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SAMPLE PATH LARGE DEVIATIONS FOR SQUARES OF STATIONARY GAUSSIAN PROCESSES

Устанавливается принцип больших уклонений для случайных ступенчатых функций вида

$$Z_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k^2,$$

где $\{X_k\}_k$ — стационарный гауссовский процесс. Рассматриваются ассоциированные случайные меры $\nu_n = (1/n) \sum_{k=1}^n X_k^2 \delta_{k/n}$. В доказательствах используется теорема Сегё для обобщенных тёплицевых матриц, приводимая в приложении и аналогичная результату работы [10]. Мы также рассматриваем ломаную, построенную по $Z_n(t)$, и изучаем умеренные уклонения для обоих случайных семейств.

Ключевые слова и фразы: гауссовские процессы, большие уклонения, теорема Сегё, тёплицевы матрицы.

1. Introduction. The aim of this paper is to provide a large deviation principle (LDP) for random functions of type

$$Z_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k^2, \tag{1}$$

and the associated polygonal line

$$\tilde{Z}_n(t) = Z_n(t) + \left(t - \frac{[nt]}{n}\right) X_{[nt]+1}^2, \tag{2}$$

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where $\{X_n\}_n$ is a stationary Gaussian process having spectral density f defined on the torus $\mathbf{T} =]-\pi, \pi]$. We assume f is continuous positive on $\mathbf{T}$.

Large deviations for random measures date back to Sanov [19] who showed an LDP for the family of empirical measures

$$\frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \quad (3)$$

where X_i are i.i.d. random variables.

Then, the first results on large deviations for random paths were given by Borovkov [2] and Varadhan [20]. In [2], Borovkov provides an LDP for the random polygonal line joining the points $(k/n, S_k/x)$, where $S_k = \sum_{i=1}^k X_i$ and $x = x(n)$ is in the range

$$\limsup_{n \rightarrow \infty} \frac{x}{n} < \infty, \quad \lim_{n \rightarrow \infty} \frac{x}{\sqrt{n \ln n}} = \infty. \quad (4)$$

He also showed large deviations for the paths $\eta(nt)/x$, where $0 \leq t \leq 1$ and η is a separable process with independent increments. The large deviations are given in the spaces $\mathcal{C}([0, 1])$ (the set of continuous functions on $[0, 1]$) or $\mathcal{D}([0, 1])$ (the set of càdàg functions on $[0, 1]$) endowed with the uniform metric. Meanwhile, Varadhan [20] proved functional large deviations in $\mathcal{D}([0, 1])$ for the random step functions

$$S_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} X_i, \quad (5)$$

where $t \in [0, T]$ and $[nt]$ denotes the integer part of nt . Later on, Mogulskii [13] improved these results: he proved large deviations for the polygonal line $(k/n, S_k/x)$ in the range

$$\limsup_{n \rightarrow \infty} \frac{x}{n} < \infty, \quad \lim_{n \rightarrow \infty} \frac{x}{\sqrt{n}} = \infty \quad (6)$$

in the space $\mathcal{D}([0, 1])$ endowed with the Skorokhod metric. For more general results on large deviations for processes with independent increments, see also [11], [3], and [14].

The results of [2], [20], [13] concerning step functions and continuous random polygonal lines built on sums of i.i.d. random variables can be found in the books of Dupuis and Ellis [6] and Dembo and Zeitouni [5].

In our paper, to derive the large deviations, we consider the distribution derivative of $t \rightarrow Z_n(t)$ and $t \rightarrow \tilde{Z}_n(t)$. Therefore we deal with the random measures ν_n and $\tilde{\nu}_n$ given by

$$\langle \nu_n, h \rangle = \frac{1}{n} \sum_{k=1}^n X_k^2 h\left(\frac{k}{n}\right) \quad (7)$$

and

$$\langle \tilde{\nu}_n, h \rangle = \sum_{k=1}^n X_k^2 \int_{(k-1)/n}^{k/n} h(s) ds, \quad (8)$$

for h in $\mathcal{C}([0, 1])$. Let $\mathcal{M}([0, 1])$ be the set of positive bounded measures on $[0, 1]$ endowed with the weak topology. Therefore ν_n and $\tilde{\nu}_n$ are a.s. in $\mathcal{M}([0, 1])$.

Analogous random measures have been investigated before by Dembo and Zeitouni [4], and Gamboa and Gassiat [7]. Previous works on LDP for this kind of random functions can be found in [8] and [16] for stationary Gaussian processes, and in [15] and [12] for i.i.d. sequences. We provide here a functional LDP for $\{\nu_n\}$ and $\{\tilde{\nu}_n\}$, and derive the associated LDP for $\{Z_n\}$ and $\{\tilde{Z}_n\}$. We also prove moderate deviations. The central limit theorem is known. Although part of this work was already presented in [21] the present work provide a full version with proofs and some extensions.

