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**A CENTRAL LIMIT RESULT IN THE WAVELET DOMAIN
FOR MINIMUM CONTRAST ESTIMATION
OF FRACTAL RANDOM FIELDS¹⁾**

В статье [36] был получен критерий минимального контраста для состоятельных оценок пространственной фрактальной размерности для области вейвлетов для класса полупараметрических фракталов гауссовых случайных полей, содержащих в качестве частных случаев фрактальное броуновское движение и процессы Рисса–Бесселя. В предлагаемой статье асимптотическая нормальность оценки минимального контраста получена с помощью центральной предельной теоремы для многомерных стохастических интегралов (см. [29]).

Ключевые слова и фразы: асимптотическая нормальность, оценка минимального контраста, область вейвлетов, винеровский хаос.

1. Introduction and examples. The most well-known and widely used minimum contrast functional in the spectral domain is the Whittle or Gauss–Whittle discrepancy (see [39] for more examples of spectral contrast processes). For instance, [16] and [11] propose a minimum contrast estimator based on the Whittle approximation to the Gaussian log-likelihood for stationary random fields. This procedure has been extended to other contexts, involving long-range dependence, fractality or intermittency.

Whittle estimators for long-range dependent processes and fields have been studied by [22] and [20], respectively. In [20], the proposed estimator is obtained from a continuous version of the Whittle contrast function, supplied with a weight function and a similar idea has been also considered in [14], for Gaussian processes with stationary increments. In both cases, consistency and asymptotic normality of the estimators were derived. Similar results

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were also obtained by [5] for non-Gaussian processes with possible long-range dependence, based on second and third order cumulant spectra. For both long-range dependence and fractality, [4] studies a class of spectral-based minimum contrast estimators for fractional stationary stochastic processes and fields, such as the fractional Riesz–Bessel motion (fRBM) introduced in [3]. For quadratic forms of the bivariate Appell polynomials, and non-linear transformations of stationary processes displaying long-range dependence, beyond the Gaussian case, Appell expansion provides a useful tool for deriving limit results (see, for instance, [15], [38] and [8]). Asymptotic properties of quasi-likelihood parameter estimators of stochastic processes, including the long-range dependent case, are also established in [18].

The references cited above present asymptotic results in an increasing domain framework, that is, assuming that the observation domain grows with the sample size. Alternatively, properties of the periodogram have been studied under fixed-domain asymptotics by [37]. In the same setting, [21] provides similar results for cross-periodograms of multivariate stationary random fields. In the parametric context, fixed-domain asymptotic properties and distribution of tapered maximum likelihood estimators for the covariance parameters have been studied by [12]. Note that results in [12] are restricted to stationary Gaussian processes observed on a bounded interval in $\mathbf{R}$, with special attention to the Ornstein–Uhlenbeck process. Fractal dimension estimation for stationary non-Gaussian random fields has been studied in [9], considering increment-based estimators.

Wavelet-based semiparametric estimation of the memory parameter, in the time series context, is addressed in [34], for the class of nonstationary $M(d)$ processes with memory parameter $d \in \mathbf{R}$. The k th order difference of these processes is weakly stationary having spectral density factorized into a memory part and a short-range spectral density. Log-regression wavelet estimators and wavelet Whittle estimators are considered in the approximation of the memory parameter. Their asymptotic variance and asymptotic normality are derived. Fourier and wavelet approaches are compared in [13] for the estimation of the memory parameter d .

The wavelet-based minimum contrast estimation of a semiparametric class of Gaussian random fields, displaying local self-similar behavior as fractional Brownian motion, has been addressed in [36]. Weak consistency of the wavelet periodogram and the minimum contrast estimator are established for a family of Haar-type isotropic wavelet functions. An increasing domain asymptotic design is considered in the wavelet domain, with respect to the continuous translation parameter. This approach is equivalent to consider fixed domain asymptotics in the spatial domain.

In this work, the asymptotic normality of the class of wavelet-based minimum contrast estimators studied in [36] is derived. Specifically, central limit results for multiple stochastic integrals derived by [29] are applied to

prove the asymptotic normality of the minimum contrast estimator. Note that the results in [29] were extended to the multidimensional case in [30] and [32], considering stochastic integrals with respect to scattered random measures. In addition, sufficient conditions for the convergence of sequences of multiple Itô-type integrals are established in [31] and [33]. A new characterization for sequences of vectors of multiple integrals is provided in [28], using Malliavin calculus. These developments constitute a powerful tool and have been recently applied to other contexts, such as the study of fractional stochastic integrals (see [10]) or Skorokhod integrals, as in [26].

The central limit result derived in this paper in the wavelet domain is based on the identification of the Wiener chaos of the semiparametric Gaussian random field class considered with the Wiener chaos of an isonormal process related to the wavelet random field. In the derivation of the asymptotic finite variance of the limit distribution, the properties of the wavelet family selected are crucial. The central limit results derived in [29] for multiple stochastic integrals are then applied to obtain the asymptotic normality of the proposed wavelet-based minimum contrast estimator. Note that, although the elements of the class of the semiparametric Gaussian random fields studied display local self-similarity, in the second-order moment sense, they are not necessarily stationary. Hence, the contrast estimation methodologies, based on the Fourier periodogram, usually developed in the spectral domain (see, for example, [20]) can not be applied in this setting. In particular, in the special case where the corresponding Reproducing Kernel Hilbert Space (RKHS) is isomorphic to a fractional Sobolev space, this fractal semiparametric class is included in the one introduced in [35].

This paper is organized as follows. In section 2, basic definitions and notation are introduced in relation to the wavelet Gaussian random field family considered. The wavelet-based minimum contrast estimation procedure is also described in this section. Section 3 is devoted to some technical conditions and preliminary results concerning the weak consistency of the wavelet periodogram and the minimum contrast parameter estimator. The identification of the wavelet random field with an isonormal process, and the corresponding Wiener chaos is obtained in section 4. Section 5 provides the main result on the asymptotic normality of the wavelet-based minimum contrast parameter estimator. Final remarks and future open research lines are discussed in section 6.

2. Notation and background. In this section, general notation and background on random fields and on the wavelet scenario selected for the subsequent developments will be described (see also [36]).

2.1. Random fields. Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$, and associated norm $\| \cdot \|_H$. Denote by X the generalized random field (GRF) indexed by the elements of H , i.e., a centered Gaussian

family defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, given by $\{X(\phi), \phi \in H\}$.

