; \mu, \sigma^2)$ be the pdf of Y and let $x_\sigma \in (0, \infty)$ for $\sigma \in (0, \infty)$ be such that $\lim_{\sigma^2 \rightarrow 0} (x_\sigma - \mu)^4 \sigma^2 = \beta \in [0, \infty]$. Further, the pdf of $Z = (Y - \mu_0)/\sigma$ is

$$f(x; \mu, \sigma^2) = \sigma f_Y(\mu_0 + \sigma x; \mu_0 + \sigma \mu, \sigma^2).$$

Using (1), we obtain

$$f(x; \mu, \sigma^2) = \sigma a(\mu_0 + \sigma x; \sigma^2) \exp \left\{ - (2\sigma^2)^{-1} d(\mu_0 + \sigma x; \mu_0 + \sigma \mu) \right\}.$$

So, looking only at x_σ from now on, the uniform saddlepoint approximation (or the saddlepoint approximation uniform on compact sets together with the fact that x_σ belongs to some compact set as $\sigma^2 \rightarrow 0$) yields

$$f(x_\sigma; \mu, \sigma^2) \sim \{2\pi V(\mu_0 + \sigma x_\sigma)\}^{-1/2} \exp \left\{ - (2\sigma^2)^{-1} d(\mu_0 + \sigma x_\sigma; \mu_0 + \sigma \mu) \right\},$$

and the continuity of the variance function $V(\cdot)$ gives

$$f(x_\sigma; \mu, \sigma^2) \sim \{2\pi V(\mu_0)\}^{-1/2} \exp \left\{ - (2\sigma^2)^{-1} d(\mu_0 + \sigma x_\sigma; \mu_0 + \sigma \mu) \right\}. \quad (6)$$

So, (6) yields

$$\frac{f(x_\sigma; \mu, \sigma^2)}{\phi(x_\sigma; \mu, V(\mu_0))} \sim \exp \left\{ \frac{(x_\sigma - \mu)^2}{2V(\mu_0)} - (2\sigma^2)^{-1} d(\mu_0 + \sigma x_\sigma; \mu_0 + \sigma \mu) \right\}.$$

We now focus on the function

$$g_\sigma(x_\sigma) = (2\sigma^2)^{-1} d(\mu_0 + \sigma x_\sigma; \mu_0 + \sigma \mu).$$

Since the function d is of class C^4 , the Taylor series expansion up to the fourth order in the variable x_σ centering in μ yields

$$\begin{aligned} g_\sigma(x_\sigma) = (2\sigma^2)^{-1} & \left\{ d(\mu_0 + \sigma \mu; \mu_0 + \sigma \mu) \right. \\ & + \sigma(x_\sigma - \mu) \partial_y d(\mu_0 + \sigma \mu; \mu_0 + \sigma \mu) + \frac{1}{2} \sigma^2 (x_\sigma - \mu)^2 \partial_{yy}^2 d(\mu_0 + \sigma \mu; \mu_0 + \sigma \mu) \\ & + \frac{1}{6} \sigma^3 (x_\sigma - \mu)^3 \partial_{yyy}^3 d(\mu_0 + \sigma \mu; \mu_0 + \sigma \mu) \\ & \left. + \frac{1}{24} \sigma^4 (x_\sigma - \mu)^4 \partial_{yyyy}^4 d(\mu_0 + \sigma \mu; \mu_0 + \sigma \mu) + \mathcal{O}(\sigma^5 (x_\sigma - \mu)^5) \right\}. \end{aligned}$$

From properties (2) and (3), we have $d(\mu_0 + \sigma \mu; \mu_0 + \sigma \mu) = 0$ and $\partial_y d(\mu_0 + \sigma \mu; \mu_0 + \sigma \mu) = 0$. By assumption $\partial_{yyy}^3 d(\mu_0 + \sigma \mu; \mu_0 + \sigma \mu) = 0$ and from the definition of the unit variance

function $V(\cdot)$, it follows that $\partial_{yy}^2 d(\mu_0 + \sigma\mu; \mu_0 + \sigma\mu) = 2/V(\mu_0 + \sigma\mu)$. So, we obtain

$$\frac{f(x_\sigma; \mu, \sigma^2)}{\phi(x_\sigma; \mu, V(\mu_0))} \sim \exp \left\{ -\frac{1}{48} \sigma^2 (x_\sigma - \mu)^4 \partial_{yyyy}^4 d(\mu_0 + \sigma\mu; \mu_0 + \sigma\mu) + \mathcal{O}(\sigma^3 (x_\sigma - \mu)^5) \right\}.$$