The rest of the paper is organized as follows. We present in Section 2 the large and moderate deviations results. Section 3 is devoted to the proofs of theorems. Deriving the

large deviations result, we needed a Szegő type theorem for generalized Toeplitz matrices. This precise result is unknown to our knowledge and despite a very similar result has been shown in [10] (see also [9]), for seek of completeness we prove it in the Appendix. The remaining part of the Appendix gather the proofs of technical lemmas.

2. Large and moderate deviations. For any h in $\mathcal{C}([0, 1])$, define

$$\Lambda(h) = \begin{cases} -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbf{T}} \ln(1 - 2h(t)f(\theta)) d\theta dt \\ \quad \text{if for all } (t, \theta) \in [0, 1] \times \mathbf{T}, h(t)f(\theta) < 1/2, \\ +\infty \quad \text{otherwise.} \end{cases}$$

Let Λ^* be the Legendre dual of Λ . From [18], we can detail this dual function as follows.

Proposition 2.1. *Let ν be the measure in $\mathcal{M}([0, 1])$ defined for any h in $\mathcal{C}([0, 1])$ by*

$$\langle \nu, h \rangle = \frac{1}{2\pi} \int_{\mathbf{T}} f(\theta) d\theta \int_{[0,1]} h(x) dx.$$

Let $\mu \in \mathcal{M}([0, 1])$ have the following Lebesgue decomposition with respect to ν : $\mu = l\nu + \mu^\perp$, where $l \in \mathcal{C}([0, 1])$ and $\mu^\perp$ is the singular part. Then

$$\Lambda^*(\mu) = \int_{[0,1]} u^*(l(t)) \nu(dt) + \int_{[0,1]} \frac{\mu^\perp(dt)}{2M},$$

where

$$u(x) = -\frac{1}{4\pi} \int_{\mathbf{T}} \ln(1 - 2xf(\theta)) d\theta \quad \text{and} \quad M = \text{ess sup } f.$$

The function u is $\mathcal{C}^2$ on $(-\infty, 1/(2M))$, and

$$\begin{aligned} u'(x) &= \frac{1}{2\pi} \int_{\mathbf{T}} \frac{f(\theta)}{1 - 2xf(\theta)} d\theta, \\ u''(x) &= \frac{1}{\pi} \int_{\mathbf{T}} \frac{f(\theta)^2}{(1 - 2xf(\theta))^2} d\theta > 0. \end{aligned}$$

Hence u' is strictly increasing, and $\lim_{x \rightarrow -\infty} u'(x) = 0$. On the other hand, we denote $u'(1/(2M)) := \lim_{x \rightarrow +\infty} u'(x) \leq +\infty$ (e.g., if $f \in \mathcal{C}^2$, then $u'(1/(2M)) = +\infty$). The recession function (see [18, Theorem 8.5]) is

$$r(u^*; y) = \frac{y}{2M}. \quad (9)$$

2.1. Large deviations. We can now state the LDP result.

Theorem 2.1. *The families $\{\nu_n\}_{n \in \mathbf{N}}$ and $\{\tilde{\nu}_n\}_{n \in \mathbf{N}}$ satisfy an LDP in $\mathcal{M}([0, 1])$ with speed n and rate function Λ^* .*

We can carry the previous LDP to the random functions Z_n and $\tilde{Z}_n$. Following Lynch and Sethuraman [11] and de Acosta [3], we introduce some notation. Let $D([0, 1], \mathbf{R})$ be the space of càdlàg real functions on $[0, 1]$, and $\text{bv}([0, 1], \mathbf{R}) \subset D([0, 1], \mathbf{R})$ the space of bounded variation functions. We can identify $\text{bv}([0, 1], \mathbf{R})$ with $\mathcal{M}([0, 1])$: to h in $\text{bv}([0, 1], \mathbf{R})$ corresponds μ_h in $\mathcal{M}([0, 1])$ characterized by $\mu_h([0, t]) = h(t)$. Up to this identification, the topological dual of $\text{bv}([0, 1], \mathbf{R})$ is the set $\mathcal{C}([0, 1])$. We endow $\text{bv}([0, 1], \mathbf{R})$ with the w^* -topology written σ , i.e., the topology induced by $\mathcal{C}([0, 1])$ on $\mathcal{M}([0, 1])$. Now, let us define the rate function associated to Z_n and $\tilde{Z}_n$: let h be in $\text{bv}([0, 1], \mathbf{R})$ and μ_h the associated measure in $\mathcal{M}([0, 1])$; let $\mu_h = (\mu_h)_a + (\mu_h)_s$ be the Lebesgue decomposition of μ_h in absolutely continuous and singular terms with respect to the Lebesgue measure on $[0, 1]$; let $h_a(t) = (\mu_h)_a([0, t])$ and $h_s(t) = (\mu_h)_s([0, t])$. Set

$$\Phi(h) = \int_{[0,1]} u^*(h'_a(t)) \nu(dt) + rh_s(1),$$

where u^* and r are defined in Proposition 2.1 and in (9), respectively.