Since our interest is focused on statistical inference on the fractal parameter characterizing the local mean quadratic variation of X , H is considered to be a fractional Sobolev space on a compact domain $D \subset \mathbf{R}^d$, namely $\overline{H}^{-\alpha}(D)$, for certain positive α . Here,

$$\overline{H}^{-\alpha}(D) = \{g \in H^{-\alpha}(\mathbf{R}^d) : \text{supp } g \subseteq D\},$$

where $\text{supp } g$ denotes the support of function g , and $H^{-\alpha}(\mathbf{R}^d)$ is the fractional Sobolev space of order $(-\alpha)$ on $\mathbf{R}^d$. This space is constituted by the tempered distributions g such that

$$(1 + \|\lambda\|^2)^{-\alpha/2} \widehat{g}(\lambda) \in L^2(\mathbf{R}^d), \quad \lambda \in \mathbf{R}^d,$$

with $\widehat{g}$ being the weak-sense Fourier transform of g defined by duality, and $L^2(\mathbf{R}^d)$ denoting the space of square-integrable functions on $\mathbf{R}^d$. The dual Hilbert space H^* of H is then given by $H^\alpha(D)$, the space of distributions f on D such that there exists a function $F \in H^\alpha(\mathbf{R}^d)$, whose restriction to domain D coincides with f (see, for example, [40], for more details). The isomorphic relationship of the RKHS of X , denoted by $\mathcal{H}(X)$, with the fractional Sobolev space $H^\alpha(D) = [\overline{H}^{-\alpha}(D)]^*$ implies that there exist positive constants $0 < C_1 \leq C_2$ such that

$$C_1 \|f\|_{\overline{H}^{-\alpha}(D)} \leq \|f\|_{[\mathcal{H}(X)]^*} = B_X(f, f) \leq C_2 \|f\|_{\overline{H}^{-\alpha}(D)} \quad \forall f \in \overline{H}^{-\alpha}(D). \quad (2.1)$$

In particular, for $\alpha > d/2$, from embedding theorems between fractional Besov spaces (c.f., [40]), the functions in the RKHS of X are Hölder continuous of order $(\alpha - d/2)$. Therefore, the Dirac delta distribution belongs to $\overline{H}^{-\alpha}(D)$, which defines the support of the covariance operator R_X of X , given by,

$$R_X(\phi)(\psi) = B_X(\phi, \psi) \quad \forall \phi, \psi \in \overline{H}^{-\alpha}(D).$$

Thus, we can univocally define a zero-mean ordinary random field, $\{\tilde{X}(x), x \in D \subseteq \mathbf{R}^d\}$, in terms of the following identity:

$$\tilde{X}(x) = X(\delta_x) = \int_D \tilde{X}(y) \delta_x(y) dy \quad \forall x \in D,$$

where the covariance kernel B_X^α of $\tilde{X}$ is determined from

$$B_X^\alpha(x, y) = B_X(\delta_x, \delta_y) = \int_D \int_D B_X^\alpha(z, v) \delta_x(z) \delta_y(v) dz dv \quad \forall x, y \in D. \quad (2.2)$$

Here, $\delta_u(\cdot)$ denotes the Dirac delta distribution at point $u \in D$. The mean-square continuity of random field $\tilde{X}$ follows from the continuity of B_X^α , since, as commented, from equation (2.1), $\alpha > d/2$, the functions in $\mathcal{H}(X)$ are

Hölder continuous of order $(\alpha - d/2)$, and, by definition of $B_{\tilde{X}}^\alpha$ in equation (2.2), $B_{\tilde{X}}^\alpha \in \mathcal{H}(X) \otimes \mathcal{H}(X)$. Therefore, equation (2.2) univocally determines the ordinary random field $\tilde{X}$, due to its continuity. In particular, X admits the integral representation

$$X(\phi) = \int_D \tilde{X}(x) \phi(x) dx \quad \forall \phi \in \overline{H}^{-\alpha}(D), \quad (2.3)$$

as well as its covariance function B_X

$$B_X(\phi, \psi) = \int_D \int_D B_{\tilde{X}}(x, y) \phi(y) \psi(x) dx dy \quad \forall \phi, \psi \in \overline{H}^{-\alpha}(D), \quad (2.4)$$

in terms of $B_{\tilde{X}}$ satisfying $B_{\tilde{X}}(x, y) = \mathbf{E}[\tilde{X}(x)\tilde{X}(y)]$, for $x, y \in D$. In the stationary case, for $\tilde{X}$ defined on $\mathbf{R}^d$, and $\alpha \in (d/2, d)$, the inequality in (2.1) means that

$$C_1(1 + |\lambda|^2)^{-\alpha} \leq f_{\tilde{X}}(\lambda) \leq C_2(1 + |\lambda|^2)^{-\alpha},$$

where $f_{\tilde{X}} = \mathcal{F}(B_{\tilde{X}})$ denotes the Fourier transform of $B_{\tilde{X}}$ (i.e., the spectral density of $\tilde{X}$).

The estimation problem considered here is associated with a semiparametric family of centered Gaussian random fields with covariance function $\{B_{\tilde{X}}^\alpha(\cdot, \cdot) = B_{\tilde{X}}(\alpha, \cdot, \cdot), \alpha \in (d/2, d)\}$, satisfying

$$\frac{B_{\tilde{X}}^\alpha(x, y)}{\|x - y\|^{2\alpha - d}} \longrightarrow C \quad \text{as } \|x - y\| \longrightarrow 0, \quad (2.5)$$

for a certain positive constant C . Specifically, the asymptotic normality of the wavelet-based minimum contrast estimator of α , proposed in [36], is derived. The estimation results formulated in [36] allow the consistent approximation of the fractal parameter $(\alpha - d/2)$ characterizing the local regularity of the sample-paths (see [1]), as well as of the functions in the associated RKHS. In particular, when an isomorphic identification between the RKHS and the fractional Sobolev space $H^\alpha(D)$ of order α can be established, we obtain the minimum contrast estimation of the parameter α characterizing the class of generalized random fields introduced in [35].