The continuity of the fourth derivative together with the fact that $\sigma(x_\sigma - \mu)^4 \rightarrow \beta$ as $\sigma^2 \rightarrow 0$ yields

$$\sigma^2 (x_\sigma - \mu)^4 \partial_{yyyy}^4 d(\mu_0 + \sigma\mu; \mu_0 + \sigma\mu) \rightarrow \beta \partial_{yyyy}^4 d(\mu_0; \mu_0). \quad (7)$$

Moreover, the hypothesis that $\sigma^2 (x_\sigma - \mu)^4 \rightarrow \beta$ as $\sigma^2 \rightarrow 0$ also implies that

$$\mathcal{O}(\sigma^3 (x_\sigma - \mu)^5) = o(1).$$

So, from the last equation and (7), we obtain

$$\frac{f(x_\sigma; \mu, \sigma^2)}{\phi(x_\sigma; \mu, V(\mu_0))} \sim \exp \left\{ -\frac{\beta}{48} \partial_{yyyy}^4 d(\mu_0; \mu_0) \right\},$$

which concludes the proof. The rate of growth of the argument of the pdfs is in general $1/\sigma^{(k-2)/k}$ for $k = 3, 4, \dots$ depending on differentiability properties of $d(y; \mu)$ (e.g., in the Student's t case, $k = 4$).

4. Special cases. In order to apply Theorem 2 to a special distribution, we write the distribution in the form (1) and compute the third derivative, check if it is zero, or not, if so, compute the fourth derivative, check if it is zero and so on, and then apply the results. We now provide some special cases. We emphasize that every special distribution is either exponential dispersion model or proper dispersion model, in such a way that the saddlepoint approximation holds uniformly on compact sets. So, we have only to check if this approximation is uniform, or if condition (ii) in Remark 1 holds.

Student's t distribution. Let t_n be a Student's t random variable with n degrees of freedom. Then, $t_n/\sqrt{n} + \mu_0$ is a $DM(\mu_0, (n+1)^{-1})$, where the function $a(y; \sigma^2)$ does not depend on y and $d(y; \mu) = \ln\{1 + (y - \mu)^2\}$. Let $f_n(y)$ be the pdf of t_n . The saddlepoint approximation is uniform in y , since $V(y) = 1$ is constant. Here, $\partial_{yyy}^3 d(\mu_0; \mu_0) = 0$, $\partial_{yyyy}^4 d(\mu_0; \mu_0) = -12$ and taking $\mu_0 = 0$, we have

$$\frac{f_n(x_\sigma)}{\phi(x_\sigma; 0, 1)} \rightarrow \exp \left(\frac{\beta}{4} \right) \quad \text{as } \sigma \rightarrow 0,$$

a result due to Finner et al. [2].

Generalized Student's t distribution. The pdf of the generalized Student's t random variable $X_{r,s}$ with parameters $r > 0$ and $s > 0$ is given by

$$f(x; s, r) = \frac{1}{\sqrt{s}B(r/2, 1/2)} \left(1 + \frac{x^2}{s} \right)^{-(r+1)/2},$$

($B(\cdot, \cdot)$ is the beta function). Here, $a(x; \sigma^2)$ does not depend on x and the saddlepoint approximation is uniform in x , since $V(x)$ is constant. It includes Student's t_n random variable when $s = r = n$. We consider two cases: first, s is known and fixed, and then $X_{r,s} + \mu_0$ is a dispersion model with $\sigma^2 = (r+1)^{-1}$, $d(y; \mu) = \ln\{1 + (y - \mu)^2/s\}$ and $V(\mu) = s$. Since $\partial_{yyy}^3 d(\mu_0; \mu_0) = 0$ and $\partial_{yyyy}^4 d(\mu_0; \mu_0) = -12/s^2$, the pdf $f_r(x)$ of $\sqrt{r+1}(X_{r,s} - \mu_0)$ satisfies

$$\frac{f_r(x_\sigma)}{\phi(x_\sigma; 0, s)} \rightarrow \exp \left\{ \frac{\beta}{4s^2} \right\} \quad \text{as } \sigma \rightarrow 0 \text{ (or } r \rightarrow \infty).$$

