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Для получения оценки сверху заменим Δ_T на 0:

$$\begin{aligned} \mathbf{P}(\tau_T \leq T + x\sqrt{T}, \nu_T > y\sqrt{T}) &\leq \mathbf{P}(D_T, V'_1 \leq TY_1, \dots, V'_1 + \dots + V'_m \leq TY_m) \\ &= \mathbf{P}(D_T) \mathbf{P}(V'_1 \leq TY_1, \dots, V'_1 + \dots + V'_m \leq TY_m). \end{aligned}$$

Применяя к сомножителям в правой части центральную предельную теорему и лемму 1 с $c_T = 0$, выводим неравенство

$$\lim \mathbf{P}(\tau_T \leq T + x\sqrt{T}, \nu_T > y\sqrt{T}) \leq \Phi\left(\frac{x}{\sigma}\right) e^{-by^2/2}. \quad (7)$$

Объединяя (6) и (7), приходим к (5). Лемма 3, а вместе с ней и теорема доказаны.

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THE ASYMPTOTIC BERRY–ESSEEN CONSTANT FOR INTERVALS¹⁾

Показано, что асимптотическая константа в неравенстве Берри–Эссена для интервальных вероятностей равна $\sqrt{2/\pi}$.

Ключевые слова и фразы: неравенство Берри–Эссена, интервальные вероятности, асимптотическая константа.

1. Introduction. Let $\mathbf{R}$ denote the real numbers, $\mathbf{Z}$ the integers, and $\mathbf{N}$ the integers ≥ 1 . Let $\text{Prob}(\mathbf{R})$ be the set of all (probability) laws on $\mathbf{R}$, δ_x the unit mass at $x \in \mathbf{R}$, and $N_{0,1}$ the standard normal law with distribution function Φ and density $\varphi := \Phi'$.

For $P \in \text{Prob}(\mathbf{R})$ and $s \in [1, \infty]$, let $\mu(P) := \int x P(dx)$ and $\beta_s(P) := \int |x - \mu(P)|^s P(dx)$ if $\int |x| P(dx) < \infty$, $\beta_s(P) := \infty$ otherwise, $\sigma^2(P) := \beta_2(P)$, and $\alpha(P) := \int (x - \mu(P))^3 P(dx)$ if $\beta_3(P) < \infty$. If P is a lattice law, let $h(P) := \sup_{a \in \mathbf{R}} \{\eta \in]0, \infty[: P(a + \eta\mathbf{Z}) = 1\}$, otherwise set $h(P) := 0$. Let us put

$$\mathcal{P}_s := \{P \in \text{Prob}(\mathbf{R}): 0 < \beta_s(P) < \infty\}.$$

In the following, we shall consider sequences $(X_k)_{k \in \mathbf{N}}$ of independent and identically distributed real-valued random variables. P will then stand for the law of X_1 and shall, for the sake of brevity, usually be omitted in the quantities μ , σ^2 , α , β_s , and h just defined. Let $P_n \in \text{Prob}(\mathbf{R})$ be given by $P_n(A) := P^{*n}(\sigma\sqrt{n}A + n\mu)$ and set $F_n(x) := P_n([-\infty, x])$ for

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$x \in \mathbf{R}$. Thus P_n is the law of the standardized sum of $X_1, \dots, X_n$ and F_n its distribution function. Now the classical Berry–Esseen theorem states the finiteness of the *Berry–Esseen constant*

$$c_{\text{BE}} := \sup_{\substack{P \in \mathcal{P}_3, \\ n \in \mathbf{N}}} \frac{\sigma^3(P)}{\beta_3(P)} \sqrt{n} \sup_{x \in \mathbf{R}} |F_n(x) - \Phi(x)|.$$

It is known that $0.4097 < (\sqrt{10} + 3)/(6\sqrt{2\pi}) \leq c_{\text{BE}} < 0.4748$, with the lower bound due to Esseen [2] as an obvious consequence of his determination of the *asymptotic Berry–Esseen constant*

$$c_{\infty, \text{BE}} := \sup_{P \in \mathcal{P}_3} \frac{\sigma^3(P)}{\beta_3(P)} \lim_{n \rightarrow \infty} \sqrt{n} \sup_{x \in \mathbf{R}} |F_n(x) - \Phi(x)| = \frac{\sqrt{10} + 3}{6\sqrt{2\pi}},$$

and with the more recent upper bound due to Shevtsova [4].