2.2. Wavelet scenario. The minimum contrast estimation procedure for α is formulated in the wavelet domain, based on the wavelet periodogram, and some considerations concerning the wavelet basis selected must be introduced. The Continuous Discrete Wavelet Transform (CDWT) is used in our development. This transform is defined by discretizing the scale of the continuous wavelet transform (CWT) on a dyadic grid (see [24] and [25], among others). Hölder regularity can be analyzed in terms of this discrete-scale based transform (see [2], [6], [23], among others). Here, compactly

supported wavelets are considered in the definition of the CDWT. For simplicity and without loss of generality, assume that the compact domain D , where the wavelet functions are defined, is of the form $D = [-M, M]^d$, for a certain positive constant M , being then $(2M)^d$ its Lebesgue measure. In the isotropic case, we consider the CDWT defined in terms of a basic wavelet function Ψ , which generates by multidimensional translations and one-dimensional dyadic dilations or contractions a wavelet basis of $L^2(D)$. Specifically, for each resolution level j , and for every $x \in D$, let us denote by

$$\Psi_{j,b}(x) = \Psi_{j,b}(\|x\|) = 2^{jd/2} \Psi(\|2^j x - b\|), \quad b \in L_j, \quad j \in \mathbf{Z},$$

a wavelet function on $\mathbf{R}^d$, translated at the center b , for the resolution level j , where L_j is the domain of $\mathbf{R}^d$ containing the continuous translation parameter b . Indeed, since the support D of the basic wavelet functions is assumed to be a d -dimensional interval, the domain L_j can be defined as $L_j = [0, 2^j]^d$. Hence, as $j \rightarrow \infty$, the set L_j becomes the d -dimensional space $\mathbf{R}_+^d$.

Note that, in the anisotropic case, orthonormal wavelet bases on $\mathbf{R}^d$ can be generated from the tensorial product of one-dimensional wavelet functions. In general, in the anisotropic case, one basic wavelet function is considered for each one of the main directions.

The asymptotic results derived in this paper hold for an increasing resolution level, $j \rightarrow \infty$, i.e., the increasing domain asymptotic framework is adopted in the wavelet domain. This approach corresponds to a fixed domain asymptotic setting in the spatial domain D .

For each resolution level $j \in \mathbf{Z}$, associated with the dyadic scale 2^{-j} , consider the CDWT of $\tilde{X}$ in equation (2.3) given by

$$\begin{aligned} X(\Psi_{j,b}) &= \int_D \tilde{X}(x) \Psi_{j,b}(x) dx \\ &= \int_D 2^{jd/2} \tilde{X}(x) \Psi(2^j x - b) dx, \quad b \in L_j \subset \mathbf{R}_+^d. \end{aligned} \quad (2.6)$$

Consider also the semiparametric covariance family defined in equation (2.5), and denote by α_0 the true parameter value. The associated two-dimensional CDWT at scale 2^{-j} is defined as

$$\begin{aligned} B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,u}) &= \mathbf{E}[X(\Psi_{j,b})X(\Psi_{j,u})] \\ &= \int_{D \times D} B_{\tilde{X}}^{\alpha_0}(x, y) \Psi_{j,b}(x) \Psi_{j,u}(y) dx dy \\ &= \int_{D \times D} B_{\tilde{X}}^{\alpha_0}(x, y) 2^{jd/2} \Psi(2^j x - b) 2^{jd/2} \Psi(2^j y - u) dx dy, \end{aligned}$$

where $X(\Psi_{j,c})$, for $c = b, b \in L_j$, is given as in equation (2.6), for each $j \in \mathbf{Z}$.

The wavelet periodogram provides a nonparametric estimator of the diagonal of the two-dimensional $(2d-)$ wavelet transform of the covariance

function (see [24]). Specifically, in our setting, the wavelet periodogram at resolution level $j \in \mathbf{Z}$ and location $b \in L_j$ is defined as

$$S(j, b, b) = \left| \int_D \tilde{X}(x) \Psi_{j,b}(x) dx \right|^2 = \left| \int_D 2^{jd/2} \tilde{X}(x) \Psi(2^j x - b) dx \right|^2. \quad (2.7)$$

The expected value of the wavelet periodogram (2.7) is then computed as

$$\mathbf{E}(S(j, b, b)) = \mathbf{E} \left[|X(\Psi_{j,b})|^2 \right] = B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b}), \quad b \in L_j, \quad j \in \mathbf{Z}. \quad (2.8)$$

The compactly supported wavelet basis selected must be such that the above integrals, defining the two-dimensional CDWT of the covariance function, are finite. It is also assumed that a continuous and discrete version of the associated wavelet transform are available with the same values on the nodes of the corresponding regular dyadic grid associated with the discrete version.

3. Preliminary results and technical conditions. The conditions assumed and the consistency results derived in [36], as well as the conditions needed for the asymptotic normality of the minimum contrast estimator (3.11) are now introduced.

3.1. Conditions. The following assumptions are made on the Gaussian random field family and wavelet transform considered in the subsequent developments:

A.1. The covariance function of the centered Gaussian random field $\tilde{X}$ introduced in equation (2.3) satisfies (2.5).

A.2. The wavelet basis selected is an isotropic wavelet basis with the basic wavelet function Ψ satisfying

$$\Psi(\|z\|) = \begin{cases} 1, & 0 \leq \|z\| < 1/2, \\ -1, & 1/2 \leq \|z\| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The results below hold for any isotropic wavelet basis with function Ψ displaying a similar behavior to the one-dimensional Haar mother wavelet, with respect to the modulus function. That is, Ψ as a function of the modulus behaves like a linear combination of weighted indicator functions of the elements of a partition of a closed interval.

Assumptions A.1 and A.2 allow the explicit computation of the asymptotic orders, when $j \rightarrow \infty$, of the CDWT of the semiparametric covariance function family (2.5) (see [36] and Lemma 3.1 below).

For deriving the consistency of the minimum contrast estimator introduced in (3.11), the following additional conditions are considered in Proposition 2 in [36]:

A.3. The weight function $\omega \in L^1(\mathbf{R}_+^d)$ is such that

$$j \int_{L_j} |w(b)| db \longrightarrow C, \quad j \rightarrow \infty, \quad C \geq 0.$$

Note that a sufficient condition for A.3 to hold is, for example,

$$\int_{L_j} |w(b)| db = \mathcal{O}(j^{-\beta}), \quad \beta \geq 1.$$

A.4. There exists a sequence of functions $\{v_j\}$ on L_j such that the sequence

$$\left\{ h^j(\alpha, b) = \frac{v_j(b)}{B_X^\alpha(\Psi_{j,b}, \Psi_{j,b})} \right\}$$

is equicontinuous with respect to the first argument, i.e.,

$$|h^j(\alpha_1, b) - h^j(\alpha_2, b)| \leq \eta(|\alpha_1 - \alpha_2|),$$

being η the modulus of continuity, independently of the argument b . In addition, the integral

$$\int_{L_j} |S(j, b, b)| \left| \frac{\omega(b)}{v_I(b)} \right| db = C_j \xrightarrow{P_{\alpha_0}} C_\infty < \infty, \quad j \rightarrow \infty,$$

where $\xrightarrow{P_{\alpha_0}}$ denotes convergence in probability, and v_I is the inferior envelope of $\{v_j\}$, that is, $v_I = \inf_{j \in \mathbf{Z}} v_j$.

