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**THE SCHOENBERG–LÉVY KERNEL AND RELATIONSHIPS
AMONG FRACTIONAL BROWNIAN MOTION,
BIFRACTIONAL BROWNIAN MOTION, AND OTHERS¹⁾**

Начиная с обсуждения взаимосвязи между фрактальным и бифрактальным броуновскими движениями на вещественной прямой, мы устанавливаем, что фрактальное броуновское движение можно разложить в независимую сумму бифрактального броуновского движения и трифрактального броуновского движения, которое определяется в настоящей работе. Более общим образом, ортогональные разложения такого типа имеют место для широкого класса гауссовских или сферически инвариантных случайных функций, ковариационные функции которых являются ядрами Шёнберга–Леви во временной, пространственной или пространственно-временной области. Также построено много самоподобных, нестационарных (гауссовских) случайных функций и изучены свойства трифрактального броуновского движения. В частности, показано, что бифрактальное броуновское движение на $\mathbf{R}^d$ является «квазиспиралью» в смысле Кахана.

Ключевые слова и фразы: бифрактальное броуновское движение, условная отрицательная определенность, ковариационная функция, сферически инвариантная случайная функция, гауссовская случайная функция, положительная определенность, ядро Шёнберга–Леви, самоподобие, трифрактальное броуновское движение, вариограмма.

1. Introduction. A fractional Brownian motion on the real line is a Gaussian stochastic process with mean zero and covariance function

$$C(x_1, x_2) = |x_1|^\nu + |x_2|^\nu - |x_1 - x_2|^\nu, \quad x_1, x_2 \in \mathbf{R}, \quad (1.1)$$

where ν is a constant with $0 < \nu \leq 2$, and $\nu/2$ is often called the Hurst parameter. It is an intrinsically stationary process or a process with stationary increments, and has many interesting properties, such as self-similarity and

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long range dependence. This process was first introduced by Kolmogorov [18] and coined its name by Mandelbrot and Van Ness [28]. Two recent reviews of this process and its applications in various sciences can be found in [29] and [33]. As a generalization of the fractional Brownian motion, a bifractional Brownian motion [16] is a zero-mean Gaussian process with covariance function

$$C(x_1, x_2) = (|x_1|^{\nu\alpha} + |x_2|^{\nu\alpha})^{1/\alpha} - |x_1 - x_2|^\nu, \quad x_1, x_2 \in \mathbf{R}, \quad (1.2)$$

where α and ν are constants, $\alpha \geq 1$, and $0 < \nu \leq 2/\alpha$. It contains the fractional Brownian motion as a special case where $\alpha = 1$, is an intrinsically stationary process only if $\alpha = 1$, and is an infinite dimensional quasi-helix in the sense of Kahane [17]. Some properties of the bifractional Brownian motion have been studied in [1], [5], [11], [12], [16], [19], [21], [22], [30], [31], [34], and [35]. However, the permissible ranges of parameters α and ν in (1.2) have not been identified completely. For instance, as pointed out in [5], when $0 < \alpha \leq 1$ and $\nu\alpha = 1$, the permissible range of the parameter α (or ν) in (1.2) is $1/2 \leq \alpha \leq 1$ (or $1 \leq \nu \leq 2$). It is also permissible [2] if $1/2 < \alpha < 1$ and $0 < \nu < 2$.

This paper is motivated initially by searching for answers for the following two questions: what is the relationship between the fractional and bifractional Brownian motions? and what is the range of the parameter (α, ν) such that (1.2) is a valid covariance function? To answer these questions, we take the difference of two covariance functions (1.1) and (1.2), (1.1) minus (1.2), and look at what kind of property the function

$$C(x_1, x_2) = |x_1|^\nu + |x_2|^\nu - (|x_1|^{\nu\alpha} + |x_2|^{\nu\alpha})^{1/\alpha}, \quad x_1, x_2 \in \mathbf{R}, \quad (1.3)$$

has. Somewhat surprisingly, we are able to show that (1.3) is also a covariance function for a Gaussian process on the real line, once $\alpha \geq 1$ and $\nu > 0$. This means that the fractional Brownian motion can be decomposed as the sum of two independent Gaussian processes, one is the bifractional Brownian motion, and the other is the one with covariance (1.3). For convenience, let us call a zero-mean Gaussian process with covariance (1.3) as a trifractional Brownian motion. Notice that, for $\alpha > 1$, $x_1, x_2 > 0$

$$x_1 + x_2 - (x_1^\alpha + x_2^\alpha)^{1/\alpha} = \frac{1}{\Gamma(1 - 1/\alpha)} \int_0^\infty \left\{ \prod_{k=1}^2 \int_{u/x_k}^\infty v^{\alpha-2} \exp(-v^\alpha) dv \right\} du.$$

This identity suggests an interesting integral representation for a trifractional fractional Brownian motion $\{Z(x), x \in \mathbf{R}\}$,

$$Z(x) = \frac{1}{\Gamma^{1/2}(1 - 1/\alpha)} \int_0^\infty \left(\int_{u/|x|^\nu}^\infty v^{\alpha-2} \exp(-v^\alpha) dv \right) dZ_0(u), \quad x \in \mathbf{R},$$

where $\alpha > 1$, $\nu > 0$, and $\{Z_0(x), x \geq 0\}$ is Brownian motion with mean zero and covariance $\min(x_1, x_2)$, $x_1, x_2 \geq 0$.

One may wonder how is the other difference between (1.1) and (1.2), (1.2) minus (1.1), which is

$$C(x_1, x_2) = (|x_1|^{\nu\alpha} + |x_2|^{\nu\alpha})^{1/\alpha} - |x_1|^\nu - |x_2|^\nu, \quad x_1, x_2 \in \mathbf{R}. \quad (1.4)$$

In the case where $\alpha > 1$, however, $C(x_1, x_2)$ is always nonpositive, and thus cannot be positive definite. Interestingly, as Lei and Nualart [21] showed, this is indeed a covariance function for a Gaussian process on the real line, provided that $1/2 \leq \alpha \leq 1$ and $\nu > 0$. As a consequence, a bifractional Brownian motion with parameters $\nu \in (0, 2]$ and $\alpha \in [1/2, 1]$ can be decomposed as an independent sum of a fractional Brownian motion and a zero-mean Gaussian process with covariance (1.4).