For the second case, s depends on r and, also, $s(r)/r \rightarrow a > 0$ as $r \rightarrow \infty$. So, $X_{r,s}/\sqrt{s} + \mu_0$ is a DM $(\mu_0, (r+1)^{-1})$ with $d(y; \mu) = \log\{1 + (y - \mu)^2\}$ and $V(\mu) = 1$. Here, $\partial_{yyy}^3 d(\mu_0; \mu_0) = 0$ and $\partial_{yyy}^4 d(\mu_0; \mu_0) = -12$ and taking $\mu_0 = 0$, the pdf of $\sqrt{(r+1)/s} X_{r,s}$, and then the pdf $g_{r,s}(x)$ of $a^{-1} X_{r,s}$ satisfies

$$\frac{g_{r,s}(x_\sigma)}{\phi(x_\sigma; 0, 1)} \rightarrow \exp\left(\frac{\beta}{4}\right) \quad \text{as } \sigma^2 \rightarrow 0 \text{ (or as } r \rightarrow \infty).$$

Moreover, the pdf $f_{r,s}(x)$ of $X_{r,s}$ satisfies

$$\frac{f_{r,s}(ax_\sigma)}{\phi(x_\sigma; 0, 1)} \rightarrow a^{-1} \exp\left(\frac{\beta}{4}\right) \quad \text{as } r \rightarrow \infty.$$

Gamma distribution. Let X_λ be a gamma random variable with mean μ_0 and precision parameter λ , which is a DM (μ_0, λ^{-1}) with $d(y; \mu) = 2\{y/\mu - \ln(y/\mu) - 1\}$ and $V(\mu) = \mu^2$. For $\mu > 0$, the approximation is uniform on compact sets. Here, $\partial_{yyy}^3 d(\mu_0; \mu_0) = -4/\mu_0^3$, and

$$\frac{f_\lambda(x_\sigma)}{\phi(x_\sigma; 0, \mu_0^2)} \rightarrow \exp\left\{\frac{\beta}{3\mu_0^3}\right\} \quad \text{as } \sigma^2 \rightarrow 0 \text{ (or } \lambda \rightarrow \infty),$$

where $f_\lambda(x)$ is the pdf of $\sqrt{\lambda}(X_\lambda - \mu_0)$.

Reciprocal gamma distribution. The reciprocal gamma distribution with mean parameter μ_0 and precision parameter $\lambda > 0$ is a DM (μ_0, λ^{-1}) with $d(y; \mu) = 2(\mu/y - \log(\mu/y) - 1)$ and $V(\mu) = \mu^2$. For $\mu > 0$, the approximation is uniform on compact sets. We have $\partial_{yyy}^3 d(\mu_0; \mu_0) = -8/\mu_0^3$ and the pdf $f_\lambda(x)$ of $\sqrt{\lambda}(X_\lambda - \mu_0)$ satisfies

$$\frac{f_\lambda(x_\sigma)}{\phi(x_\sigma; 0, \mu_0^2)} \rightarrow \exp\left\{\frac{2\beta}{3\mu_0^3}\right\} \quad \text{as } \lambda \rightarrow \infty.$$

Log-gamma distribution. For the log-gamma random variable with parameters μ_0 and $\lambda > 0$, $a(y; \sigma^2)$ does not depend on y . It is a DM (μ_0, λ^{-1}) with $d(y; \mu) = 2\{-y + \mu + \exp(y - \mu) - 1\}$. Here, $V(\mu) = 1$ leads to a uniform approximation in y . Since $\partial_{yyy}^3 d(\mu_0; \mu_0) = 2$, the pdf $f_\lambda(x)$ of $\sqrt{\lambda}(X_\lambda - \mu_0)$ satisfies

$$\frac{f_\lambda(x_\sigma)}{\phi(x_\sigma; 0, \mu_0^2)} \rightarrow \exp\left(-\frac{\beta}{6}\right) \quad \text{as } \lambda \rightarrow \infty.$$