Theorem 2.2. *The families of random functions $\{Z_n\}$ and $\{\tilde{Z}_n\}$ satisfy an LDP on the space $(\text{bv}([0, 1], \mathbf{R}), \sigma)$, with speed n and rate function Φ .*

2.2. Moderate deviations. We can state also in this case a moderate deviation principle. We detail it for ν_n , it is the same for $\tilde{\nu}_n$. Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n \rightarrow 0$ and $na_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Set

$$Y_n = \sqrt{na_n}(\nu_n - \mathbf{E}\nu_n).$$

We have the following moderate deviations principle.

Theorem 2.3. *$\{Y_n\}$ satisfy an LDP with speed a_n^{-1} and a good rate function defined, for all $\mu \in \mathcal{M}([0, 1])$ by*

$$I(\mu) = \begin{cases} \frac{\pi}{2\bar{f}^2} \int_{[0,1]} l(x)^2 dx & \text{if } \mu(dx) = l(x) dx, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\bar{f}^2 = \frac{1}{2\pi} \int_{\mathbf{T}} f(\theta)^2 d\theta.$$

2.3. Generalizations. The previous results can be generalized to some other random functions.

2.3.1. Weighted random variables. Assume g is a continuous function on $[0, 1]$ and define

$$W_n = \frac{1}{n} \sum_{k=1}^{[nt]} g\left(\frac{k}{n}\right) X_k^2.$$

For any h in $\mathcal{C}([0, 1])$, define

$$\Lambda(h) = \begin{cases} -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbf{T}} \ln(1 - 2h(t)g(t)f(\theta)) d\theta dt \\ \quad \text{if for all } (t, \theta) \in [0, 1] \times \mathbf{T}, h(t)g(t)f(\theta) < 1/2, \\ +\infty & \text{otherwise.} \end{cases}$$

The previous large deviations results apply with the rate function Λ^* .

2.3.2. Quadratic forms built on the stationary process. We define

$$m = \text{ess inf } f$$

and assume $m > 0$. Let F be a continuous positive function on $[m, M]$. Let O be an orthonormal matrix such that $O^*T_n(f)O$ is the diagonal matrix whose i -th diagonal element is $\mu_{i,n}$ the i -th eigenvalue of $T_n(f)$. Define

$$F(T_n(f)) = OD_fO^*,$$

where D_f is the diagonal matrix whose i -th element is $F(\mu_{i,n})$. Define the following quadratic form:

$$W_n = \frac{1}{n} X^* F(T_n(f)) X = \frac{1}{n} Y^* Y,$$

where $Y = (Y_1, \dots, Y_n)$ is the vector defined by

$$Y = F(T_n(f))^{1/2} X.$$

In this case, W_n satisfies an LDP and moderate deviations theorem with rate function Λ^* , where for any h in $\mathcal{C}([0, 1])$

$$\Lambda(h) = \begin{cases} -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbf{T}} \ln[1 - 2h(t)f(\theta)F(f(\theta))] d\theta dt \\ \quad \text{if for all } (t, \theta) \in [0, 1] \times \mathbf{T}, h(t)f(\theta) < 1/2, \\ +\infty & \text{otherwise.} \end{cases}$$

3. Proof of the large and moderate deviations. We first give some asymptotic properties for the families $\{\nu_n\}$ and $\{\tilde{\nu}_n\}$.

3.1. Weak convergence of $\{\nu_n\}$ and $\{\tilde{\nu}_n\}$.

Lemma 3.1. *Let h be in $\mathcal{C}([0, 1])$. Then*

$$\langle \nu_n, h \rangle \rightarrow \langle \nu, h \rangle \quad \text{in probability as } n \rightarrow +\infty \quad (10)$$

and $\langle \tilde{\nu}_n, h \rangle \rightarrow \langle \nu, h \rangle$ in probability as $n \rightarrow +\infty$, where

$$\langle \nu, h \rangle = \bar{f} \int_{[0,1]} h(x) dx, \quad \bar{f} = \frac{1}{2\pi} \int_{\mathbf{T}} f(\theta) d\theta.$$

P r o o f. Let h be in $\mathcal{C}([0, 1])$, and consider

$$\langle \nu_n, h \rangle = \frac{1}{n} \sum_{k=1}^n X_k^2 h\left(\frac{k}{n}\right).$$