Since the parameter ν in (1.3) is allowable to be any positive number, unlike that in (1.1) or (1.2), we ask a more general question about the formulation of the covariance function (1.3): is it still a covariance function if $|x|^\nu$ in (1.3) is replaced by an arbitrary nonnegative function? The answer is «yes», provided that $\alpha \geq 1$. A similar question for (1.4) gets a positive answer when $1/2 \leq \alpha \leq 1$. This enables us to introduce a class of second-order random functions or random functions with second-order increments in Section 2 on an index set $\mathbf{D}$, which may be a temporal, spatial, or spatio-temporal domain. In particular, many self-similar (Gaussian) random functions can be formulated.

For a stochastic process or random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$ whose increments have second-order moments, its variogram or structure function $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is defined by half the variance of the increment,

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2} \text{var}(Z(\mathbf{x}_1) - Z(\mathbf{x}_2)), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}.$$

When $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$ itself has second-order moments, its variogram can be found through its covariance function, $C(\mathbf{x}_1, \mathbf{x}_2) = \text{cov}\{Z(\mathbf{x}_1), Z(\mathbf{x}_2)\}$, by

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2} \{C(\mathbf{x}_1, \mathbf{x}_1) + C(\mathbf{x}_2, \mathbf{x}_2)\} - C(\mathbf{x}_1, \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}. \quad (1.5)$$

As usual, $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$ is called a stationary random function if its mean $\mathbf{E} Z(\mathbf{x})$ is a constant and its covariance function $C(\mathbf{x}_1, \mathbf{x}_2)$ depends only on the difference $\mathbf{x}_1 - \mathbf{x}_2$ (in this case $C(\mathbf{x}_1, \mathbf{x}_2)$ will be rewritten as $C(\mathbf{x}_1 - \mathbf{x}_2)$ for simplicity). By an intrinsically stationary random function we mean that, for every $\mathbf{x}_0 \in \mathbf{D}$, $\{Z(\mathbf{x} + \mathbf{x}_0) - Z(\mathbf{x}_0), \mathbf{x} \in \mathbf{D}\}$ is a stationary random function. In this case, the variogram of the random function depends only on the difference $\mathbf{x}_1 - \mathbf{x}_2$, is rewritten as $\gamma(\mathbf{x}_1 - \mathbf{x}_2)$ for simplicity, and is called an intrinsically stationary variogram on $\mathbf{D}$.

Notice that the function (1.1) is a Schoenberg–Lévy kernel (see, e.g., [9], [13], [14], [24], [27]), whose general form is

$$C(\mathbf{x}_1, \mathbf{x}_2) = \gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2) - \gamma(\mathbf{x}_1 - \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (1.6)$$

where $\mathbf{D}$ is a group and $\gamma(\mathbf{x})$ is an intrinsically stationary variogram on $\mathbf{D}$. A centered Gaussian random function $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$ with covariance (1.6) admits a simple orthogonal decomposition [23],

$$Z(\mathbf{x}) = Z_+(\mathbf{x}) + Z_-(\mathbf{x}), \quad \mathbf{x} \in \mathbf{D},$$

where

$$Z_+(\mathbf{x}) = \frac{Z(\mathbf{x}) + Z(-\mathbf{x})}{2}, \quad Z_-(\mathbf{x}) = \frac{Z(\mathbf{x}) - Z(-\mathbf{x})}{2}, \quad \mathbf{x} \in \mathbf{D},$$

and $\{Z_+(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$ and $\{Z_-(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$ are independent. In particular, this kind of orthogonal decomposition was used in [10] to derive a series expansion of the fractional Brownian motion. Section 3 provides another orthogonal decomposition for $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$, as a generalization of the fact that the fractional Brownian motion can be decomposed as an independent sum of the bifractional and trifractional Brownian motions. Section 4 gives the proofs of our theorems.

2. General results. This section introduces a class of variograms and nonstationary covariance functions associated with Gaussian or elliptical contoured random functions [26], [27] on an index set $\mathbf{D}$, with functions (1.3) and (1.4) as special examples, and obtains a property of the class.

Theorem 2.1. *If $g(\mathbf{x})$, $\mathbf{x} \in \mathbf{D}$, is a nonnegative function and α is a constant with $\alpha \geq 1$, then*

$$C_\alpha(\mathbf{x}_1, \mathbf{x}_2) = g(\mathbf{x}_1) + g(\mathbf{x}_2) - \{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{1/\alpha}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (2.1)$$

is a covariance function associated with a second-order elliptically contoured random function on $\mathbf{D}$, and

$$\gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2) = \{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{1/\alpha} - 2^{1/\alpha-1} \{g(\mathbf{x}_1) + g(\mathbf{x}_2)\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (2.2)$$

is a variogram associated with a random function whose increments are elliptically contoured, and is bounded by

$$\begin{aligned} (1 - 2^{1/\alpha-1})|g^{1/2}(\mathbf{x}_1) - g^{1/2}(\mathbf{x}_2)|^2 &\leq \gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2) \\ &\leq \frac{1}{2} |g(\mathbf{x}_1) - g(\mathbf{x}_2)|, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}. \end{aligned} \quad (2.3)$$

It is easy to check that both the left-hand side bound and the right-hand side bound in (2.3) are variograms. The assumption $\alpha \geq 1$ is necessary in (2.1) and in (2.2). Otherwise, if $\alpha < 1$, we would get the variogram $\gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2) \leq 0$, since $t^{1/\alpha}$, $t \geq 0$, is convex, or would get the variance of the corresponding random function

$$C_\alpha(\mathbf{x}, \mathbf{x}) = (2 - 2^{1/\alpha})g(\mathbf{x}) \leq 0,$$

which is impossible when $g(\mathbf{x}) > 0$. This argument results in the following corollary.

Corollary 2.1. *Let $g(\mathbf{x})$ be a positive function on $\mathbf{D}$. Then*

(i) $\alpha \geq 1$ is a necessary and sufficient condition for (2.1) to be a covariance function on $\mathbf{D}$, and

(ii) $\alpha \geq 1$ is a necessary and sufficient condition for (2.2) to be a variogram on $\mathbf{D}$.