It should be noted that no identifiability conditions are here assumed, due to the design of the contrast function $\mathcal{V}$ (see equation (3.9) below), which is positive and has a unique minimum under assumptions A.1 and A.2, in view of Lemma 3.1.

Finally, for the asymptotic normality of the minimum contrast estimator, the following additional conditions are required.

B.1. Function $B_X^\alpha(\Psi_{j,b}, \Psi_{j,b})$, $\alpha \in \Lambda$, is twice differentiable with respect to α , for any $b \in L_j$, and $j \geq j_0$, for a certain $j_0 \in \mathbf{Z}$.

B.2. The weight function $\omega \in L^1(\mathbf{R}_+^d) \cap C(\mathbf{R}_+^d)$ is such that, for $l, m = 1, \dots, 4$,

$$I_{l,m}(j) = \int_{\mathcal{S}_{l,m}(j)} \omega(b_1) \omega(b_1 - y) \omega(b_1 - h) \omega(b_1 - (h - x)) db_1 dh dx dy \longrightarrow 0$$

as $j \rightarrow \infty$, where $\mathcal{S}_{l,m}(j) = D_{l,m}(j) \times D_h(x, y, j) \times D_{b_1}(h - x, y, h, j)$, with

$$\begin{aligned} D_h(x, y, j) &= \left\{ h \in \prod_{i=1}^d [x_i - 2^j, x_i + 2^j] \cap \prod_{i=1}^d [y_i - 2^j, y_i + 2^j] \right\} \\ &= \prod_{i=1}^d [(x_i \vee y_i) - 2^j, (x_i \wedge y_i) + 2^j], \end{aligned}$$

$$\begin{aligned}
D_{b_1}(h-x, y, h, j) &= \left\{ b_1 \in \prod_{i=1}^d [h_i - x_i, h_i - x_i + 2^j] \cap \prod_{i=1}^d [y_i, y_i + 2^j] \right. \\
&\quad \left. \cap \prod_{i=1}^d [h_i, h_i + 2^j] \right\} \\
&= \prod_{i=1}^d [h_i - x_i \vee y_i \vee h_i, (h_i - x_i \wedge y_i \wedge h_i) + 2^j], \quad (3.1)
\end{aligned}$$

$$\begin{aligned}
D_{1,1}(j) &= \left[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}} \right]^d (B_R(0))^{-1} \\
&\quad \times \left[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}} \right]^d (B_R(0))^{-1}, \\
D_{1,2}(j) &= \left[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}} \right]^d (B_R(0))^{-1} \times B_R(0), \\
D_{1,3}(j) &= B_R(0) \times \left[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}} \right]^d (B_R(0))^{-1}, \\
D_{1,4}(j) &= D_{1,4} = B_R(0) \times B_R(0), \\
D_{2,1}(j) &= D_y(j, R, d)(B_R(0))^{-1} \times \left[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}} \right]^d (B_R(0))^{-1}, \\
D_{2,2}(j) &= D_y(j, R, d)(B_R(0))^{-1} \times B_R(0), \\
D_{2,3}(j) &= B_R(0) \times \left[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}} \right]^d (B_R(0))^{-1}, \\
D_{2,4}(j) &= D_{2,4} = B_R(0) \times B_R(0), \\
D_{3,1}(j) &= \left[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}} \right]^d (B_R(0))^{-1} \times D_x(j, R, d)(B_R(0))^{-1}, \\
D_{3,2}(j) &= \left[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}} \right]^d (B_R(0))^{-1} \times B_R(0), \\
D_{3,3}(j) &= B_R(0) \times D_x(j, R, d)(B_R(0))^{-1}, \\
D_{3,4}(j) &= D_{3,4} = B_R(0) \times B_R(0), \\
D_{4,1}(j) &= D_y(j, R, d)(B_R(0))^{-1} \times D_x(j, R, d)(B_R(0))^{-1}, \\
D_{4,2}(j) &= D_y(j, R, d)(B_R(0))^{-1} \times B_R(0), \\
D_{4,3}(j) &= B_R(0) \times D_x(j, R, d)(B_R(0))^{-1}, \\
D_{4,4}(j) &= D_{4,4} = B_R(0) \times B_R(0), \\
D_z(j, R, d) &= \left\{ z \in \left[-2^{j+1} - 2^j, 2^{j+1} - \frac{R}{\sqrt{d}} \right]^d \cap \left[-2^{j+1} + \frac{R}{\sqrt{d}}, 2^{j+1} + 2^j \right]^d \right\} \\
&= \left[(-2^{j+1} - 2^j) \vee \left(-2^{j+1} + \frac{R}{\sqrt{d}} \right), \left(2^{j+1} - \frac{R}{\sqrt{d}} \right) \wedge (2^{j+1} + 2^j) \right]^d. \quad (3.2)
\end{aligned}$$

R e m a r k 3.1. Note that, under assumption A3, condition B2 holds, for example, if the weight function $\omega \in L^1(\mathbf{R}_+^d) \cap C(\mathbf{R}_+^d)$ is symmetric with respect to zero.

3.2. Some preliminary results. We include in this section a lemma providing the asymptotic properties, when the resolution level increases, of the two-dimensional CDWT of the covariance (2.5), and two propositions on the consistency of the wavelet periodogram and the minimum contrast estimator. The proofs of these results can be found in [36].

Denote by $W_{\alpha_0-d/2}$ the fractional Brownian motion with Hurst parameter $(\alpha_0-d/2)$. Under condition A.1, $W_{\alpha_0-d/2}$ displays the same local behavior as the ordinary random field $\tilde{X}$ in (2.3).

Lemma 3.1 (see [36]). (i) *Under condition A.1, as $j \rightarrow \infty$, the following asymptotic approximation holds:*

$$B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,u}) \simeq [2^{-j}]^{2\alpha_0} C_{\Psi}(\alpha_0, b, u, j), \quad b \in L_j,$$

where

$$\begin{aligned} \lim_{j \rightarrow \infty} C_{\Psi}(\alpha_0, b, u, j) &= C_{\Psi}(\alpha_0, b - u, \infty) = \int_{\mathbf{R}^d} \|h\|^{2\alpha_0-d} \gamma_{\Psi}(h - (b - u)) dh \\ &= \langle \Psi_{0,b}, \Psi_{0,u} \rangle_{[\mathcal{H}_{W_{\alpha_0-d/2}}]^*} \end{aligned} \quad (3.3)$$

with

$$\gamma_{\Psi}(h - (b - u)) = \int_{\mathbf{R}^d} \Psi(z) \Psi(z - h + (b - u)) dz,$$

and $\mathcal{H}_{W_{\alpha_0-d/2}}$ denoting the RKHS of the fractional Brownian motion $W_{\alpha_0-d/2}$.

