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J. Stoyanov, G. D. Lin, Hardy's condition in the moment problem for probability distributions, *Теория вероятн. и ее примен.*, 2012, том 57, выпуск 4, 811–820

DOI: <https://doi.org/10.4213/tvp4485>

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26 апреля 2023 г., 14:31:00

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HARDY'S CONDITION IN THE MOMENT PROBLEM FOR PROBABILITY DISTRIBUTIONS

В терминах теории вероятностей условие Харди записывается так: $\mathbf{E}[e^{c\sqrt{X}}] < \infty$, где X — неотрицательная случайная величина, $c = \text{const} > 0$. При этом условии все моменты случайной величины X конечны и однозначно определяют ее распределение. Это условие, основанное на двух работах Г. Г. Харди (1917/1918), является более слабым, чем условие Крамера, которое требует существования производящей функции моментов. Мы показываем, что в условии $\mathbf{E}[e^{cX^\alpha}] < \infty$ значение $\alpha = 1/2$ (квадратный корень) неумлучшаемо в том смысле, что это наименьшая степень X , при которой это условие обеспечивает моментную определенность X . Описывается связь условия Харди со свойствами моментов X . Используя это условие, мы устанавливаем результат о моментной определенности произвольного многомерного распределения.

Ключевые слова и фразы: распределение, моменты, проблема моментов, условие Харди, условие Крамера, условие Карлемана, условие Крейна, условие Лина.

1. Introduction. A web-search for «Hardy's condition in the moment problem for probability distributions» shows that nothing is available. The reader is invited to check and eventually confirm that this is the case.

In this paper we provide details about a not so well-known criterion for uniqueness of a probability distribution by its moments. The criterion is based on two old papers by G. H. Hardy, and to the best of our knowledge it is not explicitly described in the probabilistic literature. Hardy's condition has a simple form, often it is relatively easy to check and it allows us to derive new results or provide new proofs of known results.

Hardy's criterion is intrinsically related to other well-known classical criteria based on conditions of Cramér, Carleman, Krein (see, e.g., [20], [22]) and also to more recent developments achieved in [16], [23], [17], [18], [7], and [24].

G. H. Hardy (1877–1947), who is recognized as one of the greatest analysts of 20th century, never showed an interest in probability theory, never tried to solve a probabilistic problem and never used probabilistic language. However, in the paper [9], P. Diaconis convincingly concluded that several analytical results by Hardy have been instrumental later when building some areas of modern probability theory. Diaconis did not refer to the two papers by Hardy which are in the base of the present paper. The new criterion described here shows that in fact the analytical results in [11], [12], though not intended to be probabilistic, are, when appropriately reformulated and annotated, definitely another contribution to probability theory, in particular, to the moment problem for probability distributions.

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¹⁾ The work of G. D. Lin was partly supported by the National Science Council of R.O.C.

2. Notation, notions and preliminary facts. Suppose that X is a random variable with values on the whole real line $\mathbf{R}^1$. If F denotes the distribution function of X , we write $X \sim F$. We assume that X has all moments finite, i.e., $\mathbf{E}[|X|^k] < \infty$ for $k = 1, 2, \dots$, and use the notation m_k for the moment of order k :

$$m_k = \mathbf{E}[X^k] = \int_{\mathbf{R}^1} x^k dF(x), \quad k = 1, 2, \dots$$

Thus the moment sequence $\{m_k, k = 1, 2, \dots\}$ of X , and F , is well defined. If F is the only distribution function with the moments $\{m_k\}$, we say that F is uniquely determined by its moments, or that F (or X) is *M-determinate*. Otherwise F is nonunique, or F (or X) is *M-indeterminate* in which case there is at least one distribution function, say G , such that $F \neq G$, but $m_k(F) = m_k(G)$ for all $k = 1, 2, \dots$.

According to well-known Cramér's criterion, if X is an arbitrary random variable, $X \sim F$, whose moment generating function exists, i.e., for some $t_0 > 0$, we have $M(t) = \mathbf{E}[e^{tX}] < \infty$ for all $t \in (-t_0, t_0)$ (*Cramér's condition*), then all moments of X are finite, and F is M-determinate. If there is no moment generating function, i.e., $\mathbf{E}[e^{t_n X}] = \infty$ for some points t_n converging to zero, we say that F has *heavy tail(s)*. Still, we may have all moments m_k finite and F either M-determinate or M-indeterminate.