The assumption $g(\mathbf{x}) \geq 0$ is necessary in the formulation of the covariance function (2.1) when $\alpha > 1$. For instance, when α is an odd number, the variance of the associated random function should be nonnegative, i.e.,

$$0 \leq C_\alpha(\mathbf{x}, \mathbf{x}) = (2 - 2^{1/\alpha})g(\mathbf{x}), \quad \mathbf{x} \in \mathbf{D},$$

which implies that $g(\mathbf{x}) \geq 0$, $\mathbf{x} \in \mathbf{D}$. Also, when α is an even number, we obtain $g(\mathbf{x}) \geq 0$ from

$$0 \leq C_\alpha(\mathbf{x}, \mathbf{x}) = 2g(\mathbf{x}) - 2^{1/\alpha}|g(\mathbf{x})|, \quad \mathbf{x} \in \mathbf{D}.$$

For α being an irrational number, the nonnegative assumption of $g(\mathbf{x})$ guarantees that the process is real.

Notice that, for $1 \leq \alpha_1 < \alpha_2$,

$$(x_1^{\alpha_1} + x_2^{\alpha_1})^{1/\alpha_1} \geq (x_1^{\alpha_2} + x_2^{\alpha_2})^{1/\alpha_2}, \quad x_1, x_2 \geq 0.$$

It implies that $C_\alpha(\mathbf{x}_1, \mathbf{x}_2)$ is nonnegative, and that $C_\alpha(\mathbf{x}_1, \mathbf{x}_2)$ is increasing with respect to α , i.e.,

$$C_{\alpha_2}(\mathbf{x}_1, \mathbf{x}_2) \geq C_{\alpha_1}(\mathbf{x}_1, \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}.$$

As α tends to ∞ ,

$$\begin{aligned} C_\alpha(\mathbf{x}_1, \mathbf{x}_2) &\rightarrow g(\mathbf{x}_1) + g(\mathbf{x}_2) - \max\{g(\mathbf{x}_1), g(\mathbf{x}_2)\} \\ &= \min\{g(\mathbf{x}_1), g(\mathbf{x}_2)\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \end{aligned}$$

and

$$\begin{aligned} \gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2) &\rightarrow \max\{g(\mathbf{x}_1), g(\mathbf{x}_2)\} - \frac{1}{2}\{g(\mathbf{x}_1) + g(\mathbf{x}_2)\} \\ &= \frac{1}{2}|g(\mathbf{x}_1) - g(\mathbf{x}_2)|, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}. \end{aligned}$$

Corollary 2.2. For $1 \leq \alpha_1 < \alpha_2$,

$$0 \leq C_{\alpha_1}(\mathbf{x}_1, \mathbf{x}_2) \leq C_{\alpha_2}(\mathbf{x}_1, \mathbf{x}_2) \leq \min\{g(\mathbf{x}_1), g(\mathbf{x}_2)\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}.$$

Corollary 2.3. Let $\alpha \geq 1$ and let $g(\mathbf{x})$ be a positive function on $\mathbf{D}$. Then

$$C(\mathbf{x}_1, \mathbf{x}_2) = 1 - \frac{\{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{1/\alpha}}{g(\mathbf{x}_1) + g(\mathbf{x}_2)}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (2.4)$$

and

$$C(\mathbf{x}_1, \mathbf{x}_2) = \{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{-1/\alpha} - \{g(\mathbf{x}_1) + g(\mathbf{x}_2)\}^{-1}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (2.5)$$

are covariance functions on $\mathbf{D}$.

The function (2.4) is a covariance function, since it is the product of (2.1) and $\{g(\mathbf{x}_1) + g(\mathbf{x}_2)\}^{-1}$, which is a covariance function on $\mathbf{D}$. Similarly, (2.5) is the product of (2.1) and $\{g(\mathbf{x}_1) + g(\mathbf{x}_2)\}^{-1}\{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{-1/\alpha}$. As an example of (2.4), taking $g(x) = \exp(x)$, $x \in \mathbf{R}$, yields a stationary covariance function

$$C(x) = 1 - \frac{(1 + \exp(\alpha x))^{1/\alpha}}{1 + \exp(x)}, \quad x \in \mathbf{R}.$$

Corollary 2.4. Let $\alpha \geq 1$ and let $g(\mathbf{x})$ be a positive function on $\mathbf{D}$. Then

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \frac{\{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{1/\alpha}}{g(\mathbf{x}_1) + g(\mathbf{x}_2)} - 2^{1/\alpha-1}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D},$$

and

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \{g(\mathbf{x}_1) + g(\mathbf{x}_2)\}^{-1} - 2^{1/\alpha-1}\{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{-1/\alpha}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D},$$

are variograms on $\mathbf{D}$.

The following corollary is obtained from Theorem 1 of Székely and Rizzo [32].

Corollary 2.5. Let $\alpha > 1$ and let $g(\mathbf{x})$ be a continuous, nonnegative function on $\mathbf{D}$. Suppose X_1, X_2, Y_1 , and Y_2 are independent random variables on $\mathbf{D}$, X_1 and X_2 are identically distributed, and Y_1 and Y_2 are identically distributed. If $g(X_1)$ and $g(Y_1)$ have finite expected values, then

$$2 \mathbf{E} \{g(X_1) + g(Y_1)\}^{1/\alpha} \geq \mathbf{E} \{g(X_1) + g(X_2)\}^{1/\alpha} + \mathbf{E} \{g(Y_1) + g(Y_2)\}^{1/\alpha}. \quad (2.6)$$

When $g(\mathbf{x})$ is not a constant, equality holds in (2.6) if and only if X_1 and Y_1 are identically distributed.

As we have seen, $\alpha \geq 1$ is necessary in (2.1) and (2.2). A natural question is, what kind of change is needed for the case $0 < \alpha \leq 1$? The answer is given in the next theorem when $1/2 \leq \alpha \leq 1$, and, more interestingly, what needs to do is just to change the sign of (2.1) or (2.2) in order to get a right answer.