Set X the Gaussian vector $(X_1, X_2, \dots, X_n)$ and Δ_h the diagonal matrix

$$\begin{pmatrix} h\left(\frac{1}{n}\right) & 0 & 0 & 0 \\ 0 & h\left(\frac{2}{n}\right) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & h(1) \end{pmatrix}.$$

Therefore we can write

$$\langle \nu_n, h \rangle = \frac{1}{n} X^* \Delta_h X,$$

where X^* denote the transpose of X . By an orthonormal change of basis,

$$\langle \nu_n, h \rangle = \frac{1}{n} U_n^* T_n(f)^{1/2} \Delta_h T_n(f)^{1/2} U_n,$$

where U_n is a standard normal vector and $T_n(f)$ the order- n Toeplitz matrix associated to f . Therefore

$$\langle \nu_n, h \rangle = \frac{1}{n} \sum_{k=1}^n \lambda_{k,n} Z_{k,n}, \quad (11)$$

where $\{Z_{k,n}\}$ are independent $\chi^2(1)$ -distributed random variables, and $\{\lambda_{k,n}\}$ are the eigenvalues of $T_n(f)^{1/2} \Delta_h T_n(f)^{1/2}$.

We can write as well

$$\langle \tilde{\nu}_n, h \rangle = \frac{1}{n} \sum_{k=1}^n \tilde{\lambda}_{k,n} Z_{k,n}, \quad (12)$$

where $\{Z_{k,n}\}$ are independent $\chi^2(1)$ -distributed random variables, and $\{\tilde{\lambda}_{k,n}\}$ are the eigenvalues of $T_n(f)^{1/2} A_h T_n(f)^{1/2}$, and the matrix A_h is diagonal with k -th diagonal term

$$(A_h)_{k,k} = \int_{(k-1)/n}^{k/n} h(s) ds.$$

We have the two following results on the distributions $\{\lambda_{k,n}\}$ and $\{\tilde{\lambda}_{k,n}\}$, whose proofs are postponed to the Appendix.

Lemma 3.2. *The sequences $\{\lambda_{k,n}\}$ and $\{\tilde{\lambda}_{k,n}\}$ are bounded as follows:*

$$\forall n \in \mathbf{N}, \quad \forall 1 \leq k \leq n, \quad |\lambda_{k,n}| \leq \|h\|_{\infty} \|f\|_{\infty}, \quad |\tilde{\lambda}_{k,n}| \leq \|h\|_{\infty} \|f\|_{\infty},$$

Lemma 3.3. *For any p in $\mathbf{N}$, $p \geq 1$,*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n \lambda_{k,n}^p &= \frac{1}{2\pi} \int_{[0,1]} \int_{\mathbf{T}} (h(t)f(\theta))^p dt d\theta, \\ \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n (\tilde{\lambda}_{k,n})^p &= \frac{1}{2\pi} \int_{[0,1]} \int_{\mathbf{T}} (h(t)f(\theta))^p dt d\theta. \end{aligned}$$

With the above lemma, $\lim_{n \rightarrow +\infty} \mathbf{E} \langle \nu_n, h \rangle = \langle \nu, h \rangle$. Moreover,

$$\lim_{n \rightarrow +\infty} n \mathbf{D} \langle \nu_n, h \rangle = \frac{2}{n} \sum_{k=1}^n \lambda_{k,n}^2 = \frac{1}{\pi} \int_{[0,1]} \int_{\mathbf{T}} (h(t)f(\theta))^2 dt d\theta.$$

We do as well for $\tilde{\nu}_n$, and it ends the proof of Lemma 3.1.

3.2. Proof of Theorem 2.1. The proof follows exactly the scheme of [8]. We detail here for ν_n , it is similar for $\tilde{\nu}_n$. With the decomposition (11), we get the normalized cumulant generating function of ν_n : for any $h \in \mathcal{C}([0, 1])$,

$$\Lambda_n(h) = \frac{1}{n} \ln \mathbf{E} \left(\exp \{ n \langle \nu_n, h \rangle \} \right) = \begin{cases} -\frac{1}{2n} \sum_{k=1}^n \ln(1 - 2\lambda_{k,n}) \\ \quad \text{if for all } k, \quad \lambda_{k,n} < \frac{1}{2}, \\ +\infty \quad \text{otherwise.} \end{cases} \quad (13)$$

From Lemma 3.3, we can determine the limit of Λ_n in two cases:

(i) if for all $(t, \theta) \in [0, 1] \times \mathbf{T}$ we have $h(t)f(\theta) < 1/2$, then

$$\lim_{n \rightarrow +\infty} \Lambda_n(h) = -\frac{1}{4\pi} \int_{[0,1]} \int_{\mathbf{T}} \ln(1 - 2h(t)f(\theta)) d\theta dt = \Lambda(h);$$

(ii) if there exists $(t, \theta) \in [0, 1] \times \mathbf{T}$ such that $h(t)f(\theta) > 1/2$, then for n large enough, $\Lambda_n(h) = +\infty$ and

$$\lim_{n \rightarrow +\infty} \Lambda_n(h) = +\infty = \Lambda(h).$$

These two cases do not cover the whole set $\mathcal{C}([0, 1])$. Nevertheless, this will be sufficient for the LDP, since they contain a dense subset of exposing hyperplanes of Λ^* .

Upper bound. From [5, Theorem 4.5.3, b)] and the following lemma, which proof is postponed to the Appendix, the upper bound holds for compact sets.

Lemma 3.4. *For any $\delta > 0$ and μ in $\mathcal{M}([0, 1])$, there exists h_δ in $\mathcal{C}([0, 1])$ such that*

$$\begin{aligned} \forall (t, \theta), \quad h_\delta(t)f(\theta) &< \frac{1}{2}, \\ \int_{[0,1]} h_\delta(t) d\mu(t) - \Lambda(h_\delta) &\geq \Lambda_\delta^*(\mu), \end{aligned} \quad (14)$$

where $\Lambda_\delta^*(\mu) = \min\{\Lambda^*(\mu) - \delta, 1/\delta\}$.

Exponential tightness. Remark that for a real number a ,

$$\left\{ \sup_{\|h\|_\infty \leq 1} \langle \nu_n, h \rangle \geq a \right\} \subset \{\nu_n(1) \geq a\}.$$

If $M = \text{ess sup}_\theta f(\theta)$, then for any $y < 1/(2M)$,

$$\limsup_n \frac{1}{n} \ln \mathbf{P}(\nu_n(1) \geq a) \leq -ya - \frac{1}{4\pi} \int_{[0,1]} \int_{\mathbf{T}} \ln(1 - 2yf(\theta)) d\theta$$

and

$$\lim_{a \rightarrow +\infty} \limsup_n \frac{1}{n} \ln \mathbf{P}(\nu_n(1) \geq a) = -\infty.$$

Hence the sequence (ν_n) is exponentially tight, and the upper bound holds for any closed subset of $\mathcal{M}([0, 1])$.

Lower bound. We study the set of exposed points of Λ^* (see [5]). Let

$$\mathcal{H} = \left\{ \mu \in \mathcal{M}([0, 1]); \mu = l\nu, \ 0 < l < u' \left(\frac{1}{2M} \right), \ l \text{ is continuous on } [0, 1] \right\}.$$

The following two lemmas, whose proofs are postponed to the Appendix, show that $\mathcal{H}$ is a dense subset of the exposed points of Λ^* , which concludes the proof of Theorem 2.1.

Lemma 3.5. *Let $\mu = l\nu$ be in $\mathcal{H}$. There exists h_l in $\mathcal{C}([0, 1])$ such that*

$$\begin{aligned} \forall (t, \theta) \in [0, 1] \times \mathbf{T}, \quad h_l(t)f(\theta) &< 1/2, \\ \forall \xi \in \mathcal{M}([0, 1]), \quad \Lambda^*(\mu) - \Lambda^*(\xi) &< (\mu - \xi)(h_l). \end{aligned} \quad (15)$$

Furthermore, there exists $\gamma > 1$ such that $\Lambda(\gamma l) < +\infty$.

Hence μ is an exposed point of Λ^* with exposing hyperplane h_l .

Lemma 3.6. *Let μ be in $\mathcal{M}([0, 1])$ such that $\Lambda^*(\mu) < +\infty$. There exists a sequence $(\mu_n) \in \mathcal{H}$ such that $\mu_n \Rightarrow \mu$ and $\lim_{n \rightarrow +\infty} \Lambda^*(\mu_n) = \Lambda^*(\mu)$.*

3.3. Proof of Theorem 2.3. The normalized cumulant generating function of Y_n is given for any h in $\mathcal{C}([m, M])$ by

$$\begin{aligned} \Lambda_n(h) &= a_n \ln \mathbf{E} \left(\exp \left\{ \sqrt{\frac{n}{a_n}} (\langle \nu_n, h \rangle - \mathbf{E}(\langle \nu_n, h \rangle)) \right\} \right) \\ &= -\frac{a_n}{2} \sum_{k=1}^n \ln \left(1 - \frac{2}{\sqrt{na_n}} \lambda_{k,n} \right) + \frac{2}{\sqrt{na_n}} \lambda_{k,n}. \end{aligned}$$