The Kolmogorov distance used above is the supremum distance over the system of all unbounded intervals. We propose to use instead the system $\mathfrak{I}$ of all intervals whatsoever, which is a special case of the system of all convex sets used in the multidimensional setting. Due to the continuity properties of probability measures, it suffices to consider intervals of the form $]a, b]$, that is, we have

$$\sup_{I \in \mathfrak{I}} |P_n(I) - N_{0,1}(I)| = \sup_{x, y \in \mathbf{R}} |F_n(x) - F_n(y) - \Phi(x) + \Phi(y)|. \quad (1)$$

An obvious upper bound for the right-hand side is $2c_{\text{BE}}\beta_3/(\sigma^3\sqrt{n})$. While we do not know whether here $2c_{\text{BE}}$ can be replaced by a smaller constant, we will determine below the *asymptotic Berry–Esseen constant for interval probabilities* as

$$c_{\infty, \text{BE}}(\mathfrak{I}) := \sup_{P \in \mathcal{P}_3} \frac{\sigma^3(P)}{\beta_3(P)} \lim_{n \rightarrow \infty} \sqrt{n} \sup_{I \in \mathfrak{I}} |P_n(I) - N_{0,1}(I)| = \frac{2}{\sqrt{2\pi}}, \quad (2)$$

which is strictly smaller than $2c_{\infty, \text{BE}}$. Let us note that $c_{\infty, \text{BE}}(\mathfrak{I})$ happens to be twice the modified asymptotic Berry–Esseen constant due to Rogozin [3], and a consideration of the symmetric Bernoulli case shows that a Rogozin type modification of $c_{\infty, \text{BE}}(\mathfrak{I})$ yields the same value $\sqrt{2/\pi}$ as above.

2. Results. For $P \in \mathcal{P}_3$, we shall use the following asymptotic expansion due to Esseen [1]:

$$F_n(x) = \Phi(x) + \frac{h}{\sigma\sqrt{n}} \psi_n(x) \varphi(x) - \frac{\alpha}{6\sigma^3\sqrt{n}} \varphi''(x) + o\left(\frac{1}{\sqrt{n}}\right), \quad n \rightarrow \infty \quad (3)$$

uniformly in $x \in \mathbf{R}$, where, if $h > 0$, ψ_n is a certain $h/(\sigma\sqrt{n})$ -periodic function: If $a \in \mathbf{R}$ is such that $P(a + h\mathbf{Z}) = 1$, then $\psi_n(x) = 1/2 - \text{frac}(x\sigma\sqrt{n}/h - an/h)$, where $\text{frac}(t) := t - [t]$ for $t \in \mathbf{R}$. As observed in [2], it then follows that

$$\lim_{n \rightarrow \infty} \sqrt{n} \sup_{x \in \mathbf{R}} |F_n(x) - \Phi(x)| = \frac{1}{\sqrt{2\pi}} \left(\frac{h}{2\sigma} + \frac{|\alpha|}{6\sigma^3} \right).$$

The analogous result for interval probabilities is the following theorem.