Theorem 2.2. *If $g(\mathbf{x})$, $\mathbf{x} \in \mathbf{D}$, is a nonnegative function and α is a constant with $1/2 \leq \alpha \leq 1$, then*

$$C(\mathbf{x}_1, \mathbf{x}_2) = \{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{1/\alpha} - \{g(\mathbf{x}_1) + g(\mathbf{x}_2)\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (2.7)$$

is a covariance function associated with a second-order elliptically contoured random function on $\mathbf{D}$, and

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = 2^{1/\alpha-1} \{g(\mathbf{x}_1) + g(\mathbf{x}_2)\} - \{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{1/\alpha}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (2.8)$$

is a variogram associated with a random function whose increments are elliptically contoured, and is bounded by

$$\begin{aligned} (2^{1/\alpha-1} - 1) |g^{1/2}(\mathbf{x}_1) - g^{1/2}(\mathbf{x}_2)|^2 &\leq \gamma(\mathbf{x}_1, \mathbf{x}_2) \\ &\leq |g^{1/2}(\mathbf{x}_1) - g^{1/2}(\mathbf{x}_2)|^2, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}. \end{aligned} \quad (2.9)$$

One may wonder why the parameter α in Theorem 2.2 is restricted in the interval $[1/2, 1]$. The reason is, as Example 2.1 below indicates, Theorem 2.2 is no longer true if α is in the interval $(0, 1/2)$.

Corollary 2.6. *Let $1/2 \leq \alpha \leq 1$ and let $g(\mathbf{x})$ be a positive function on $\mathbf{D}$. Then*

$$C(\mathbf{x}_1, \mathbf{x}_2) = \frac{\{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{1/\alpha}}{g(\mathbf{x}_1) + g(\mathbf{x}_2)} - 1, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D},$$

and

$$C(\mathbf{x}_1, \mathbf{x}_2) = \{g(\mathbf{x}_1) + g(\mathbf{x}_2)\}^{-1} - \{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{-1/\alpha}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D},$$

are covariance functions, and

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = 2^{1/\alpha-1} - \frac{\{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_2)\}^{1/\alpha}}{g(\mathbf{x}_1) + g(\mathbf{x}_2)}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D},$$

is a variogram on $\mathbf{D}$.

The functions in the following corollary are formulated from a rescaled sum of (2.2) and (2.8), or a sum of (2.1) and (2.7).

Corollary 2.7. *If $1/2 \leq \alpha_1 \leq 1 \leq \alpha_2$, then*

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \left\{ \frac{g^{\alpha_2}(\mathbf{x}_1) + g^{\alpha_2}(\mathbf{x}_2)}{2} \right\}^{1/\alpha_2} - \left\{ \frac{g^{\alpha_1}(\mathbf{x}_1) + g^{\alpha_1}(\mathbf{x}_2)}{2} \right\}^{1/\alpha_1}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D},$$

is a variogram on $\mathbf{D}$, and

$$C(\mathbf{x}_1, \mathbf{x}_2) = \{g^{\alpha_1}(\mathbf{x}_1) + g^{\alpha_1}(\mathbf{x}_2)\}^{1/\alpha_1} - \{g^{\alpha_2}(\mathbf{x}_1) + g^{\alpha_2}(\mathbf{x}_2)\}^{1/\alpha_2}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (2.10)$$

is a covariance function on $\mathbf{D}$.

In particular, when a nonnegative function $g(\mathbf{x})$, $\mathbf{x} \in \mathbf{D}$, is homogeneous of order ν on a group $\mathbf{D}$ in the sense that it satisfies

$$g(a\mathbf{x}) = |a|^\nu g(\mathbf{x}), \quad \text{for all } a \in \mathbf{R} \text{ and } \mathbf{x} \in \mathbf{D},$$

each of (2.1), (2.7), and (2.10) is also homogeneous of order ν , so that the corresponding zero-mean Gaussian random function $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$ is self-similar with index $\nu/2$,

$$\{Z(a\mathbf{x}), \mathbf{x} \in \mathbf{D}\} \stackrel{d}{=} \{|a|^{\nu/2} Z(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}, \quad \text{for all } a \in \mathbf{R},$$

where $\nu > 0$, and $\stackrel{d}{=}$ denotes the equality of the finite-dimensional distributions [15]. The following example illustrates the important correspondence between self-similar and stationary processes on the real line through Lamperti's transformation [20], [33], [34].

Example 2.1. A trifractional Brownian motion $\{Z_0(x), x \in \mathbf{R}\}$ with mean zero and covariance function (1.3) is self-similar with index $\nu/2$. By Lamperti's transformation, we obtain a stationary process $\{Z(x) = 2^{-1/2} e^{-(\nu/2)x} Z_0(e^x), x \in \mathbf{R}\}$ with covariance function

$$\begin{aligned} \text{cov}(Z(x_1), Z(x_2)) &= \cosh\left(\frac{\nu}{2}(x_2 - x_1)\right) \\ &\quad - 2^{1/\alpha-1} \cosh^{1/\alpha}\left(\frac{\nu\alpha}{2}(x_2 - x_1)\right), \quad x_1, x_2 \in \mathbf{R}, \end{aligned}$$

where $\alpha \geq 1$.

Similarly, when $1/2 \leq \alpha \leq 1$,

$$C(x) = 2^{1/\alpha-1} \cosh^{1/\alpha}\left(\frac{\nu\alpha}{2}x\right) - \cosh\left(\frac{\nu}{2}x\right) \quad (2.11)$$

is a stationary covariance function on $\mathbf{R}$. However, the permissible range of α in (2.11) cannot cover any point of the interval $(0, 1/2)$. To see this,

notice that (2.11) is twice differentiable on $\mathbf{R}$ with $C''(0) = 0$ and

$$\begin{aligned} \nu^{-2}C''(x) &= 2^{1/\alpha-3}(1-\alpha)\cosh^{1/\alpha-2}\left(\frac{\nu\alpha}{2}x\right)\sinh^2\left(\frac{\nu\alpha}{2}x\right) \\ &\quad + 2^{1/\alpha-3}\alpha\cosh^{1/\alpha}\left(\frac{\nu\alpha}{2}x\right) - \frac{1}{4}\cosh\left(\frac{\nu\alpha}{2}x\right), \quad x \in \mathbf{R}. \end{aligned}$$

If (2.11) is a stationary covariance function on $\mathbf{R}$, then so is $-C''(x)$, by Slutsky's theorem (see [8, Theorem 1.4]). This implies that $-C''(0) = \nu^2(2^{1/\alpha}\alpha - 2)/8$ must be nonnegative, and thus, $\alpha \leq 2^{1-1/\alpha}$. Combining the last inequality with $0 < \alpha \leq 1$ we obtain that $\alpha \in [1/2, 1]$ is a necessary condition for (2.11) or (1.4) to be a covariance function on $\mathbf{R}$. This also explains why $\alpha \in [1/2, 1]$ is required in Theorem 2.2.