We recall that $\{\lambda_{k,n}\}$ are the eigenvalues of the matrix $T_n(f)^{1/2} \Delta_h T_n(f)^{1/2}$. We can assert

$$\Lambda_n(h) = \frac{1}{n} \sum_{k=1}^n \lambda_{k,n}^2 + O \left(\frac{1}{n\sqrt{na_n}} \sum_{k=1}^n |\lambda_{k,n}|^3 \right).$$

Then from (10) we find that

$$\lim_{n \rightarrow +\infty} \Lambda_n(h) = \Lambda = \overline{f^2} \int_{[0,1]} h(x)^2 dx. \quad (16)$$

This function is defined on all $\mathcal{C}([0, 1])$, then the rate function is the Legendre dual of Λ which is, from [18],

$$I(\mu) = \frac{\pi}{2\overline{f^2}} \int_{[0,1]} l(x)^2 dx,$$

where $d_\mu(t) = l(x) dx$.

4. Appendix.

4.1. A Szegő theorem for generalized Toeplitz matrices. In this section we show a result on the distribution of eigenvalues of some kind of generalized Toeplitz matrices. Suppose g is a real function defined on $[0, 1] \times \mathbf{T}$ such that for any $x \in [0, 1]$ we have $g(x, \cdot) \in L^1(\mathbf{T})$. Define

$$\widehat{g}_k(x) = \frac{1}{2\pi} \int_{\mathbf{T}} g(x, \theta) e^{-ik\theta} d\theta$$

and

$$T_n^{\text{gen}}(g)_{k,l} = \widehat{g}_{l-k} \left(\frac{k}{n} \right). \quad (17)$$

Denote

$$\|\widehat{g}_k\|_\infty = \sup_{x \in [0,1]} |\widehat{g}_k(x)|.$$

Theorem 4.1. *If*

$$M := \sum_k \|\widehat{g}_k\|_\infty < \infty, \quad (18)$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{tr} (T_n^{\text{gen}}(g))^p = \frac{1}{2\pi} \int_0^1 \int_{\mathbf{T}} g(x, \theta)^p d\theta dx. \quad (19)$$

P r o o f. This proof is analogous to the one of [10]. Let $\varepsilon > 0$ be fixed and $m \in \mathbf{N}$ chosen such that

$$\sum_{|k|>m} \|\widehat{g}_k\|_\infty < \varepsilon.$$

Consider the trigonometric polynom of degree m :

$$g^m(x, \theta) = \sum_{k=-m}^m \widehat{g}_k(x) e^{ik\theta}. \quad (20)$$

Let $T_n^{\text{gen}}(g^m)$ be the generalized Toeplitz matrix associated to g^m as in (17). Therefore

$$T_n^{\text{gen}}(g) = T_n^{\text{gen}}(g^m) + R$$

and the sum of the moduli of the elements of any row of R is less than ε . Hence the same is true for the eigenvalues of R , i.e., for the eigenvalues of $T_n^{\text{gen}}(g) - T_n^{\text{gen}}(g^m)$. From the Weyl–Courant lemma, we can, therefore, bound $|\lambda_{k,n} - \lambda_{k,n}^m| \leq \varepsilon$, where $\{\lambda_{k,n}\}_k$ and $\{\lambda_{k,n}^m\}_k$ are the eigenvalues of $T_n^{\text{gen}}(g)$ and $T_n^{\text{gen}}(g^m)$, respectively, which are nondecreasingly ordered. From assumption (18), $|\lambda_{k,n}| \leq M$, $|\lambda_{k,n}^m| \leq M$. Hence for any positive integer s

$$|(\lambda_{k,n})^s - (\lambda_{k,n}^m)^s| \leq \varepsilon s M^{s-1}.$$

We can bound similarly $|g(x, \theta)^s - g^m(x, \theta)^s|$, and, therefore, to show (19) it is enough to consider the polynomial g^m . We derive

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{tr}(T_n^{\text{gen}}(g^m))^p = \sum_{D_p} \sum_{j=1}^m \widehat{g}_{l_1} \left(\frac{j+l_1}{n} \right) \widehat{g}_{l_2} \left(\frac{j+l_1+l_2}{n} \right) \cdots \widehat{g}_{l_p} \left(\frac{j}{n} \right),$$