Lemma 1. *For any nonnegative random variable X and any fixed constant $c > 0$ the function $g(\delta) := \mathbf{E}[\exp(cX^\delta)]$ is nondecreasing in $\delta > 0$.*

Introduce the number $\delta_0 = \sup\{\delta: \mathbf{E}[\exp(cX^\delta)] < \infty\}$, for which the term *Cramér index of X* is appropriate. Then $\mathbf{E}[\exp(cX^\delta)] = \infty$ for any $\delta > \delta_0$, while $\mathbf{E}[\exp(cX^\delta)] < \infty$ for $\delta \in [0, \delta_0)$. Now the claim in Lemma 1 easily follows from the properties of the power function and the exponential function.

There are criteria for uniqueness and for nonuniqueness of a distribution in terms of the moments. They depend on the range of values of X , $X \sim F$, equivalently, on the support of F . Two cases are of importance, when the support is a subset of the positive semi-real line $\mathbf{R}^+ = [0, \infty)$ (Stieltjes moment problem), and when the support is a subset of the whole real line $\mathbf{R}^1 = (-\infty, \infty)$ (Hamburger moment problem).

It is possible that a distribution F on $\mathbf{R}^+$ is unique for the Stieltjes moment problem, but nonunique for the Hamburger moment problem. This can eventually happen only for discrete distributions with unbounded support which includes the origin; for details see [5, p. 481]. In this paper we focus our attention on the Stieltjes moment problem.

3. Hardy's criterion and comments on the proof. We are in a position now to formulate one of the main results discussed in this paper.

Theorem 1 (Hardy's criterion). *Let X be a nonnegative random variable with distribution function F . Suppose that the following condition is satisfied for some $c > 0$:*

$$\mathbf{E}[e^{c\sqrt{X}}] < \infty \quad (\text{Hardy's condition}). \quad (1)$$

Then all positive integer order moments of X are finite, i.e., $m_k = \mathbf{E}[X^k] = \int_0^\infty x^k dF(x) < \infty$, $k = 1, 2, \dots$, and moreover, F is M-determinate, i.e., F is the only distribution function with the moment sequence $\{m_k, k = 1, 2, \dots\}$.

Remark 1. Let us emphasize that the required condition is on $\sqrt{X}$, i.e., we require the (1/2)-Cramér's condition for X , or Cramér's condition for $\sqrt{X}$. This implies that $\sqrt{X}$ is M-determinate. However, the conclusion in the theorem is about X itself. Equivalently, under Hardy's condition, F is the only distribution function with the moment sequence $\{m_k, k = 1, 2, \dots\}$.

Proof. We found that the arguments of Hardy are deep and transparent and we hope the reader also to find them interesting. We present below the three major steps of his proof. We do not give all technical details; they can be seen in the original papers.

Hardy's condition is introduced for an arbitrary distribution function F . Let us concentrate on the absolutely continuous case. If $f(x) = F'(x)$, $x \in \mathbf{R}^+$, is the density of the random variable $X \sim F$, we assume that for some $c > 0$ one has

$$\int_0^\infty e^{c\sqrt{x}} f(x) dx < \infty. \quad (2)$$

Our goal is to determine the function $f(x)$, $x > 0$, which satisfies the relations

$$\int_0^\infty x^k f(x) dx = m_k, \quad k = 1, 2, \dots \quad (3)$$

Recall that $\{m_k, k = 1, 2, \dots\}$ is the moment sequence of X , also of F .

Step 1. Introduce the function $\varphi(z)$ for complex z , as follows:

$$\begin{aligned} \varphi(z) &:= \sum_{k=0}^\infty \frac{(-1)^k m_k z^{2k}}{(2k)!} = \int_0^\infty f(x) \sum_{k=0}^\infty \frac{(-1)^k x^k z^{2k}}{(2k)!} dx \\ &= \int_0^\infty f(x) \cos(z\sqrt{x}) dx = 2 \int_0^\infty u f(u^2) \cos(zu) du. \end{aligned}$$

The integrability of the function $f(x)e^{c\sqrt{x}}$ guarantees that the function $\varphi(z)$ is well defined for z with $|z| < c$. Also, $\varphi(z)$ is an even function which is analytic in the strip $\{z = x + iy, -c < y < c\}$ with $\varphi(z) \rightarrow 0$ as $x \rightarrow \pm\infty$.

Step 2. The convergence of the integral $\int_0^\infty f(x) \operatorname{ch}(|z|\sqrt{x}) dx$ allows us to invert the equality $\varphi(z) = 2 \int_0^\infty u f(u^2) \cos(zu) du$, thus getting

$$u f(u^2) = \frac{1}{\pi} \lim_{\lambda \rightarrow \infty} \int_0^\lambda \left(1 - \frac{z}{\lambda}\right) \varphi(z) \cos(zu) dz,$$

and this relation holds for almost all $u > 0$. This gives the value of $f(x)$ for any $x > 0$,

$$f(x) = \frac{1}{\pi\sqrt{x}} \lim_{\lambda \rightarrow \infty} \int_0^\lambda \left(1 - \frac{z}{\lambda}\right) \varphi(z) \cos(z\sqrt{x}) dz. \quad (4)$$

The properties of $\varphi(z)$ and the applicability of an inversion imply that $f(x)$, $x > 0$, is a density function which is unique eventually up to the values on a set of measure zero.