A property of the random function with second-order increments whose variogram is (2.2) or (2.8) is presented in the next theorem.

Theorem 2.3. *Suppose that a random function $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$ has second-order increments with variogram $\gamma(\mathbf{x}_1, \mathbf{x}_2)$. If $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is either of the form (2.2) or of the form (2.8), then*

$$\text{cov}\{Z(\mathbf{x}_2) - Z(\mathbf{x}_1), Z(\mathbf{x}_4) - Z(\mathbf{x}_3)\} \geq 0, \quad (2.12)$$

whenever $g(\mathbf{x}_1) \leq g(\mathbf{x}_2) \leq g(\mathbf{x}_3) \leq g(\mathbf{x}_4)$, $\mathbf{x}_k \in \mathbf{D}$, $k = 1, \dots, 4$.

In particular, when $\mathbf{D} = \mathbf{R}$ and $g(x)$, $x \in \mathbf{R}$, is an increasing and nonnegative function, (2.12) means that nonoverlapping increments of the process $\{Z(x), x \in \mathbf{R}\}$ are nonnegatively correlated. In such a case, the mean square derivative of the process does not exist or exists as a constant almost surely [3].

Example 2.2. Let $\alpha > 1$, $\nu > 0$, and $\|\mathbf{x}\| = (\sum_{k=1}^d x_k^2)^{1/2}$, $\mathbf{x} \in \mathbf{R}^d$. A random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d\}$ with covariance function $\|\mathbf{x}_1\|^\nu + \|\mathbf{x}_2\|^\nu - (\|\mathbf{x}_1\|^{\nu\alpha} + \|\mathbf{x}_2\|^{\nu\alpha})^{1/\alpha}$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^d$, has the property that

$$\text{cov}(Z(\mathbf{x}_2) - Z(\mathbf{x}_1), Z(\mathbf{x}_4) - Z(\mathbf{x}_3)) \geq 0$$

whenever $\|\mathbf{x}_1\| \leq \dots \leq \|\mathbf{x}_4\|$.

3. Decomposition for a Schoenberg–Lévy kernel. Suppose that $\gamma(\mathbf{x})$, $\mathbf{x} \in \mathbf{D}$, is an intrinsically stationary variogram on a group $\mathbf{D}$. In this section we consider a decomposition of the Schoenberg–Lévy kernel, $\gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2) - \gamma(\mathbf{x}_1 - \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}$, similar to that of the fractional Brownian motion.

The proposed decomposition for the Schoenberg–Lévy kernel is

$$\begin{aligned} \gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2) - \gamma(\mathbf{x}_1 - \mathbf{x}_2) &= \left[\gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2) - \{\gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2)\}^{1/\alpha} \right] \\ &\quad + \left[\{\gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2)\}^{1/\alpha} - \gamma(\mathbf{x}_1 - \mathbf{x}_2) \right], \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \end{aligned}$$

where $\alpha \geq 1$ and the first term on the right-hand side is a covariance function according to Theorem 2.1. Just like (1.1), this decomposition makes sense if we are able to show that the second term on the right-hand side is also a covariance function on $\mathbf{D}$. A result is presented in Part (ii) of the following theorem, under an additional assumption that $\gamma^\alpha(\mathbf{x})$ is an intrinsically stationary variogram on $\mathbf{D}$, while Part (i) is relevant to another type of decomposition that decomposes the Schoenberg–Lévy kernel as the difference of two terms when $1/2 \leq \alpha \leq 1$. The integral decomposition of a form that is more general than (3.3) is presented in [1]. For the permissibility of the power of an intrinsically stationary variogram see [25].

Theorem 3.1. *Assume that $\gamma(\mathbf{x})$ is an intrinsically stationary variogram on a group $\mathbf{D}$ and α is a positive constant.*

(i) *If $1/2 \leq \alpha \leq 1$, then*

$$\begin{aligned} \gamma(\mathbf{x}_1, \mathbf{x}_2) = & 2^{1/\alpha-1} \{ \gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2) \} - \{ \gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2) \}^{1/\alpha} \\ & + \gamma(\mathbf{x}_1 - \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \end{aligned} \quad (3.1)$$

is a variogram associated with a random function whose increments are elliptically contoured, and is bounded by

$$\gamma(\mathbf{x}_1 - \mathbf{x}_2) \leq \gamma(\mathbf{x}_1, \mathbf{x}_2) \leq 2\gamma(\mathbf{x}_1 - \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (3.2)$$

and

$$C(\mathbf{x}_1, \mathbf{x}_2) = \{ \gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2) \}^{1/\alpha} - \gamma(\mathbf{x}_1 - \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (3.3)$$

is the covariance function corresponding to a second-order elliptically contoured random function on $\mathbf{D}$.

(ii) *If $\alpha > 1$ and $\gamma^\alpha(\mathbf{x})$ is an intrinsically stationary variogram on $\mathbf{D}$, then (3.1) is a variogram associated with a random function whose increments are elliptically contoured, and is bounded by*

$$\gamma(\mathbf{x}_1 - \mathbf{x}_2) - \frac{1}{2} |\gamma(\mathbf{x}_1) - \gamma(\mathbf{x}_2)| \leq \gamma(\mathbf{x}_1, \mathbf{x}_2) \leq \gamma(\mathbf{x}_1 - \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \quad (3.4)$$

and (3.3) is the covariance function corresponding to a second-order elliptically contoured random function on $\mathbf{D}$.