where $D_p = \{(l_1, \dots, l_p) \in \mathbf{Z}^p; \sum l_i = 0\}$ and the second sum on the right-hand side above is on j such that $j + \sum_{i=1}^k l_i$, for k from 1 to p , is in the range $1, \dots, n$, i.e., $sp \leq j \leq n - sp$. Therefore we have to suppress at most $2sp + 1$ terms. From classical results on Riemann sums,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{D_p} \sum_{j=1}^m \widehat{g}_{l_1} \left(\frac{j+l_1}{n} \right) \widehat{g}_{l_2} \left(\frac{j+l_1+l_2}{n} \right) \cdots \widehat{g}_{l_p} \left(\frac{j}{n} \right) \\ &= \sum_{D_p} \int_0^1 \widehat{g}_{l_1}(x) \widehat{g}_{l_2}(x) \cdots \widehat{g}_{l_p}(x) dx \\ &= \sum_{(l_1, \dots, l_p) \in \mathbf{Z}^p} \frac{1}{2\pi} \int_{\mathbf{T}} e^{i(l_1+l_2+\dots+l_p)\theta} d\theta \int_0^1 g_{l_1}(x) \widehat{g}_{l_2}(x) \cdots \widehat{g}_{l_p}(x) dx \\ &= \frac{1}{2\pi} \int_0^1 \int_{\mathbf{T}} g(x, \theta)^p d\theta dx. \end{aligned}$$

4.2. Proof of Proposition 2.1. This proposition is a consequence of Theorem 5 of Rockafellar [18]. For the sake of clarity, we recall the framework of that paper. Let h be in $\mathcal{C}([m, M])$, and

$$\Lambda(h) = \int_{[m, M]} u(t, h(t)) d\nu(t),$$

where $u(t, x): [m, M] \times \mathbf{R} \rightarrow \mathbf{R}$ is a function convex in x , and ν a nonnegative σ -finite measure. For any μ in $\mathcal{M}([m, M])$ admitting (with respect to ν) the Lebesgue decomposition $\mu = l\nu + \mu^\perp$, where $l \in \mathcal{C}([m, M])$ and $\mu^\perp$ is the singular part, we have

$$\Lambda^*(\mu) = \int_{[m, M]} u^*(t, l(t)) d\nu(t) + \int_{[m, M]} r(u^*(t, \cdot); d\mu^\perp/d\eta(t)) d\eta(t), \quad (21)$$

where η is any nonnegative measure of $\mathcal{M}([m, M])$ with respect to which $\mu^\perp$ is absolutely continuous, and $u^*(t, \cdot)$ is the dual function of $u(t, \cdot)$:

$$\forall t, \quad u^*(t, y) = \sup_{x \in \mathbf{R}} \{xy - u(t, x)\}.$$

Applying the result of (21) to $u(t, x) = -(1/t) \ln(1 - 2tx)$, we have the formula of Proposition 2.1

4.3. Proof of Lemma 3.2. From [1, Proposition V.1.8 and Theorem X.1.1], since $T_n(f)$ is a Hermitian positive matrix, we deduce the inequality

$$\|T_n(f)^{1/2} \Delta_h T_n(f)^{1/2}\| \leq \|T_n(f)\| \|\Delta_h\|. \quad (22)$$

From [9, Chap. 5, Section 5.2] it follows that $\|T_n(f)\| \leq \|f\|_\infty$. In addition,

$$\|\Delta_h\| \leq \sup_k \sum_s |(\Delta_h)_{ks}| \leq \|h\|_\infty. \quad (23)$$

Putting together inequalities (22) and (23), we get the result.

4.4. Proof of Lemma 3.3. This lemma is a direct consequence of Theorem 4.1 above, for both random measures.

4.5. Proof of Lemma 3.4. From the definition of Λ^* , for any $\delta > 0$, there exists h_δ in $\mathcal{C}([0, 1])$ such that inequality (14) holds. In case we only have

$$\forall (t, \theta) \in [0, 1] \times \mathbf{T}, \quad h_\delta(t) f(\theta) \leq \frac{1}{2},$$

we choose h_ε with $\varepsilon > 0$ such that

$$\int_{[0,1]} h_\varepsilon(t) d\mu(t) - \Lambda(h_\varepsilon) \geq \Lambda_\delta^*(\mu) - \varepsilon$$

(this is possible, since Λ is continuous in a neighborhood of h_δ).

Then (14) holds with another δ . From assumption on f , $f > 0$, it then follows that $h_\varepsilon f < 1/2$.

4.6. Proof of Lemma 3.5. For all $0 < y < 1/u'(1/(2M))$, there exists a unique x_y in $(-\infty, 1/(2M))$ such that $y = u'(x_y)$. For such a pair (y, x_y) ,

$$u^*(y) = yx_y - u(x_y).$$

Since u' is strictly increasing and continuous, u^* is strictly convex on $0 < y < u'(1/(2M))$. For such an y and $z > 0$, $z \neq y$,

$$u^*(y) - u^*(z) < (y - z)x_y \quad (24)$$

(then y is an exposed point of u^* with exposing hyperplane x_y). If $\mu = l\nu$ and $\xi = \tilde{l}\nu + \xi^\perp$, then we apply inequality (24) with $y = l(t)$ and $z = \tilde{l}(t)$, and after that we integrate over $[0, 1]$ against ν . We obtain inequalities (15) with $h_l(t) = x_{l(t)}$.