Step 3. The last step is to show that the function $f(x)$, $x > 0$, in formula (4), is the solution of our problem, i.e., that all relations (3) are indeed satisfied. This can be done by a further inversion and using a variant of Parseval's formula; see [25, Section 11.9].

Theorem 1 is proved.

R e m a r k 2. The above arguments are given in the absolutely continuous case when everything is expressed in terms of the density. Hardy mentioned that his result is true for arbitrary distribution function F on $\mathbf{R}^+$. In this general case Hardy's condition is written as $\int_0^\infty e^{c\sqrt{x}} dF(x) < \infty$ for some $c > 0$. However the two inversions and the limit need more delicate reasoning based on properties of Stieltjes integrals. We do not give details here. Instead, see Remark 5 below for an alternative proof.

Corollary 1. *If the random variable $X \geq 0$ satisfies Cramér's condition $\mathbf{E}[e^{cX}] < \infty$ for some $c > 0$, then X^2 is M -determinate.*

Corollary 2. *If $X \geq 0$ is a random variable with the Cramér index equal to δ_0 , then the transformed random variable $X^{2\delta}$ is M -determinate for each $\delta \in (0, \delta_0)$.*

4. Illustrative examples. Let us present some examples showing how Hardy's condition works. We either obtain new statements or provide another proof of a statement which is known in the literature.

Example A. Let $Z \sim N(0, 1)$. Cramér's condition holds, hence Z , or $N(0, 1)$, is M-determinate, a well-known fact. This implies that the half-normal distribution, that of $|Z|$, satisfies Cramér's condition and hence it is M-determinate. By Hardy's result, the square Z^2 is M-determinate, which is also a known fact, since $Z^2 = \chi_1^2$ has a moment generating function, i.e., Cramér's condition holds. Apply Hardy's result again to conclude more, that the fourth degree Z^4 is M-determinate. This property is known; see, e.g., [2]. It can also be proved by using Theorem 4 in the paper [16, p. 89, 205] as done in [23]. Applying Hardy's theorem we have another proof of this statement.

Example B. Start with a random variable $X \sim \text{Exp}(1)$. There is a moment generating function, i.e., Cramér's condition holds, so X is M-determinate. However, Cramér's condition for X implies, by Hardy's criterion, that the square X^2 is M-determinate. Notice that X^2 has a heavy-tailed distribution. We can apply Carleman's condition mentioned above, or use again Theorem 4 in [16], to conclude that X^r is M-determinate for all $r \in [0, 2]$. Krein's condition easily implies the M-indeterminacy of X^r for $r > 2$. Hardy's criterion implies, e.g., that if $r > 2$, then $X^{r/2}$ does not have a moment generating function.

We consider now a quite general example which is available in the literature (see, e.g., [13], [22], and [20]). Our goal is to show different methods for proving that a distribution is M-determinate or M-indeterminate. It will be clear that in some cases the neatest method is to apply Hardy's condition.

Example C. Let $\alpha, \beta > 0$ and $X_{\alpha, \beta} \sim F$ with density

$$f_{\alpha, \beta}(x) = K_{\alpha, \beta} e^{-\beta x^\alpha}, \quad x > 0; \quad K_{\alpha, \beta} = \frac{\alpha \beta^{1/\alpha}}{\Gamma(1/\alpha)}.$$

We claim that $X_{\alpha, \beta}$ is M-determinate if and only if $\alpha \geq 1/2$.

Sufficiency. Note first that for $c > 0$, we have

$$\mathbf{E}[e^{c\sqrt{X_{\alpha, \beta}}}] = K_{\alpha, \beta} \int_0^\infty e^{c\sqrt{x}} e^{-\beta x^\alpha} dx.$$

If $\alpha \geq 1/2$, take $c = \beta/2$, then

$$\mathbf{E}[e^{c\sqrt{X_{\alpha, \beta}}}] \leq K_{\alpha, \beta} \int_0^\infty e^{-\beta\sqrt{x}/2} dx = \frac{8}{\beta^2} K_{\alpha, \beta} < \infty.$$

Hence, by Theorem 1, $X_{\alpha, \beta}$ is M-determinate for $\alpha \geq 1/2$.