Theorem. Let $P \in \mathcal{P}_3$. Then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sqrt{n} \sup_{I \in \mathfrak{I}} |P_n(I) - N_{0,1}(I)| \\ &= \frac{1}{\sqrt{2\pi}} \begin{cases} \frac{h}{\sigma} & \text{if } |\alpha| \leq h\sigma^2, \\ \frac{h}{2\sigma} + \frac{|\alpha|}{6\sigma^3} + \frac{|\alpha|}{3\sigma^3} \exp\left(-\frac{3}{2}\left(1 - \frac{h\sigma^2}{|\alpha|}\right)\right) & \text{if } |\alpha| > h\sigma^2. \end{cases} \end{aligned}$$

P r o o f. We may assume $\alpha \geq 0$, $\sigma = 1$, and $\mu = 0$. On the right of (1), we may omit the absolute value signs, since for a function f satisfying $f(x, y) = -f(y, x)$, we have $\sup_{x, y} |f(x, y)| = \sup_{x, y} f(x, y) \vee (-f(x, y)) = \sup_{x, y} f(x, y) \vee \sup_{x, y} f(y, x) = \sup_{x, y} f(x, y)$. Thus, using (3), we get (reading backwards to prove existence of the limits)

$$\begin{aligned} & \lim_{n \rightarrow \infty} 2\sqrt{n} \sup_{I \in \mathcal{I}} |P_n(I) - N_{0,1}(I)| \\ &= \lim_{n \rightarrow \infty} \sup_{x, y \in \mathbf{R}} \left(2h \left(\psi_n(x) \varphi(x) - \psi_n(y) \varphi(y) \right) - \frac{\alpha}{3} \left(\varphi''(x) - \varphi''(y) \right) \right) \\ &= \sup_{x, y \in \mathbf{R}} \left(h\varphi(x) - \frac{\alpha}{3} \varphi''(x) + h\varphi(y) + \frac{\alpha}{3} \varphi''(y) \right) \\ &= \sup_{y \in \mathbf{R}} f(y). \end{aligned}$$

where

$$f(y) := \frac{1}{\sqrt{2\pi}} (h + \alpha/3) + h\varphi(y) + \frac{\alpha\varphi''(y)}{3}$$

and where, if $h > 0$, the second equality above is true since $\sup_{x \in \mathbf{R}} \psi_n(x) = 1/2$, $\inf_{x \in \mathbf{R}} \psi_n(x) = -1/2$, the period length of ψ_n tends to zero, and the function occurring on the right is continuous.

If $\alpha = 0$, then f is maximized at $y = 0$ with the value $f(0) = 2h/\sqrt{2\pi}$. Let now $\alpha > 0$. Then $f'(y) = h\varphi'(y) + \alpha\varphi'''(y)/3 = -\alpha y\varphi(y)(y^2 - 3 + 3h/\alpha)/3$. If also $\alpha \leq h$, then $\{f' > 0\} =]-\infty, 0[$, and f is again maximized at $y = 0$, with the same value as above. If instead $\alpha > h$, then $\{f' > 0\} =]-\infty, -y_0[\cup]0, y_0[$ with $y_0 := \sqrt{3 - 3h/\alpha}$, and the maximal value of f is

$$f(\pm y_0) = \frac{1}{\sqrt{2\pi}} \left(h + \frac{\alpha}{3} + \frac{2\alpha}{3} \exp \left(-\frac{3}{2} \left(1 - \frac{h}{\alpha} \right) \right) \right).$$

The theorem is proved.

For completeness we present a short proof of a classical fact needed below.

Lemma (von Mises [5]). *Let $P \in \text{Prob}(\mathbf{R})$ be discrete and let $\eta \in [0, \infty[$ be such that $0 < |x - y| < \eta$ implies $P(\{x\})P(\{y\}) = 0$. Let $s \in [1, \infty[$. Then*

$$\eta\beta_s \leq 2\beta_{s+1}. \quad (4)$$

Finite equality $\eta\beta_s = 2\beta_{s+1} < \infty$ holds if and only if $s = 1$ and $P = \lambda\delta_x + (1 - \lambda)\delta_{x+\eta}$ for some $\lambda \in [0, 1]$ and $x \in \mathbf{R}$, or $s > 1$ and $P \in \bigcup_{x \in \mathbf{R}} \{\delta_x, (\delta_x + \delta_{x+\eta})/2\}$.