Generalized hyperbolic secant distribution. The pdf of the generalized hyperbolic secant (GHS) distribution is given by

$$f(y; \mu_0, \lambda) = \lambda c^*(\lambda y; \lambda) \exp\left\{\lambda[y \operatorname{arctg}(\mu_0) - 0.5 \log(1 + \mu_0^2)]\right\},$$

where

$$c^*(z; \lambda) = \frac{2^{\lambda-2} |\Gamma(\lambda/2 + iz/2)|^2}{\pi \Gamma(\lambda)},$$

and μ_0 and λ are the mean and the precision parameters, respectively. So, X_λ follows a GHS distribution, which is a DM (μ_0, λ^{-1}) with $d(y; \mu) = 2y(\arctan y - \arctan \mu) + \log\{(1 + \mu^2)/(1 + y^2)\}$ and $V(\mu) = 1 + \mu^2$. So, for $\mu > 0$, the saddlepoint approximation is uniform on compact sets. Further, $\partial_{yyy}^3 d(\mu_0; \mu_0) = -4\mu_0/(1 + \mu_0^2)^2$, which implies that the pdf $f_\lambda(x)$ of $\sqrt{\lambda}(X_\lambda - \mu_0)$ satisfies

$$\frac{f_\lambda(x_\sigma)}{\phi(x_\sigma; 0, 1 + \mu_0^2)} \rightarrow \exp\left\{\frac{\beta \mu_0}{3(1 + \mu_0^2)^2}\right\} \quad \text{as } \sigma^2 \rightarrow 0 \text{ (or } \lambda \rightarrow \infty).$$

Inverse Gaussian distribution. It is a dispersion model $DM(\mu_0, \sigma^2)$ with $d(y; \mu) = (y - \mu)^2 / (y\mu^2)$. Here, $V(\mu) = \mu^3$ and the saddlepoint approximation is exact. We obtain $\partial_{yyy}^3 d(\mu_0; \mu_0) = -6\mu_0^{-4}$, so that the pdf $f(x)$ of $(X - \mu_0)/\sigma$ satisfies

$$\frac{f(x_\sigma)}{\phi(x_\sigma; 0, \mu_0^3)} \rightarrow \exp\{\beta/(2\mu_0^4)\}$$

when $\sigma^2 \rightarrow 0$.

Reciprocal inverse Gaussian distribution. It is a $DM(\mu_0, \lambda^{-1})$ with $d(y; \mu) = (y - \mu)^2 / y$ and $V(\mu) = \mu$. The saddlepoint approximation is exact. We have $\partial_{yyy}^3 d(\mu_0; \mu_0) = -6\mu_0^{-2}$ and the pdf $f(x)$ of $(X - \mu_0)/\sigma$ satisfies

$$\frac{f(x_\sigma)}{\phi(x_\sigma; 0, \mu_0)} \rightarrow \exp\left\{\frac{\beta}{2\mu_0^2}\right\} \quad \text{as } \sigma^2 \rightarrow 0.$$

Hyperbola distribution. The pdf (with parameters μ_0 and λ) is

$$f(y; \mu_0, \lambda) = \frac{\exp(-\lambda)}{2K_0(\lambda)} y^{-1} \exp\left\{-\frac{\lambda(y - \mu_0)^2}{2y\mu_0}\right\},$$

where K_0 is a Bessel function. It is a $DM(\mu_0, \lambda^{-1})$, $d(y; \mu) = (y - \mu)^2 / (y\mu)$ and $V(\mu) = \mu^2$. For $\mu > 0$, the approximation is uniform on compacts. Further, $\partial_{yyy}^3 d(\mu_0; \mu_0) = -6\mu_0^{-3}$ which implies that

$$\frac{f(x_\sigma)}{\phi(x_\sigma; 0, \mu_0^2)} \rightarrow \exp\left\{\frac{\beta}{2\mu_0^3}\right\} \quad \text{as } \sigma^2 \rightarrow 0,$$

where $f(x)$ is the pdf of $\sqrt{\lambda}(X - \mu_0)$.