Necessity. If $0 < \alpha < 1/2$, with reference to [1], [15], [16], and [23], we have to check that the logarithmic normalized integral of the density $f_{\alpha, \beta}$, also known as Krein quantity, is finite:

$$\begin{aligned} K[f_{\alpha, \beta}] &= \int_0^\infty \frac{-\ln f_{\alpha, \beta}(x^2)}{1+x^2} dx \\ &= -\ln K_{\alpha, \beta} \int_0^\infty \frac{1}{1+x^2} dx + \beta \int_0^\infty \frac{x^{2\alpha}}{1+x^2} dx < \infty. \end{aligned}$$

Therefore, the random variable $X_{\alpha, \beta}$ is M-indeterminate for any $\alpha \in (0, 1/2)$. Let us mention that one of the results to which we refer here is Theorem 1 of [16] and its proof uses some fundamental results on Hardy spaces.

Remark 3. The sufficiency part in Example C can be established in two other ways, either applying Carleman's criterion, or using Theorem 4 from [16].

If we want to check Carleman's condition, we first need the moments, and a calculation shows that

$$m_k = \mathbf{E}[X_{\alpha,\beta}^k] = \frac{\Gamma((k+1)/\alpha)}{\beta^{k/\alpha}\Gamma(1/\alpha)}, \quad k = 1, 2, \dots$$

Using Stirling's formula for the gamma function (see [8] for an elementary proof), we find that

$$(m_k)^{1/(2k)} \approx (\alpha\beta e)^{-1/(2\alpha)} k^{1/(2\alpha)} \text{ as } k \rightarrow \infty \implies \sum_{k=1}^{\infty} (m_k)^{-1/(2k)} = \infty$$

for $\alpha \geq 1/2$. Since the moments of $X_{\alpha,\beta}$ satisfy Carleman's condition, we conclude that $X_{\alpha,\beta}$ is M-determinate for $\alpha \geq 1/2$.

Another way to get the same statement is first to note that for the density of $X_{\alpha,\beta}$ the Krein quantity $K[f_{\alpha,\beta}]$ equals ∞ if $\alpha \geq 1/2$. Then we need to check that the density $f_{\alpha,\beta}(x)$ is «away from zero» at the origin, i.e., that $\lim_{x \rightarrow 0+} f_{\alpha,\beta}(x) = K_{\alpha,\beta} > 0$, and finally, check Lin's condition (see [16, Theorem 4 and its slight modification on p. 205]):

$$\frac{-x f'_{\alpha,\beta}(x)}{f_{\alpha,\beta}(x)} = \alpha\beta x^\alpha \nearrow \infty \text{ as } x \rightarrow \infty.$$

This implies that $X_{\alpha,\beta}$ is M-determinate for $\alpha \geq 1/2$.

R e m a r k 4. There is another way to prove the necessity part in Example C. It is based on constructing an explicit Stieltjes class, as proposed in [24]. This is an infinite family of distributions, which in terms of densities is written as $\{f_{\alpha,\beta}(1 + \varepsilon h), \varepsilon \in [-1, 1]\}$, and they all have the same moments as $X_{\alpha,\beta}$. To do this, take, e.g., the following perturbation function:

$$h(x) = \sin(x^\alpha \beta \operatorname{tg}(\alpha\pi)), \quad x > 0,$$

where $\alpha \in (0, 1/2)$ and $\beta > 0$. Then we have

$$\begin{aligned} \int_0^\infty x^k f_{\alpha,\beta}(x) h(x) dx &= K_{\alpha,\beta} \int_0^\infty x^k e^{-\beta x^\alpha} \sin(x^\alpha \beta \operatorname{tg}(\alpha\pi)) dx \\ &= \frac{1}{\alpha} K_{\alpha,\beta} \Gamma\left(\frac{k+1}{\alpha}\right) \left(\frac{\cos(\alpha\pi)}{\beta}\right)^{(k+1)/\alpha} \sin((k+1)\pi) \\ &= 0 \quad \text{for any integer } k > 0 \end{aligned}$$

(see [20, Chap. II, Section 12.9] or [10, formula 3.944(9)]). Thus we conclude that all density functions $f_{\alpha,\beta}$ and $f_{\alpha,\beta}(1 + \varepsilon h)$, $\varepsilon \in [-1, 1]$, have the same moment sequence $\{m_k, k = 1, 2, \dots\}$ for $\alpha \in (0, 1/2)$ and $\beta > 0$.

5. Properties of Hardy's condition. We now present results showing a property of Hardy's condition and its relationship with the moments.