Corollary 3.1. *A second-order random function on a group $\mathbf{D}$ whose covariance function is a Schoenberg–Lévy kernel, $\gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2) - \gamma(\mathbf{x}_1 - \mathbf{x}_2)$, admits an orthogonal decomposition as the sum of two independent second-order random functions, one having the covariance $\gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2) - \{ \gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2) \}^{1/\alpha}$, and the other having the covariance $\{ \gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2) \}^{1/\alpha} - \gamma(\mathbf{x}_1 - \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}$, where $\alpha \geq 1$ and $\gamma^\alpha(\mathbf{x})$ is an intrinsically stationary variogram on $\mathbf{D}$.*

Clearly, the function (1.2) is an example of (3.3) with $\gamma(x) = |x|^\nu$, $x \in \mathbf{R}$, where ν is a positive constant. Since it is homogeneous of order ν , by Lamperti's transformation, (1.2) is a covariance function on $\mathbf{R}$ if and only if

$$C(x) = \left\{ \exp\left(\frac{\nu\alpha}{2}x\right) + \exp\left(-\frac{\nu\alpha}{2}x\right) - \left| \exp\left(\frac{1}{2}x\right) - \exp\left(-\frac{1}{2}x\right) \right|^\nu \right\}^{1/\alpha}, \quad x \in \mathbf{R}, \quad (3.5)$$

is a stationary covariance function on $\mathbf{R}$. For this to be true, a necessary condition is that ν is at most 2. Otherwise, if $\nu > 2$, (3.5) would be twice differentiable at the origin and thus

$$0 \leq \lim_{x \rightarrow 0+} \frac{C(0) - C(2x/(\nu\alpha))}{x^2/2} = -\frac{2^{1/\alpha}}{\alpha},$$

which is a contradiction. Also, when $\nu = 2$, it follows from

$$0 \leq \lim_{x \rightarrow 0+} \frac{C(0) - C(2x/(\nu\alpha))}{x^2/2} = -\frac{2^{1/\alpha}}{\alpha} + \frac{2}{\alpha^2}$$

that α is necessarily in the interval $[1/2, 1]$. By Theorem 3.1, (1.2) is the covariance function for a bifractional Brownian motion if the parameter (α, ν) belongs to the set

$$\left\{ (\alpha, \nu): \frac{1}{2} \leq \alpha \leq 1, 0 < \nu \leq 2 \right\} \cup \left\{ (\alpha, \nu): \alpha \geq 1, 0 < \nu \leq \frac{2}{\alpha} \right\}.$$

In (3.3) choosing $\gamma(\mathbf{x}) = \|\mathbf{x}\|^\nu$, $\mathbf{x} \in \mathbf{R}^d$, we obtain a bifractional Brownian motion in $\mathbf{R}^d$ with covariance

$$C(\mathbf{x}_1, \mathbf{x}_2) = (\|\mathbf{x}_1\|^{\nu\alpha} + \|\mathbf{x}_2\|^{\nu\alpha})^{1/\alpha} - \|\mathbf{x}_1 - \mathbf{x}_2\|^\nu, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^d,$$

where $1/2 \leq \alpha \leq 1$ and $0 < \nu \leq 2$, or $\alpha \geq 1$ and $0 < \nu \leq 2/\alpha$. When $1/2 \leq \alpha \leq 1$ and $0 < \nu \leq 2$, inequality (3.2) becomes

$$\|\mathbf{x}_1 - \mathbf{x}_2\|^\nu \leq \gamma(\mathbf{x}_1, \mathbf{x}_2) \leq 2\|\mathbf{x}_1 - \mathbf{x}_2\|^\nu, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^d,$$

which, in the sense of Kahane [17], means that the bifractional Brownian motion in $\mathbf{R}^d$ is an infinite dimensional quasi-helix. This is also the case when $\alpha \geq 1$ and $0 < \nu \leq 2/\alpha$, since inequality (3.4) becomes

$$\frac{1}{2} \|\mathbf{x}_1 - \mathbf{x}_2\|^\nu \leq \gamma(\mathbf{x}_1, \mathbf{x}_2) \leq \|\mathbf{x}_1 - \mathbf{x}_2\|^\nu, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^d.$$

The particular case where $d = 1$ and $\alpha \geq 1$ was derived in [16].

4. Proofs.

4.1. Proof of Theorem 2.1. It suffices to consider the case where $\alpha > 1$, since $\alpha = 1$ corresponds to a degenerate case with $\gamma_1(\mathbf{x}_1, \mathbf{x}_2) \equiv 0$ or $C_1(\mathbf{x}_1, \mathbf{x}_2) \equiv 0$.

To show that (2.1) is a covariance function associated with a zero-mean Gaussian or elliptically contoured random function, we need to prove that it is positive definite [26], [36], [37]; that is, the inequality

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j C_\alpha(\mathbf{x}_i, \mathbf{x}_j) \geq 0$$

holds for every positive integer n , all $\mathbf{x}_k \in \mathbf{D}$ and real numbers a_k ($k = 1, \dots, n$). For this purpose, we need the following formula:

$$\begin{aligned} x_1^{1/\alpha} + x_2^{1/\alpha} - (x_1 + x_2)^{1/\alpha} &= \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right) \frac{1}{\Gamma(\beta + 1)} \\ &\times \int_0^\infty \left\{ \prod_{k=1}^2 \int_0^{x_k} \exp(-u_k t^{1/\beta}) du_k \right\} dt, \quad x_1, x_2 \geq 0, \end{aligned} \quad (4.1)$$

where $\beta = 2 - 1/\alpha$. Using this formula we obtain

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n a_i a_j C_\alpha(\mathbf{x}_i, \mathbf{x}_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \left[\{g^\alpha(\mathbf{x}_i)\}^{1/\alpha} + \{g^\alpha(\mathbf{x}_j)\}^{1/\alpha} - \{g^\alpha(\mathbf{x}_i) + g^\alpha(\mathbf{x}_j)\}^{1/\alpha} \right] \\ &= \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right) \frac{1}{\Gamma(\beta + 1)} \int_0^\infty \left\{ \sum_{i=1}^n a_i \int_0^{g^\alpha(\mathbf{x}_i)} \exp(-ut^{1/\beta}) du \right\}^2 dt \geq 0. \end{aligned}$$

When a random function is a second-order one with covariance (2.1), its variogram can easily be found via formula (1.5) and this coincides with (2.2).