(ii) Moreover, under condition A.2, for $\|b - u\| \gg 1$,

$$\begin{aligned} C_{\Psi}(\alpha_0, b, u, \infty) &= \int_{\mathbf{R}^d} \|z\|^{2\alpha_0-d} \left[\int_{\mathbf{R}^d} \Psi(x) \Psi(x - z + (b - u)) dx \right] dz \\ &\sim \mathcal{O}(\|b - u\|^{2\alpha_0-3d}). \end{aligned} \quad (3.4)$$

Otherwise,

$$C_{\Psi}(\alpha_0, b, u, \infty) = \frac{1}{2\alpha_0(2\alpha_0 + d)} \xi_v(b - u) \quad (3.5)$$

with

$$\begin{aligned} \xi_v(b - u) &= \|b - u - v\|^{2\alpha_0+d} - 4(\|b - u - v/2\|)^{2\alpha_0+d} \\ &\quad + 6\|b - u\|^{2\alpha_0+d} - 4\|b - u + v/2\|^{2\alpha_0+d} \\ &\quad + \|b - u + v\|^{2\alpha_0+d}, \end{aligned} \quad (3.6)$$

for $v \in \mathbf{R}_+^d$ such that $\|v\| = 1$. In particular,

$$\lim_{j \rightarrow \infty} C_{\Psi}(\alpha_0, b, j) = C_{\Psi}(\alpha_0, \infty) = \tilde{L}_d(\alpha_0) = \frac{1 - 2^{-(2\alpha_0+d-2)}}{\alpha_0(2\alpha_0 + d)}. \quad (3.7)$$

Lemma 3.1 will also be applied in the proof of the main result of this paper, Theorem 5.1.

Lemma 3.2 (see [36]). *Assume that conditions A.1 and A.2 hold. Let ω be a weight function such that the integral in (3.8) below is well defined. Then, as $j \rightarrow \infty$, for $\omega \in L^1(\mathbf{R}_+^d)$, the following limit holds in probability:*

$$\int_{L_j} \omega(b) [S(j, b, b) - B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})] db \longrightarrow 0. \quad (3.8)$$

The proof is derived in [36] from the Hermite expansion of the wavelet periodogram applying Lemma 3.1.

3.3. Minimum contrast estimation. For the class of random fields considered, and in the wavelet scenario previously established, the authors in [36] propose a wavelet-based minimum contrast estimator of α_0 . Following the results in [17], they define the contrast function:

$$\mathcal{V}(\alpha, \alpha_0) = - \int_{\mathbf{R}^d} \left[\ln \left(\frac{\tilde{L}_d(\alpha_0)}{\tilde{L}_d(\alpha)} \right) - \frac{\tilde{L}_d(\alpha_0)}{\tilde{L}_d(\alpha)} + 1 \right] \omega(b) db, \quad (3.9)$$

where ω is a weight function assumed to be in $L^1(\mathbf{R}_+^d)$. It can be seen that from definition of function $\tilde{L}_d$ on $(d/2, d)$ as in equation (3.7), the contrast function $\mathcal{V}$ in (3.9) is positive and has a unique minimum in $\alpha = \alpha_0$.

The sequence of random variables $\{U_j(\alpha), j \in \mathbf{Z}\}$ given, for all $\alpha \in (d/2, d)$, by

$$U_j(\alpha) = \int_{L_j} \left[\ln \left(B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right) + \frac{S(j, b, b)}{B_X^\alpha(\Psi_{j,b}, \Psi_{j,b})} \right] \omega(b) db, \quad j \in \mathbf{Z}_+, \quad (3.10)$$

defines a contrast process for the contrast function $\mathcal{V}$, since

$$U_j(\alpha) - U_j(\alpha_0) = - \int_{L_j} \left(\ln \frac{B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})}{B_X^\alpha(\Psi_{j,b}, \Psi_{j,b})} - \frac{S(j, b, b)}{B_X^\alpha(\Psi_{j,b}, \Psi_{j,b})} + \frac{S(j, b, b)}{B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})} \right) \omega(b) db$$

converges in probability to the contrast function, as $j \rightarrow \infty$ (see [36]). The minimum contrast estimator $\tilde{\alpha}_j$ is then formulated, for each resolution level j , as

$$\tilde{\alpha}_j = \arg \min_{\alpha \in (d/2, d)} U_j(\alpha). \quad (3.11)$$

The weak-consistency of the Fourier periodogram has been obtained in [14] for nonstationary Gaussian processes with stationary increments. Lemma 1 in [4] also provides parallel results in the Fourier spectral domain from continuous observations (for both random processes and fields). In addition, Proposition 1 in [20] gives an analogous result, in terms of the Fourier

periodogram, for strictly stationary random fields. Lemma 3.2 establishes a consistency result for the wavelet periodogram. In all the cited references, the weak-consistency of the periodogram leads to the weak-consistency of the minimum contrast estimator proposed, under suitable conditions on the weight function family. For instance, in the spectral domain, [14] obtain the weak consistency of a minimum contrast estimator for a process with stationary increments. Analogous results were derived by [4] and [20] for Gaussian processes and fields. Finally, we refer to Lemma 3.3 below, obtained in [36], on the weak consistency of the wavelet-based minimum contrast estimator (3.11).

Lemma 3.3 (see [36]). *Under conditions A.1–A.4, as $j \rightarrow \infty$,*

$$\tilde{\alpha}_j \longrightarrow \alpha_0, \quad \text{in probability.}$$

4. Generalized Gaussian random fields and isonormal processes. The methodological lines established in [19] are followed in the derivation of the main result of this section, Lemma 4.3. This lemma provides the identification of the wavelet Gaussian random field family previously introduced with an isonormal process family. The associated Wiener chaos are also connected. For a detailed review on Wiener chaos and stochastic integrals, see [27].

Let us first consider the following definitions.

Definition 4.1. Let H be a separable Hilbert space. The set of random variables $\{\eta(h): h \in H\}$ is said to be an isonormal process on H if η is a centered H -indexed Gaussian family defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and it satisfies

$$\mathbf{E}[\eta(h)\eta(g)] = \langle h, g \rangle_H,$$

for every $h, g \in H$.

For $n \geq 1$, let $H^{\otimes n}$ be the n th tensor product of H , and let $H^{\odot n}$ be the n th symmetric tensor product of H . Denote by H_n the n th Hermite polynomial.

Definition 4.2. For an isonormal process η , the n th Wiener chaos $\mathcal{H}_n$ is defined as the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbf{P})$ generated by the random variables $\{H_n(\eta(h)), h \in H, \|h\|_H = 1\}$. The space $L^2(\Omega, \mathcal{G}, \mathbf{P})$ can then be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{H}_n$:

$$L^2(\Omega, \mathcal{G}, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where $\mathcal{G}$ denotes the σ -field generated by the random variables $\{\eta(h), h \in H\}$.