Theorem 2. *The constant $1/2$ (square root) in Hardy's condition (1) is the best possible. In other words, for each $\rho \in (0, 1/2)$, there exists a random variable $X \geq 0$ such that $\mathbf{E}[e^{cX^\rho}] < \infty$ for some $c > 0$, however X is M-indeterminate.*

P r o o f. Consider $X_{\alpha,\beta}$ defined in Example C. For $\rho \in (0, 1/2)$ and $c > 0$, take $\alpha = \rho$ and $\beta = 2c > 0$. Then we have

$$\mathbf{E}[e^{cX_{\alpha,\beta}^\rho}] = K_{\alpha,2c} \int_0^\infty e^{-cx^\alpha} dx = \frac{K_{\alpha,2c}}{K_{\alpha,c}} < \infty.$$

However, $X_{\alpha,\beta}$ is M-indeterminate for $\alpha \in (0, 1/2)$, as shown above.

Theorem 3. Let $a > 0$ and $X \sim F$ be a nonnegative random variable.

(i) If $\mathbf{E}[e^{cX^a}] < \infty$ for some constant $c > 0$, then $m_k \leq \Gamma(k/a + 1)c_0^k$, $k = 1, 2, \dots$, for some constant $c_0 > 0$ (independent of k).

(ii) Conversely, if, in addition, $a \leq 1$ and $m_k \leq \Gamma(k/a + 1)c_0^k$, $k = 1, 2, \dots$, for some constant $c_0 > 0$ (independent of k), then $\mathbf{E}[e^{cX^a}] < \infty$ for some $c > 0$.

P r o o f. (i) Suppose $\mathbf{E}[e^{cX^a}] < \infty$ for some $c > 0$. Then

$$\begin{aligned} m_k &= \int_0^\infty x^k dF(x) = \int_0^\infty x^k e^{-cx^a} e^{cx^a} dF(x) \\ &\leq \max_{0 \leq x < \infty} \{x^k e^{-cx^a}\} \int_0^\infty e^{cx^a} dF(x) = \left(\frac{k}{ace}\right)^{k/a} \mathbf{E}[e^{cX^a}]. \end{aligned}$$

Applying the inequality for the gamma function:

$$\sqrt{2\pi x} \left(\frac{x}{e}\right)^x \leq \Gamma(x+1) \leq \sqrt{2\pi x} \left(\frac{x}{e}\right)^x e^{1/(12x)} \quad \text{for } x > 0 \quad (5)$$

(see, e.g., [26, Part II, Chap. 12.33]), we have, by letting $x = k/a$,

$$\begin{aligned} m_k &\leq \left(\frac{1}{c}\right)^{k/a} \Gamma\left(\frac{k}{a} + 1\right) \frac{\mathbf{E}[e^{cX^a}]}{\sqrt{2\pi k/a}} \leq \Gamma\left(\frac{k}{a} + 1\right) (c^{-1/a})^k \frac{\mathbf{E}[e^{cX^a}]}{\sqrt{2\pi/a}} \\ &\leq \Gamma\left(\frac{k}{a} + 1\right) c_0^k, \quad k = 1, 2, \dots \end{aligned}$$

Here $c_0 = c^{-1/a} c_*$ and $c_* = \max\{1, \mathbf{E}[e^{cX^a}]/\sqrt{2\pi/a}\}$.

(ii) Suppose $a \in (0, 1]$ and $m_k \leq \Gamma(k/a + 1)c_0^k$, $k = 1, 2, \dots$, for some $c_0 > 0$. We want to show that $\mathbf{E}[e^{cX^a}] < \infty$ for some $c > 0$. With $c \in (0, a/c_0^a)$, we have

$$\begin{aligned} \mathbf{E}[e^{cX^a}] &= \int_0^\infty e^{cx^a} dF(x) = \int_0^\infty \sum_{k=0}^\infty \frac{(cx^a)^k}{k!} dF(x) = \sum_{k=0}^\infty \frac{c^k}{k!} \int_0^\infty x^{ak} dF(x) \\ &= \sum_{k=0}^\infty \frac{c^k}{k!} \mathbf{E}[X^{ak}] \leq \sum_{k=0}^\infty \frac{c^k}{k!} (\mathbf{E}[X^k])^a = \sum_{k=0}^\infty \frac{c^k}{k!} m_k^a, \end{aligned}$$

by Lyapunov's inequality. On the other hand, by the assumption on m_k and the inequality (5), we have

$$m_k \leq \Gamma\left(\frac{k}{a} + 1\right) c_0^k \leq \sqrt{\frac{2\pi k}{a}} \left(\frac{k}{ae}\right)^{k/a} e^{1/(12k/a)} c_0^k, \quad (6)$$

and hence

$$m_k^a \leq e \left(\frac{2\pi k}{a}\right)^{a/2} \left(\frac{k}{ae}\right)^k c_0^{ak},$$

because $a^2/(12k) \leq 1$. Applying (5) once again, we obtain

$$\frac{m_k^a}{k!} \leq \frac{e(2\pi k/a)^{a/2} \left(\frac{k}{ae}\right)^k c_0^{ak}}{\sqrt{2\pi k} \left(\frac{k}{e}\right)^k} \leq A k^{(a-1)/2} \left(\frac{c_0^a}{a}\right)^k \leq A \left(\frac{c_0^a}{a}\right)^k,$$

where $A = e(2\pi)^{(a-1)/2} a^{-a/2} > 0$. Therefore, since $c c_0^a < a$, we have

$$\mathbf{E}[e^{cX^a}] \leq \sum_{k=0}^\infty \frac{c^k}{k!} m_k^a \leq \sum_{k=0}^\infty A \left(\frac{c c_0^a}{a}\right)^k = \frac{A}{1 - c c_0^a/a} < \infty.$$

Theorem 3 is proved.