Hyperbolic distribution. The pdf (for $y \in \mathbf{R}$) with parameters μ_0 and λ is

$$f(y; \mu_0, \lambda) = \frac{1}{2aK_1(\lambda)} \exp\left(-\lambda[a\{1 + (y - \mu_0)^2\}^{1/2} - b(y - \mu_0)]\right),$$

where K_1 is a Bessel function and $a^2 = 1 + b^2$. It is a $DM(\mu_0, \lambda^{-1})$ for fixed b , $d(y; \mu) = a\{1 + (y - \mu)^2\}^{1/2} - b(y - \mu) - a$ and $V(\mu) = 2/a$, which implies that the approximation is uniform in y . We have $\partial_{yyy}^3 d(\mu_0; \mu_0) = 0$ and $\partial_{yyy}^4 d(\mu_0; \mu_0) = -3a$. So, the pdf $f(x)$ of $\sqrt{\lambda}(X - \mu_0)$ satisfies

$$\frac{f(x_\sigma)}{\phi(x_\sigma; 0, \mu_0)} \rightarrow \exp\left(\frac{a\beta}{4}\right) \quad \text{as } \sigma \rightarrow 0.$$

Simplex distribution. The pdf of the simplex distribution [1] with parameters $\mu_0 \in (0, 1)$ and $\sigma^2 > 0$ is

$$f(y; \mu_0, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2\{y(1-y)\}^3}} \exp\left\{-\frac{(y - \mu_0)^2}{2\sigma^2 y(1-y)\mu_0^2(1-\mu_0)^2}\right\}.$$

It is a $DM(\mu_0, \sigma^2)$ with $d(y; \mu) = (y - \mu)^2 / \{y(1-y)\mu^2(1-\mu)^2\}$. The variance function is $V(\mu) = \mu^3(1-\mu)^3$ and the approximation is exact. We obtain $\partial_{yyy}^3 d(\mu_0; \mu_0) = 6(2\mu_0 - 1)/\{\mu_0(1-\mu_0)\}^4$ and then the

$$\frac{f(x_\sigma)}{\phi(x_\sigma; 0, V(\mu_0))} \rightarrow \exp\left[-\frac{\beta(\mu_0 - 0.5)}{\{\mu_0(1-\mu_0)\}^4}\right] \quad \text{as } \sigma^2 \rightarrow 0,$$

where $f(x)$ is the pdf of $(X - \mu_0)/\sigma$.

von-Mises distribution. The pdf of the von Mises distribution with $\mu_0 \in [0, 2\pi)$ and $\sigma^2 > 0$ can be expressed as

$$f(y; \mu_0, \sigma^2) = \frac{\exp(\sigma^{-2})}{2\pi I_0(\sigma^{-2})} \exp\left\{-\frac{1 - \cos(y - \mu_0)}{\sigma^2}\right\}$$

for $0 \leq y \leq 2\pi$, where $I_0(\cdot)$ denotes the modified Bessel function. This distribution is a $DM(\mu_0, \sigma^2)$ with $d(y; \mu) = 2\{1 - \cos(y - \mu)\}$ and $V(\mu) = 1$, which yields an approximation uniform in y . Since $\partial_{yyy}^3 d(\mu_0; \mu_0) = 0$, $\partial_{yyy}^4 d(\mu_0; \mu_0) = -2$, the pdf $f(x)$ of $(X - \mu_0)/\sigma$ satisfies

$$\frac{f(x_\sigma)}{\phi(x_\sigma; 0, 1)} \rightarrow \exp\left(\frac{\beta}{24}\right) \quad \text{as } \sigma^2 \rightarrow 0,$$

Leipnik distribution. The pdf of the Leipnik distribution (for $y, \mu_0 \in (-1, 1)$ and $\lambda > 0$) is

$$f(y; \mu_0, \lambda) = \frac{(1 - y^2)^{-1/2}}{B((\lambda + 1)/2, 1/2)} \exp\left\{-\frac{\lambda}{2} \ln \frac{1 - 2y\mu_0 + \mu_0^2}{1 - y^2}\right\}.$$

It is a $DM(\mu_0, \lambda^{-1})$ with $\sigma^2 = \lambda^{-1}$, $d(y; \mu) = \log\{(1 - 2y\mu + \mu^2)/(1 - y^2)\}$ and $V(\mu) = 1 - \mu^2$, which implies that the approximation is uniform on compacts for $0 < \mu < 1$. From $\partial_{yyy}^3 d(\mu_0; \mu_0) = 12\mu_0/\{(1 - \mu_0^2)^2\}$, the pdf $f(x)$ of $\sqrt{\lambda}(X - \mu_0)$ satisfies

$$\frac{f(x_\sigma)}{\phi(x_\sigma, 0, 1 - \mu_0^2)} \rightarrow \exp\left\{-\frac{\beta\mu_0}{(1 - \mu_0^2)^2}\right\} \quad \text{as } \lambda \rightarrow \infty.$$