The multiple Wiener–Itô stochastic integral of order n with respect to an isonormal process η , denoted by I_n^η , provides an isometry between the n th Wiener chaos associated to η and $H^{\odot n}$. For every $n \geq 1$, the norm considered in $H^{\odot n}$ is given by $\sqrt{n!} \cdot \|\cdot\|_{H^{\otimes n}}$. The following result in [29] will be used for obtaining the asymptotic distribution of the minimum contrast estimator.

Lemma 4.1 (see [29, Theorem 1]). *Assume that η is an isonormal process on a separable Hilbert space H . Denote by $I_n^\eta(f)$ the multiple Wiener–Itô stochastic integral (of order n) of f with respect to η , and fix $n \geq 2$. For any sequence of elements $\{f_m, m \geq 1\}$ such that $f_m \in H^{\odot n}$, for every $m \geq 1$, and*

$$\lim_{m \rightarrow \infty} n! \|f_m\|_{H^{\otimes n}}^2 = \lim_{m \rightarrow \infty} \mathbf{E}[I_n^\eta(f_m)^2] = \sigma_n^2$$

the following conditions are equivalent:

- (i) $\lim_{m \rightarrow \infty} \mathbf{E}[I_n^\eta(f_m)^4] = 3[\sigma_n^2]^2$;
- (ii) $\lim_{m \rightarrow \infty} \|f_m \otimes_p f_m\|_{H^{\otimes 2(n-p)}}^2 = 0$, for $p = 1, \dots, n-1$, with $H^{\otimes 2(n-p)}$ denoting, as before, the $2(n-p)$ th tensor product of H , and $f_m \otimes_p f_m$ being the p th self-contraction of f_m ;
- (iii) the sequence $\{I_n^\eta(f_m), m \geq 1\}$ converges, as $m \rightarrow \infty$, to a centered Gaussian random variable with variance σ_n^2 .

In order to derive the main result in this paper (Theorem 5.1), some statements on the identification of Gaussian generalized random fields with isonormal processes are needed. These results, as well as some other statements for their respective Wiener chaos are now introduced (see [41, Lemmas 1.1 and 1.2]) and [19].

Theorem 4.1. *Let $\xi: \Omega \rightarrow \mathcal{B}$ be a centered Gaussian random element, which maps a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ into a real separable Banach space $\mathcal{B}$, and $\text{supp } \xi = \mathcal{B}$. Then, there exists a separable Hilbert space H and an isonormal process $\{\eta(h): h \in H\}$ such that*

$$\langle \xi, x^* \rangle \in \{\eta(h): h \in H\},$$

for every $x^ \in \mathcal{B}^*$, where $\mathcal{B}^*$ denotes the dual space of $\mathcal{B}$.*

P r o o f. Let $B: \mathcal{B}^* \rightarrow \mathcal{B}$ be a symmetric positive operator. Then, there exists a Hilbert space H , and a linear continuous operator $A: \mathcal{B}^* \rightarrow H$ such that $B = A^* \cdot A$, and $\overline{A(\mathcal{B}^*)} = H$, $A^*(H) \subset \mathcal{B}$.

It is known that H is a separable Hilbert space if $\mathcal{B}$ is a separable one (see [41]). Let $B: \mathcal{B}^* \rightarrow \mathcal{B}$ be the covariance operator of the random element ξ , then

$$\mathbf{E} \langle \xi, x^* \rangle \langle \xi, y^* \rangle = \langle Bx^*, y^* \rangle = \langle Ax^*, Ay^* \rangle. \quad (4.1)$$

For all $h \in H$ of the form $h = Ax^*$, we define a random variable

$$\eta(h) := \langle \xi, x^* \rangle. \quad (4.2)$$

Note that $\eta(h)$, for $h \in H$, is a centered Gaussian random variable with

$$\mathbf{E} \eta^2(h) = \langle Ax^*, Ax^* \rangle = \|Ax^*\|_H^2 = \|h\|_H^2,$$

that is

$$\eta(h) \sim N(0, \|h\|_H^2), \quad h \in A(\mathcal{B}^*).$$

The map $h \rightarrow \eta(h)$, $h \in A(\mathcal{B}^*) \subset H$, $\eta(h) \in L^2(\Omega, \mathcal{F}, \mathbf{P})$, can be extended by continuity to the isometry between $H = \overline{A(\mathcal{B}^*)}$ and the closure of the set of random variables $\{\eta(h), h \in A(\mathcal{B}^*)\}$ in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Thus, $\{\eta(h): h \in H\}$ is an isonormal process. Theorem 4.1 is proved.

R e m a r k 4.1. If $\text{supp } \xi \neq \mathcal{B}$, then $\widetilde{\mathcal{B}} = \text{supp } \xi \subset \mathcal{B}$ is a separable Banach space, and Theorem 4.1 can be reformulated for this separable Banach space $\widetilde{\mathcal{B}}$.

Let now $\mathcal{B} = H_0$ be a separable Hilbert space, and let $B: H_0 \rightarrow H_0$ be a symmetric positive operator. By Theorem 4.1, there exists a Hilbert space H and a linear and continuous operator $A: H_0 \rightarrow H$ such that $A^*A = B$. One can choose $H = H_0$, and $A: H_0 \rightarrow H_0$ identical to $B^{1/2}$, the positive square root of B . Note that $B^{1/2}(H_0)$ is dense in H_0 (see [41]).

In a similar way to (4.1), we obtain that if B is the covariance operator of the random element ξ on the separable Hilbert space H_0 , then, for $x, y \in H_0$

$$\mathbf{E} \langle \xi, x \rangle_{H_0} \langle \xi, y \rangle_{H_0} = \langle Bx, y \rangle_{H_0} = \langle B^{1/2}x, B^{1/2}y \rangle_{H_0} = \langle h, g \rangle_{H_0}. \quad (4.3)$$

Then,

$$\begin{aligned} \mathbf{E} [\eta(h_1)\eta(h_2)] &= \mathbf{E} \langle \xi, x_1 \rangle_{H_0} \langle \xi, x_2 \rangle_{H_0} \\ &= \langle Bx_1, x_2 \rangle_{H_0} = \langle B^{1/2}x_1, B^{1/2}x_2 \rangle_{H_0} = \langle h_1, h_2 \rangle_{H_0}. \end{aligned} \quad (4.4)$$

As above, the map $h \rightarrow \eta(h)$, $h \in B^{1/2}(H_0) \subset H_0$, $\eta(h) \in L^2(\Omega, \mathcal{F}, \mathbf{P})$, can be extended by continuity to the isometry between $H_0 = \overline{B^{1/2}(H_0)}$ and the closure of the set of random variables $\{\eta(h), h \in B^{1/2}(H_0)\}$ in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Thus, $\{\eta(h): h \in H_0\}$ is an isonormal process.