Corollary 3. Let $X \sim F$ be a nonnegative random variable. Then:

- (i) $\mathbf{E}[e^{c\sqrt{X}}] < \infty$ for some $c > 0$ if and only if $m_k \leq (2k)! c_0^k$, $k = 1, 2, \dots$, for some $c_0 > 0$;
- (ii) $\mathbf{E}[e^{cX}] < \infty$ for some $c > 0$ if and only if $m_k \leq k! c_0^k$, $k = 1, 2, \dots$, for some $c_0 > 0$.

R e m a r k 5. Hardy's condition (1) is sufficient, but not necessary, for a random variable $X \geq 0$ to be M-determinate. Notice that this is on $\mathbf{R}^+$. Indeed, let us mention first that the moment condition in part (i) of Corollary 3 implies that Carleman's condition holds by (5), i.e., $\sum_{k=1}^{\infty} (m_k)^{-1/(2k)} = \infty$. The second point is that there exists an $X \geq 0$ which is M-determinate, but fails to satisfy Carleman's condition (see [13, p. 93]), and hence fails to satisfy Hardy's condition (1). We modify Heyde's arguments as follows. Suppose $\{m_k\}_{k=1}^{\infty}$ is the moment sequence of M-determinate random variable X on $\mathbf{R}^+$. Then it is possible to add a mass $\mu > 0$ at the origin such that the resulting sequence of moments $\{m_k(1+\mu)^{-1}\}_{k=1}^{\infty}$ corresponds to an M-indeterminate X^* on $\mathbf{R}^+$. Thus, the Carleman quantity for the moments of X^* is necessarily finite. If Carleman's condition were necessary for X to be M-determinate, then we would have $\sum_{k=1}^{\infty} (m_k)^{-1/(2k)} = \infty$. However, this is clearly not possible because $\sum_{k=1}^{\infty} (m_k)^{-1/(2k)} (1+\mu)^{1/(2k)} < \infty$. Therefore, in general, Carleman's condition is not necessary for an $X \geq 0$ to be M-determinate on $\mathbf{R}^+$.

R e m a r k 6. For $a \in (0, 1]$, we give in Theorem 3 a characteristic property of $\mathbf{E}[\exp(cX^a)] < \infty$, where $c > 0$, in terms of the moments $\{m_k\}_{k=1}^{\infty}$ of X . For a general $a > 0$, we still can obtain a similar result; however, the condition is expressed in terms of the moments $\{m_{ak}\}_{k=1}^{\infty}$ of X , where $m_{ak} = \mathbf{E}[X^{ak}]$ may involve fractional moments. Precisely, for $a > 0$, we have the result: $\mathbf{E}[\exp(cX^a)] < \infty$ for some $c > 0$ if and only if $m_{ak} \leq k! c_0^k$, $k = 1, 2, \dots$, for some $c_0 > 0$ (independent of k). In particular, we have $\mathbf{E}[\exp(X^2)] < \infty$ if and only if $m_{2k} \leq k! c_0^k$, $k = 1, 2, \dots$, for some constant $c_0 > 0$ (independent of k). (In this case, the growth rate of m_{2k} is required to be less than that in the case $\mathbf{E}[\exp(X)] < \infty$.) We note that the above *general* result is actually a consequence of Corollary 3(ii) by taking X^a instead of X .

The moment condition in Theorem 3 has the following equivalent form.

Theorem 4. Let $a > 0$ and $X \sim F$ be a nonnegative random variable. Then $\limsup_{k \rightarrow \infty} m_k^{a/k}/k < \infty$ if and only if $m_k \leq \Gamma(k/a + 1) c_0^k$, $k = 1, 2, \dots$, for some $c_0 > 0$ (independent of k).

P r o o f. Sufficiency. Suppose that $m_k \leq \Gamma(k/a + 1) c_0^k$, $k = 1, 2, \dots$, for some $c_0 > 0$ (independent of k). Then we have, by (6),

$$\frac{m_k^{a/k}}{k} \leq \frac{c_0^a}{ae} \left(\frac{2\pi k}{a} \right)^{a/(2k)} e^{a^2/(12k^2)} \rightarrow \frac{c_0^a}{ae} \quad \text{as } k \rightarrow \infty.$$

Therefore, $\limsup_{k \rightarrow \infty} m_k^{a/k}/k < \infty$.