When a random function has second-order increments only, its covariance may not be well defined, so that formula (1.5) cannot be used to conclude that (2.2) is the variogram of the random function. Nevertheless, (2.2) is nonnegative with $\gamma_\alpha(\mathbf{x}, \mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbf{D}$, and is conditionally negative definite, in the sense that the inequality

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j \gamma_\alpha(\mathbf{x}_i, \mathbf{x}_j) \leq 0$$

holds for every integer $n \geq 2$, all $\mathbf{x}_k \in \mathbf{D}$ and real numbers a_k ($k = 1, \dots, n$) subject to $\sum_{k=1}^n a_k = 0$. In fact, the last inequality follows by using formula (4.1),

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j \gamma_\alpha(\mathbf{x}_i, \mathbf{x}_j) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j \{g^\alpha(\mathbf{x}_i) + g^\alpha(\mathbf{x}_j)\}^{1/\alpha}$$

$$\begin{aligned}
&= - \sum_{i=1}^n \sum_{j=1}^n a_i a_j \left[\{g^\alpha(\mathbf{x}_i)\}^{1/\alpha} + \{g^\alpha(\mathbf{x}_j)\}^{1/\alpha} - \{g^\alpha(\mathbf{x}_i) + g^\alpha(\mathbf{x}_j)\}^{1/\alpha} \right] \\
&= - \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right) \frac{1}{\Gamma(\beta+1)} \int_0^\infty \left\{ \sum_{i=1}^n a_i \int_0^{g^\alpha(\mathbf{x}_i)} \exp(-ut^{1/\beta}) du \right\}^2 dt \leq 0.
\end{aligned}$$

Thus, (2.2) can be regarded as the variogram of a random function with second-order increments [26].

To verify (2.3), we rewrite $\gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2)$ as

$$\gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2) = -C_\alpha(\mathbf{x}_1, \mathbf{x}_2) + (1 - 2^{1/\alpha-1})\{g(\mathbf{x}_1) + g(\mathbf{x}_2)\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}.$$

By the Cauchy–Schwarz inequality, $C_\alpha(\mathbf{x}_1, \mathbf{x}_2) \leq 2(1 - 2^{1/\alpha-1})\sqrt{g(\mathbf{x}_1)g(\mathbf{x}_2)}$. Thus,

$$\gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2) \geq (1 - 2^{1/\alpha-1})|g^{1/2}(\mathbf{x}_1) - g^{1/2}(\mathbf{x}_2)|^2, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}.$$

Since $\gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2)$ is increasing with respect to α , we obtain

$$\gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2) \leq \lim_{\alpha \rightarrow \infty} \gamma_\alpha(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2} |g(\mathbf{x}_1) - g(\mathbf{x}_2)|, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}.$$

4.2. Proof of Theorem 2.2. It suffices to consider the case where $1/2 \leq \alpha < 1$, just like in the proof of Theorem 2.1.

Notice that formula (4.1) still holds when $1/2 \leq \alpha < 1$. Using this formula we obtain

$$\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^n a_i a_j C(\mathbf{x}_i, \mathbf{x}_j) \\
&= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \left[\{g^\alpha(\mathbf{x}_i) + g^\alpha(\mathbf{x}_j)\}^{1/\alpha} - \{g^\alpha(\mathbf{x}_i)\}^{1/\alpha} - \{g^\alpha(\mathbf{x}_j)\}^{1/\alpha} \right] \\
&= \frac{1}{\alpha} \left(\frac{1}{\alpha} - 1\right) \frac{1}{\Gamma(\beta+1)} \int_0^\infty \left\{ \sum_{i=1}^n a_i \int_0^{g^\alpha(\mathbf{x}_i)} \exp(-ut^{1/\beta}) du \right\}^2 dt \geq 0;
\end{aligned}$$

that is, (2.7) is positive definite. Clearly, $C(\mathbf{x}_1, \mathbf{x}_2)$ is nonnegative and is not greater than $(2^{1/\alpha} - 2)\sqrt{g(\mathbf{x}_1)g(\mathbf{x}_2)}$ according to the Cauchy–Schwarz inequality.

Just like in the proof of Theorem 2.1, we can show that the function (2.8) is the variogram associated with a random function with second-order increments on $\mathbf{D}$. Similar to that in the proof of inequality (2.3), the left-hand side inequality of (2.9) can be obtained. Since (2.8) is decreasing with respect to α in the interval $[1/2, 1]$, we obtain

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) \leq |g^{1/2}(\mathbf{x}_1) - g^{1/2}(\mathbf{x}_2)|^2, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}.$$

4.3. Proof of Theorem 2.3. We give a proof when $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is of the form (2.2) only. In this case, using the identity

$$x_1^{1/\alpha} + x_2^{1/\alpha} - (x_1 + x_2)^{1/\alpha} = \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right) \int_0^{x_1} \int_0^{x_2} (u+v)^{1/\alpha-2} du dv, \quad x_1, x_2 \geq 0,$$

we obtain, when $g(\mathbf{x}_1) \leq g(\mathbf{x}_2) \leq g(\mathbf{x}_3) \leq g(\mathbf{x}_4)$,

$$\begin{aligned} & \text{cov}\{Z(\mathbf{x}_2) - Z(\mathbf{x}_1), Z(\mathbf{x}_4) - Z(\mathbf{x}_3)\} \\ &= \gamma_\alpha(\mathbf{x}_1, \mathbf{x}_4) + \gamma_\alpha(\mathbf{x}_2, \mathbf{x}_3) - \gamma_\alpha(\mathbf{x}_1, \mathbf{x}_3) - \gamma_\alpha(\mathbf{x}_2, \mathbf{x}_4) \\ &= \left[g(\mathbf{x}_1) + g(\mathbf{x}_3) - \{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_3)\}^{1/\alpha} \right] \\ & \quad + \left[g(\mathbf{x}_2) + g(\mathbf{x}_4) - \{g^\alpha(\mathbf{x}_2) + g^\alpha(\mathbf{x}_4)\}^{1/\alpha} \right] \\ & \quad - \left[g(\mathbf{x}_2) + g(\mathbf{x}_3) - \{g^\alpha(\mathbf{x}_2) + g^\alpha(\mathbf{x}_3)\}^{1/\alpha} \right] \\ & \quad - \left[g(\mathbf{x}_1) + g(\mathbf{x}_4) - \{g^\alpha(\mathbf{x}_1) + g^\alpha(\mathbf{x}_4)\}^{1/\alpha} \right] \\ &= \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right) \int_{g(\mathbf{x}_1)}^{g(\mathbf{x}_2)} \int_{g(\mathbf{x}_3)}^{g(\mathbf{x}_4)} (u+v)^{1/\alpha-2} du dv \geq 0, \end{aligned}$$

under the assumption that $\alpha \geq 1$.