P r o o f. We may assume $\eta > 0$ and $P \in \mathcal{P}_{s+1} \setminus \{(\delta_x + \delta_y)/2: x, y \in \mathbf{R}\}$, all other cases being trivial, and then also $\mu = 0$. Let X, Y be independent with law P . Then $|X - \mu| = |X|$ is not constant almost surely, and hence the map $[1, s+1] \ni t \mapsto \ln \beta_t \in \mathbf{R}$ is strictly convex by Hölder's inequality, yielding $\beta_{s+1}/\beta_s \geq \beta_2/\beta_1$, with equality if and only if $s = 1$. We also have $2\beta_2 = \mathbf{E}(X - Y)^2 \geq \eta \mathbf{E}|X - Y| \geq \eta \mathbf{E}|X| = \eta\beta_1$, with equality in the first inequality if and only if P is of the form $\lambda\delta_x + (1 - \lambda)\delta_{x+\eta}$, in which case equality holds throughout. Hence (4) holds, with the discussion of equality.

The lemma is proved.

Corollary. *Identity (2) is true, with the $\sup_{P \in \mathcal{P}_3}$ attained precisely for $P \in \{(\delta_x + \delta_y)/2: x, y \in \mathbf{R}, x \neq y\}$.*

P r o o f. Let $P \in \mathcal{P}_3$ and let L denote the limit in Theorem. If $|\alpha| \leq h\sigma^2$, then Lemma with $s = 2$ and $\eta = h$ yields $\sqrt{2\pi}L = h/\sigma \leq 2\beta_3/\sigma^3$, with equality in the last step if and only if P is of the form $(\delta_x + \delta_y)/2$. If $|\alpha| > h\sigma^2$, then $\sqrt{2\pi}L < h/(2\sigma) + |\alpha|/(6\sigma^3) + |\alpha|/(3\sigma^3) < |\alpha|/\sigma^3 \leq \beta_3/\sigma^3$.

Corollary is proved.

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OPTIMAL MULTIPLE STOPPING WITH SUM-PAYOFF

Рассматривается задача оптимальной остановки независимой последовательности $X_1, \dots, X_n$ (время дискретно), где разрешено m остановок. Функция выплат — это сумма значений в моменты остановок. В предположении, что соответствующие вложенные точечные процессы сходятся к пуассоновскому процессу на плоскости, мы находим приближенно оптимальные моменты остановки и значения в эти моменты. Решения получаются из системы m дифференциальных уравнений первого порядка. В качестве применения рассматривается случай $X_i = c_i Z_i + d_i$, где (Z_i) — независимые одинаково распределенные величины из области притяжения распределения экстремальных значений. Мы получаем явные результаты для значений в моменты остановки и аппроксимирующих правил оптимальной остановки.

Ключевые слова и фразы: оптимальная многократная остановка, проблема наилучшего выбора, экстремальные значения, вложенный пуассоновский процесс.

1. Introduction. For discrete time sequences $X_1, \dots, X_n$ we consider the optimal stopping problem with m allowed stops $1 \leq T_1 < \dots < T_m \leq n$. The aim is to choose the stopping times T_i such that the expectation of the sum of the stopped values is maximized, i.e.,

$$\mathbf{E} \sum_{i=1}^m X_{T_i} = \sup \mathbf{E} \sum_{i=1}^m X_{\tau_i} \quad (1.1)$$

over all stopping times $1 \leq \tau_1 < \dots < \tau_m \leq n$.

It is essential to assume that $\tau_1 < \tau_2 < \dots < \tau_m$. The case where inequality constraints $\tau_1 \leq \dots \leq \tau_m$ are assumed, reduces this problem to the one-stopping problem. The optimal stopping times are then given by $T_1 = \dots = T_m$, where T_1 is optimal for the one-stopping problem.

In some instances, problem (1.1) has been solved in [12], [16], [17] while an extension of the problem to vector offers is considered in [14]–[16], [3], [2]. A solution of problem

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