Necessity. Suppose that $\limsup_{k \rightarrow \infty} m_k^{a/k}/k < \infty$. Then there exists a constant $M > 0$ such that $m_k^{a/k}/k < M$ for $k = 1, 2, \dots$. Hence, by (5), we have

$$m_k \leq (aM)^{k/a} \left(\frac{k}{a} \right)^{k/a} \leq \Gamma\left(\frac{k}{a} + 1\right) e^{k/a} (aM)^{k/a} \left(\frac{2\pi}{a} \right)^{-1/2} \leq \Gamma\left(\frac{k}{a} + 1\right) c_0^k,$$

where $c_0 = (eaM)^{1/a} c_*$ and $c_* = \max\{1, (2\pi/a)^{-1/2}\}$.

Theorem 4 is proved.

Corollary 4. Let $a \in (0, 1]$ and $X \sim F$ be a nonnegative random variable. Then $\mathbf{E}[e^{cX^a}] < \infty$ for some $c > 0$ if and only if $\limsup_{k \rightarrow \infty} m_k^{a/k}/k < \infty$.

Corollary 5. Let $X \sim F$ be a nonnegative random variable whose moments satisfy the condition $\limsup_{k \rightarrow \infty} m_k^{1/k}/k < \infty$. Then X^2 is M-determinate.

The following surprising result demonstrates the power and usefulness of Hardy's condition.

Theorem 5. Let $Y, Y \sim G$, be an arbitrary random variable on $\mathbf{R}^1$ satisfying Cramér's condition, i.e., the moment generating function of Y exists. Then:

- (i) The random variable Y^2 is M-determinate on $\mathbf{R}^+$.
- (ii) If, in addition, G does not have a mass at the origin, the random variable Y^2 is M-determinate on $\mathbf{R}^1$.

P r o o f. Since Y satisfies Cramér's condition, there exists an $r > 0$ such that $\mathbf{E} e^{tY} < \infty$ for all $t \in (-r, r)$, which implies that $\mathbf{E}[\exp(t|Y|)] < \infty$ for $t \in (0, r)$. Hence, by Theorem 1, Y^2 is M-determinate on $\mathbf{R}^+$. This proves part (i). Part (ii) follows from the assumption that G does not have a mass at the origin and the result by Heyde [13] already used above.

Theorem 5 is proved.

Corollary 6. If $X > 0$ satisfies Cramér's condition, then X^2 is M-determinate on $\mathbf{R}^+$ and also on $\mathbf{R}^1$.

R e m a r k 7. Cramér's condition is sufficient, but not necessary, for an $X > 0$ to have M-determinate X^2 on $\mathbf{R}^+$. For example, let X be a discrete random variable with $\mathbf{P}[X = n] = C \exp(-n/\ln n)$ for $n = 3, 4, \dots$, where C is the norming constant such that $\sum_{n=3}^{\infty} \mathbf{P}[X = n] = 1$. Then X does not have a moment generating function, but $\sum_{k=1}^{\infty} (m_{2k})^{-1/(2k)} = \infty$ (see [6, p. 304] or [22, p. 112]). This means that X^2 satisfies Carleman's condition ($\sum_{k=1}^{\infty} (\mathbf{E}[(X^2)^k])^{-1/(2k)} = \infty$) and hence is M-determinate on $\mathbf{R}^+$ and also on $\mathbf{R}^1$ because $X^2 > 0$.

R e m a r k 8. Let $Y \sim G$ be a random variable on $\mathbf{R}^1$. As before $m_k = \mathbf{E}[Y^k]$ and we need the notation $\mu_k = \mathbf{E}[|Y|^k]$ for the absolute moment of order k . Then in summary, the following statements are equivalent:

- (a) Y satisfies Cramér's condition: $\mathbf{E}[\exp(tY)] < \infty$, $t \in (-r, r)$, for some $r > 0$;
- (b) $\mathbf{E}[\exp(t|Y|)] < \infty$, $t \in (0, r)$, for some $r > 0$;
- (c) $\limsup_{k \rightarrow \infty} \mu_k^{1/k}/k < \infty$;
- (d) $\limsup_{k \rightarrow \infty} m_{2k}^{1/2}/(2k) < \infty$;
- (e) $\mu_k \leq k! c_0^k$, $k = 1, 2, \dots$, for some $c_0 > 0$ (independent of k);
- (f) $m_{2k} \leq (2k)! c_0^k$, $k = 1, 2, \dots$, for some $c_0 > 0$ (independent of k).