4.4. Proof of Theorem 3.1. (i) In this case (3.3) can be decomposed, using the Schoenberg–Lévy kernel as a component, as

$$\begin{aligned} C(\mathbf{x}_1, \mathbf{x}_2) &= [\gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2) - \gamma(\mathbf{x}_1 - \mathbf{x}_2)] \\ & \quad + \left[\{ \gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2) \}^{1/\alpha} - \gamma(\mathbf{x}_1) - \gamma(\mathbf{x}_2) \right], \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \end{aligned}$$

where, by Theorem 2.2, the second-term on the right-hand side is a covariance function. Thus, $C(\mathbf{x}_1, \mathbf{x}_2)$ is the covariance function corresponding to a (Gaussian) random function on $\mathbf{D}$. Also, as the sum of variograms (2.8) and $\gamma(\mathbf{x}_1 - \mathbf{x}_2)$, (3.1) is a variogram on $\mathbf{D}$.

Since $\gamma(\mathbf{x})$ is an intrinsically stationary variogram on $\mathbf{D}$, we have (see, e.g., [6], [23])

$$|\gamma^{1/2}(\mathbf{x}_1) - \gamma^{1/2}(\mathbf{x}_2)| \leq \gamma^{1/2}(\mathbf{x}_1 - \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D},$$

and obtain inequality (3.2) from (2.9) and from

$$\begin{aligned} 0 &\leq 2^{1/\alpha-1} \{ \gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2) \} - \{ \gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2) \}^{1/\alpha} \\ &\leq (\gamma^{1/2}(\mathbf{x}_1) - \gamma^{1/2}(\mathbf{x}_2))^2 \leq \gamma(\mathbf{x}_1 - \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}. \end{aligned}$$

(ii) Since $\gamma^\alpha(\mathbf{x})$ is an intrinsically stationary variogram on $\mathbf{D}$, the Schoenberg–Lévy kernel $\gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2) - \gamma^\alpha(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}$, is a covariance function on $\mathbf{D}$. So are

$$\exp\{\gamma^\alpha(\mathbf{x}_1)u + \gamma^\alpha(\mathbf{x}_2)u - \gamma^\alpha(\mathbf{x}_1 - \mathbf{x}_2)u\} - 1, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D},$$

and

$$\begin{aligned} & \exp\{-\gamma^\alpha(\mathbf{x}_1)u - \gamma^\alpha(\mathbf{x}_2)u\} - \exp\{-\gamma^\alpha(\mathbf{x}_1 - \mathbf{x}_2)u\} \\ &= \exp\{-\gamma^\alpha(\mathbf{x}_1)u\} \cdot \left[\exp\{\gamma^\alpha(\mathbf{x}_1)u + \gamma^\alpha(\mathbf{x}_2)u - \gamma^\alpha(\mathbf{x}_1 - \mathbf{x}_2)u\} - 1 \right] \\ & \times \exp\{-\gamma^\alpha(\mathbf{x}_2)u\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \end{aligned}$$

for any nonnegative number u . For every integer $n \geq 2$, any $\mathbf{x}_k \in \mathbf{D}$ and real numbers a_k ($k = 1, \dots, n$) subject to $\sum_{k=1}^n a_k = 0$, using the formula

$$t^{1/\alpha} = \left\{ \alpha \Gamma\left(1 - \frac{1}{\alpha}\right) \right\}^{-1} \int_0^\infty \{1 - \exp(-tu)\} u^{-1-1/\alpha} du, \quad t \geq 0, \quad (4.2)$$

we obtain

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^n a_i a_j \gamma(\mathbf{x}_i, \mathbf{x}_j) \\ &= - \sum_{i=1}^n \sum_{j=1}^n a_i a_j \left[\{\gamma^\alpha(\mathbf{x}_i) + \gamma^\alpha(\mathbf{x}_j)\}^{1/\alpha} - \gamma(\mathbf{x}_i - \mathbf{x}_j) \right] \\ &= - \frac{1}{\alpha \Gamma(1 - 1/\alpha)} \int_0^\infty \sum_{i=1}^n \sum_{j=1}^n a_i a_j \left[\exp\{-\gamma^\alpha(\mathbf{x}_i)u - \gamma^\alpha(\mathbf{x}_j)u\} \right. \\ & \quad \left. - \exp\{-\gamma^\alpha(\mathbf{x}_i - \mathbf{x}_j)u\} \right] u^{-1-1/\alpha} du \leq 0, \end{aligned}$$

so that (3.1) is a variogram on $\mathbf{D}$ associated with a random function $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{D}\}$, say. Its increment $\{Z(\mathbf{x}) - Z(\mathbf{0}), \mathbf{x} \in \mathbf{D}\}$ has second-order moments with covariance

$$\begin{aligned} & \text{cov}(Z(\mathbf{x}_1) - Z(\mathbf{0}), Z(\mathbf{x}_2) - Z(\mathbf{0})) \\ &= \gamma(\mathbf{x}_1, \mathbf{0}) + \gamma(\mathbf{x}_2, \mathbf{0}) - \gamma(\mathbf{x}_1, \mathbf{x}_2) - \gamma(\mathbf{0}, \mathbf{0}) \\ &= \{\gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2)\}^{1/\alpha} - \gamma(\mathbf{x}_1 - \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \end{aligned}$$

which coincides with (3.3).

It follows from (2.3) that

$$\begin{aligned} \gamma(\mathbf{x}_1, \mathbf{x}_2) &= \gamma(\mathbf{x}_1 - \mathbf{x}_2) - \left[\{\gamma^\alpha(\mathbf{x}_1) + \gamma^\alpha(\mathbf{x}_2)\}^{1/\alpha} - 2^{1/\alpha-1} \{\gamma(\mathbf{x}_1) + \gamma(\mathbf{x}_2)\} \right] \\ &\geq \gamma(\mathbf{x}_1 - \mathbf{x}_2) - \frac{1}{2} |\gamma(\mathbf{x}_1) - \gamma(\mathbf{x}_2)|, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}, \end{aligned}$$

and $\gamma(\mathbf{x}_1, \mathbf{x}_2) \leq \gamma(\mathbf{x}_1 - \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}$.

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