4.7. Proof of Lemma 3.6. Following the sketch of proof of [8], we proceed in 4 steps. Assume $u'(1/(2M)) = +\infty$.

Step 1. Let $\mu = l\nu + \mu^\perp$ be in $\mathcal{M}([0, 1])$ such that $\Lambda^*(\mu) < \infty$ with l continuous and $l \in (0, u'(1/(2M)))$, and such that $\mu^\perp$ is in $L^1([0, 1])$. Since ν has full support on $[0, 1]$, there exists a sequence of continuous positive functions on $[0, 1]$ such that $h_n d\nu \Rightarrow \mu^\perp$. From the lower semi-continuity of Λ^* ,

$$\liminf_{n \rightarrow +\infty} \Lambda^*((l + h_n)\nu) \geq \Lambda^*(\mu).$$

Since u^* is a convex function, from [17], for any $y > 0$ and $z \geq 0$,

$$u^*(y + z) \leq u^*(y) + \frac{z}{2M}.$$

Therefore

$$\Lambda^*((l + \tilde{l})\nu) \leq \Lambda^*(l\nu) + \frac{1}{2M} \int \tilde{l}(t) d\nu(t). \quad (25)$$

The inequality above implies that

$$\Lambda^*((l + h_n)\nu) \leq \Lambda^*(l\nu) + \frac{1}{2M} \int_{[0,1]} h_n d\nu,$$

and then

$$\liminf_{n \rightarrow +\infty} \Lambda^*((l + h_n)\nu) \leq \Lambda^*(\mu).$$

We now show that the lemma is true if $\mu = l\nu$ with l which is ν -a.s. in $(0, u'(1/(2M)))$ and integrable.

Step 2. We prove the result for $\mu = l\nu$ assuming that l is in $(0, u'(1/(2M)))$, integrable and that for some $\varepsilon > 0$, the inequality $l > \varepsilon$ holds ν -a.s. There exists a sequence (l_n) of continuous positive functions such that l_n converges both in $L^1(\nu)$ and ν -a.s. to l and $l_n > \varepsilon/2$. Since on $(\varepsilon/2, u'(1/(2M)))$ the function u^* is Lipschitzian, the lemma holds.

Step 3. Define $l_\varepsilon := l\mathbf{1}_{l>\varepsilon} + \varepsilon\mathbf{1}_{l\leq\varepsilon}$. Apply second step and inequality (25) noticing that l_ε converges in $L^1(\nu)$ to l and that $l_\varepsilon \geq l$.

Step 4. For $\mu = l\nu + \eta$, combine first and third steps.

If $u'(1/(2M)) < +\infty$, then we have to modify the second and third steps, introducing an additional truncation at level $u'(1/(2M)) - \varepsilon$.

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A UNIFIED «BANG-BANG» PRINCIPLE WITH RESPECT TO $\mathcal{R}$ -INVARIANT PERFORMANCE BENCHMARKS¹⁾

В последнее время появился ряд работ, посвященных нахождению оптимального момента продажи акции — так, чтобы ожидаемое отношение цены продажи к некоторому эталону (т.е. к последней наивысшей цене акции) при конечном временном горизонте было максимальным. И хотя задача формулировалась в различных постановках, оптимальное решение в результате имело вид «bang-bang», который был изначально найден в работе А. Н. Ширяева, Ц. Сюя и С. И. Чжоу [12] и который можно буквально интерпретировать как «Купить-и-держатель» или «Продать-сразу-же» — в зависимости от качества акции. В настоящей статье мы сначала предлагаем три алгебраических условия на класс бенчмарков и называем любой бенчмарк, удовлетворяющий этим трем условиям, $\mathcal{R}$ -инвариантным характеристическим бенчмарком. Мы показываем, что если F есть $\mathcal{R}$ -инвариантный характеристический бенчмарк, то соответствующая задача оптимальной остановки имеет оптимальное решение типа «bang-bang». Предлагаемый нами подход позволяет дать единое доказательство всех похожих задач для броуновского движения, рассматриваемых в литературе, а также получить новые результаты; в частности, в п. 3.2 решена оставшаяся часть — не получившая освещения в литературе — задачи, первоначально сформулированной А. Н. Ширяевым (см. [11]).

Ключевые слова и фразы: оптимальная остановка, правило «Купить-и-держатель», правило «Продать-сразу-же», $\mathcal{R}$ -инвариантный характеристический бенчмарк.

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