Transformed Leipnik distribution. The pdf of the transformed Leipnik distribution for $y, \mu_0 \in (0, 1)$ and $\lambda > 0$ is

$$f(y; \mu_0, \lambda) = \frac{\{y(1 - y)\}^{-1/2}}{B((\lambda + 1)/2, 1/2)} \exp\left[-\frac{\lambda}{2} \ln\left\{1 + \frac{(y - \mu_0)^2}{y(1 - y)}\right\}\right].$$

It is a $DM(\mu_0, \lambda^{-1})$ with $\sigma^2 = \lambda^{-1}$, $d(y; \mu) = \log[(1 + y - \mu)^2/\{y(1 - y)\}]$ and $V(\mu) = \mu(1 - \mu)$. For $0 < \mu < 1$, the approximation is uniform on compact sets. Further, $\partial_{yyy}^3 d(\mu_0; \mu_0) = 6(2\mu_0 - 1)/\{\mu_0(1 - \mu_0)\}^2$ implies that the pdf $f(x)$ of $\sqrt{\lambda}(X - \mu_0)$ satisfies

$$\frac{f(x_\sigma)}{\phi(x_\sigma, 0, \mu_0(1 - \mu_0))} \rightarrow \exp\left\{-\frac{\beta(2\mu_0 - 1)}{12\{\mu_0(1 - \mu_0)\}^2}\right\} \quad \text{as } \lambda \rightarrow \infty.$$

5. Conclusions. Under the assumption of uniform saddlepoint approximation, the dispersion models are approximately normal for small values of the dispersion parameter σ^2 . This type of approximation is called small dispersion asymptotics. This fact inspired us to generalize the result of [3] specific for the Student's t distribution. We show, under certain regularity conditions, that this asymptotic property for the Student's t distribution is of fairly general nature and holds to a much wider class of dispersion models. Moreover, in a much more flexible fashion, since we allow the derivatives of the unit deviance function to vanish any finite number of times. Finally, we apply this result to a number of concrete distributions that can be written in the form of a dispersion model.

Acknowledgments. The authors would like to thank the Editor and the referee for carefully reading and for their comments which greatly improved the paper.

REFERENCES

1. Barndorff-Nielsen O. E., Jørgensen B. Some parametric models on the simplex. — J. Multivariate Anal., 1991, v. 38, № 1, p. 106–116.
2. Finner H., Dickhaus T., Roters M. Dependency and false discovery rate: asymptotics. — Ann. Statist., 2007, v. 35, № 4, p. 1432–1455.
3. Finner H., Dickhaus T., Roters M. Asymptotic tail properties of Student's t -distribution. — Comm. Statist. — Theory Methods, 2008, v. 37, № 1–2, p. 175–179.

4. *Fisher N. I.* Statistical Analysis of Circular Data. Cambridge: Cambridge Univ. Press, New York, 1993, 277 p.
5. *Jørgensen B.* Exponential dispersion models. — J. Roy. Statist. Soc., 1987, B, v. 49, № 2, p. 127–162.
6. *Jørgensen B.* Small dispersion asymptotics. — Rebrapе, 1987, v. 1, № 1, p. 59–90.
7. *Jørgensen B.* Proper dispersion models. — Rebrapе, 1997, v. 11, № 2, p. 1–89.
8. *Jørgensen B.* The Theory of Dispersion Models. London: Chapman and Hall, 1997, 237 p.
9. *Мардуа К.* Статистический анализ угловых наблюдений. М.: Наука, 1978, 239 с.

Поступила в редакцию
15.VI.2008

Исправленный вариант
17.VII.2010

© 2011 г.

ИНФОРМАЦИЯ О НАУЧНОЙ ЖИЗНИ

Редакционная коллегия журнала «Теория вероятностей и ее применения» поздравляет **Альберта Николаевича Ширяева** с избранием в действительные члены Российской академии наук.