As in Remark 4.1, if $\text{supp } \xi \neq H_0$, then $\tilde{H}_0 = \text{supp } \xi \subset H_0$ is a separable Hilbert space, and Theorem 4.1 can be reformulated for $\tilde{H}_0$.

Let $\mathcal{B}$ be a real separable Banach space, and let ξ be a centered Gaussian element in $\mathcal{B}$. We denote $\mathcal{H}_0(\xi) = \mathbf{R}^1$, $\mathcal{H}_1(\xi)$ a closed linear subspace of a set of r.v.'s $\{\langle \xi, x^* \rangle, x^* \in \mathcal{B}^*\}$ in the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{P}_n(\xi)$, $n \geq 2$, be a set of polynomials of degree at least n of r.v.'s $\langle \xi, x_1^* \rangle, \dots, \langle \xi, x_n^* \rangle$, $x_1^*, \dots, x_n^* \in \mathcal{B}^*$.

Denote by $\mathcal{H}_n(\xi)$ the closure of the linear subspace $\{\mathcal{P}_n(\xi)\} \oplus \{\mathcal{H}_0(\xi)\} \oplus \dots \oplus \mathcal{H}_{n-1}(\xi)\}$, $n \geq 2$. The set of random variables $\mathcal{H}_n(\xi)$, $n \geq 1$, is said to be the n th Wiener chaos associated with ξ .

Proposition 4.1. Suppose that there exists an isonormal process $\eta = \{\eta(h), h \in H\}$ such that $\langle \xi, x^* \rangle \in \{\eta(h), h \in H\}$, for every $x^* \in \mathcal{B}^*$. Then $\mathcal{H}_n(\xi) \subset \mathcal{H}_n(\eta)$, for every $n \geq 1$.

Remark 4.2. Let $\{\eta(h), h \in H\}$ be the isonormal process considered in Theorem 4.1. Then, from (4.2), we obtain that

$$\mathcal{H}_n(\xi) = \mathcal{H}_n(\eta), \quad n \geq 1.$$

The following proposition (see [19]) is a reformulation of Theorem 1 of [30] (see also [29]).

Proposition 4.2. Let $\{\eta(h), h \in H\}$ be an isonormal process. Consider the natural numbers $1 \leq n_1 < n_2 < \dots < n_K < \infty$, $K \geq 2$, and the set of r.v.'s $\pi_{j,n_k}(\eta) \in \mathcal{H}_{n_k}(\eta)$, $k = 1, \dots, K$, $j \geq 0$, such that

(a)

$$\lim_{j \rightarrow \infty} \mathbf{E} \pi_{j,n_k}^2(\eta) = 1, \quad k = 1, \dots, K, \quad (4.5)$$

(b)

$$\lim_{j \rightarrow \infty} \mathbf{E} \pi_{j,n_k}(\eta) \pi_{j,n_p}(\eta) = 0, \quad k \neq p, \quad k, p \in \{1, \dots, K\}, \quad (4.6)$$

(c)

$$\lim_{j \rightarrow \infty} \mathbf{E} \pi_{j,n_k}^4(\eta) = 3, \quad k = 1, \dots, K. \quad (4.7)$$

Then, the vector $\pi_{j,K}(\eta) = (\pi_{j,n_1}(\eta), \dots, \pi_{j,n_K}(\eta))$ converges in distribution, as $j \rightarrow \infty$, to a multidimensional standard Gaussian vector $\pi_K \sim N(0, \mathbf{I}_K)$.

The proof follows from Lemma 4.1.

Propositions 4.1 and 4.2 are now applied to obtain the following result.

Theorem 4.2. Let ξ and η be as in Proposition 4.1. If $\pi_{j,n_k}(\xi) \in \mathcal{H}_{n_k}(\xi)$ such that

(a)

$$\lim_{j \rightarrow \infty} \mathbf{E} \pi_{j,n_k}^2(\xi) = 1, \quad k = 1, \dots, K, \quad (4.8)$$

(b)

$$\lim_{j \rightarrow \infty} \mathbf{E} \pi_{j,n_k}(\xi) \pi_{j,n_p}(\xi) = 0, \quad k \neq p, \quad k, p \in \{1, \dots, K\}, \quad (4.9)$$

(c)

$$\lim_{j \rightarrow \infty} \mathbf{E} \pi_{j,n_k}^4(\xi) = 3, \quad k = 1, \dots, K. \quad (4.10)$$

Then, the vector

$$\pi_{j,K}(\xi) = (\pi_{j,n_1}(\xi), \dots, \pi_{j,n_K}(\xi))$$

converges in distribution, as $j \rightarrow \infty$, to a K -dimensional Gaussian distribution $\pi_K \sim N(0, \mathbf{I}_K)$.

Obviously, for every $x \in \mathbf{R}_+^d$, $H_n(\xi(x)) \in \mathcal{H}_n(\xi)$, with $n \geq 1$, under the assumption of mean-square continuity of the centered Gaussian random field ξ .

Lemma 4.2. *Let $\varphi \in C(\mathbf{R}_+^d)$, then, for $j > 0$,*

$$S_n(j) = \int_{[0,2^j]^d} \varphi(\mathbf{x}) H_n(\xi(\mathbf{x})) d\mathbf{x} \in \mathcal{H}_n(\xi), \quad n \geq 1.$$

P r o o f. For a fixed $j > 0$, and $n \geq 1$, and $0 \leq x_0^k < x_1^k < \dots < x_N^k = 2^j$, $\Delta x_i^k = x_{i+1}^k - x_i^k = 2^j / (N)^{1/d}$, for $k = 1, \dots, d$, and $i = 0, \dots, N-1$. It is easy to see that, for every $N \geq 1$,

$$\Sigma_n(N, j) = \sum_{i=0}^{N-1} \varphi(\mathbf{x}_i) H_n(\xi(\mathbf{x}_i)) \Delta \mathbf{x}_i \in \mathcal{H}_n(\xi).$$