6. Moment uniqueness of a multivariate distribution. While the problem of moments for one-dimensional distributions is in general well developed and several results are available in the literature, very little is done for multivariate distributions. The paper [14] summarizes known analytical results and contains some new results. Now, having to hand Hardy's criterion, we can establish a new and quite general result.

Suppose $X = (X_1, \dots, X_n)$ is an arbitrary random vector in n -dimensional Euclidean space $\mathbf{R}^n$ with all multi-indexed moments

$$m_{k_1, \dots, k_n} = \mathbf{E}[X_1^{k_1} \cdots X_n^{k_n}], \quad k_j = 1, 2, \dots, \quad j = 1, 2, \dots, n,$$

finite. We are interested in the determinacy of the vector $X \in \mathbf{R}^n$ by the multi-indexed moment sequence $\{m_{k_1, \dots, k_n}, k_j = 1, 2, \dots, j = 1, 2, \dots, n\}$. Since the distribution of X is arbitrary, this is a Hamburger moment problem.

Denote by $\|X\|$ the Euclidean length of X , i.e.,

$$\|X\| = \sqrt{\|X\|^2} = \sqrt{X_1^2 + \cdots + X_n^2}.$$

Theorem 6. *With the notation introduced above, suppose that the one-dimensional nonnegative random variable $\|X\|$ satisfies Cramér's condition:*

$$\mathbf{E}[e^{c\|X\|}] < \infty \quad \text{for some } c > 0.$$

Then the random vector $X \in \mathbf{R}^n$, or equivalently, its n -dimensional distribution function, is uniquely determined by the set of multi-indexed moments $\{m_{k_1, \dots, k_n}, k_j = 1, 2, \dots, j = 1, 2, \dots, n\}$.

P r o o f. Cramér's condition for $\|X\|$ implies that $\|X\|$ itself, as a nonnegative random variable is M-determinate. This is the Stieltjes moment problem.

However, Cramér's condition for $\|X\|$ means that Hardy's condition (1) holds for the square $\|X\|^2$, and by Theorem 1, $\|X\|^2$ is M-determinate. This is also a Stieltjes moment problems.

Finally it remains to apply a result from [19] stating that if the square $\|X\|^2$ is M-determinate, this is a Stieltjes moment problem, then $X \in \mathbf{R}^n$ is M-determinate, as a Hamburger moment problem.

Theorem 6 is proved.

7. Additional comments. Let us give some details on Hardy's criterion.

(a) The arguments in the proof of Theorem 1 are those used by Hardy [11] and they are given here as summarized in [25, Section 11.9]. Just to mention that in his second paper [12], Hardy presented yet another proof of the same result. It involves more analytics and we do not give details here.

(b) In his paper [12] Hardy noted without proof that if the density f of X is *positive* and satisfies (1), then the moments behave quite well in the sense that the moment m_k of order k is such that $m_k < (2k)!c_0^k$ for some $c_0 > 0$; see also the above Corollary 3 in which we do not assume any smoothness condition on the distribution function of X . As noted in Remark 5, these bounds imply that Carleman's condition is satisfied: $C(\{m_k\}) = \sum_{k=1}^{\infty} (m_k)^{-1/(2k)} = \infty$. Thus the distribution, say F , corresponding to the density f , is M-determinate. This condition could not be used by Hardy in 1917: it was not simply available at the time. Just to mention that Carleman's condition for uniqueness was given in purely complex analysis terms and published only in 1926; see [3]. Interestingly, in [3] there is no reference to the two papers by Hardy. Even more interesting is that G. H. Hardy and T. Carleman published a joint paper in Fourier analysis (see [4]), with no reference to the Hardy papers [11], [12].

(c) De Jeu [7] considered the multidimensional moment problem by using the class of the so-called quasi-analytic functions. Perhaps the idea goes back partly to [3]. De Jeu [7] established some general results and in particular, Hardy's criterion can be derived as a consequence of his Theorem 1.3. In [7] there is no reference to the papers by Hardy [11], [12].

(d) The book [25] contains a brief description of the Hardy results, but everything is analysis. Fundamental sources in the problem of moments are the books by Shohat and Tamarkin [21] and Akhiezer [1]. In both books the papers by Hardy are mentioned only with brief comments.

(e) As far as we know, [18] is the only paper published in a stochastics journal with a reference to the Hardy papers [11], [12]. The authors of [18] use the Hardy result as an intermediate step when proving one of their results.

Acknowledgments. We would like to express our thanks to Professor Nick Bingham for his constructive comments and suggested additional references. We are grateful to the referees and the editors for the useful suggestions taken into account when preparing the final version of the paper.

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Поступила в редакцию
19.VI.2012