We have

$$\begin{aligned} & \mathbf{E} (S_n(j) - \Sigma_n(N, j))^2 \\ & \leq 2N \sum_{i=0}^{N-1} \mathbf{E} \left(\int_{\prod_{k=1}^d [x_i^k, x_{i+1}^k]} \varphi(\mathbf{x}) [H_n(\xi(\mathbf{x})) - H_n(\xi(\mathbf{x}_i))] d\mathbf{x} \right)^2 \\ & \quad + 2N \sum_{i=0}^{N-1} \mathbf{E} \left(\int_{\prod_{k=1}^d [x_i^k, x_{i+1}^k]} (\varphi(\mathbf{x}) - \varphi(\mathbf{x}_i)) H_n(\xi(\mathbf{x}_i)) d\mathbf{x} \right)^2 \\ & = \Sigma_1(j) + \Sigma_2(j). \end{aligned}$$

By the continuity of the covariance function of ξ , for every $\varepsilon > 0$, there exists N_0 such that, for $N > N_0$,

$$\max_{\mathbf{x} \in \prod_{k=1}^d [x_i^k, x_{i+1}^k]} \mathbf{E} [H_n(\xi(\mathbf{x})) - H_n(\xi(\mathbf{x}_i))]^2 < \varepsilon, \quad i = 0, \dots, N-1.$$

If

$$\max_{\mathbf{x} \in \prod_{k=1}^d [x_i^k, x_{i+1}^k]} |\varphi(\mathbf{x}) - \varphi(\mathbf{x}_i)| < \varepsilon, \quad i = 0, \dots, N-1,$$

then

$$\Sigma_1(j) \leq 2 \max_{\mathbf{x} \in [0,2^j]^d} \varphi^2(\mathbf{x}) 2^{2dj} \varepsilon, \quad \Sigma_2(j) \leq 2n! 2^{2jd} \varepsilon^2.$$

Thus,

$$\lim_{N \rightarrow \infty} \mathbf{E} (S_n(j) - \Sigma_n(N, j))^2 = 0,$$

and hence, $S_n(j) \in \mathcal{H}_n(\xi)$, $n \geq 1$, for every $j > 0$. Lemma 4.2 is proved.

From Theorem 4.1 and Lemma 4.2, we obtain the following corollary.

Corollary 4.1. *If $\varphi_{j,k} \in C(\mathbf{R}_+^d)$, $k = 1, \dots, K$, $j > 0$, and conditions (a) and (c) of Theorem 4.2 are satisfied for r.v.'s*

$$\pi_{j,n_k}(\xi) = \int_{[0,2^j]^d} \varphi_{j,k}(\mathbf{x}) H_{n_k}(\xi(\mathbf{x})) d\mathbf{x}, \quad (4.11)$$

then the vector

$$\pi_{j,K}(\xi) = \left(\int_{[0,2^j]^d} \varphi_{j,1}(\mathbf{x}) H_{n_1}(\xi(\mathbf{x})) d\mathbf{x}, \dots, \int_{[0,2^j]^d} \varphi_{j,K}(\mathbf{x}) H_{n_K}(\xi(\mathbf{x})) d\mathbf{x} \right) \quad (4.12)$$

converges in distributions, as $j \rightarrow \infty$, to a standard Gaussian vector $\pi_K \sim N(0, \mathbf{I}_K)$.

Indeed, for random variables (4.11), condition (b) of Theorem 4.2 is satisfied by the orthogonality property of Hermite polynomials.

Lemma 4.3. (i) *Under condition A.1, the wavelet random field can be identified with an isonormal process $\eta_{X_W^N}$;*

(ii) *for the normalized wavelet random field $\{X_W^N(j, b), b \in L_j, j \in \mathbf{Z}\}$, given by*

$$X_W^N(j, u) = \frac{X(\Psi_{j,u})}{[B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{1/2}}, \quad u \in L_j, \quad (4.13)$$

the integral functionals $\int_{L_j} \varphi(b) H_k(X_W^N(b)) db$ belong to the Wiener chaos of $\eta_{X_W^N}$, for $\varphi \in C(\mathbf{R}_+^d)$.

P r o o f. (i) The proof follows by applying Theorem 4.1 with $\xi = \tilde{X}$ and $H = L^2(D) = \overline{\text{span}}^{L^2(D)} \{\Psi_{j,b}, b \in L_j \cap \mathbf{Z}^d; j \in \mathbf{Z}\}$. The isonormal process η is defined as

$$\eta(h) = X(f) = \int_D f(x) \tilde{X}(x) dx,$$

where $h = \mathcal{T}_X(f)$, with

$$R_X(f)(g) = B_X(f, g) = \langle \mathcal{T}_X(f), \mathcal{T}_X(g) \rangle_{L^2(D)} \quad \forall f, g \in L^2(D).$$

Here, B_X is defined as in equation (2.4).

From equation (2.6), the wavelet random field is given by

$$\begin{aligned} X(\Psi_{j,b}) &= \int_D \tilde{X}(x) \Psi_{j,b}(x) dx = \langle \tilde{X}, \Psi_{j,b} \rangle_{L^2(D)} \\ &= \eta(\mathcal{T}_X(\Psi_{j,b})), \quad b \in L_j \subset \mathbf{R}_+^d, \end{aligned}$$

which proves the first assertion.

(ii) Finally, since the n th Wiener chaos $\mathcal{H}_n$ is defined as the closed linear subspace, in $L^2(\Omega, \mathcal{F}, \mathbf{P})$, generated by the random variables

$\{H_n(\eta(h)), h \in H, \|h\|_H = 1\}$, in our case, from equation (4.13),

$$\begin{aligned} X_W^N(j, b) &= \left\langle \tilde{X}, \frac{\Psi_{j,b}}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} \right\rangle_{L^2(D)} \\ &= \eta \left(\mathcal{T}_X^{\alpha_0} \left(\frac{\Psi_{j,b}}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} \right) \right) \end{aligned} \quad (4.14)$$

with

$$\left\| \mathcal{T}_X^{\alpha_0} \left(\frac{\Psi_{j,b}}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} \right) \right\|_{L^2(D)} = 1,$$

due to the fact

$$\|\mathcal{T}_X^{\alpha_0}(\Psi_{j,b})\|_{L^2(D)} = [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}.$$

Then, from Lemma 4.2 the second assertion follows. Lemma 4.3 is proved.

R e m a r k 4.3. Note that, as we have assumed that the discrete version provided by $\{\Psi_{j,b}, b \in L_j \cap \mathbf{Z}^d, j \in \mathbf{Z}\}$ of our CDWT coincides with the continuous one over the nodes of the dyadic regular grid defined by discrete translations at each scale, the following identities then hold:

$$L^2(D) = \overline{sp}^{L^2(D)}\{\Psi_{j,b}, b \in L_j \cap \mathbf{Z}^d, j \in \mathbf{Z}\} \subseteq \overline{sp}^{L^2(D)}\{\Psi_{j,b}, b \in L_j, j \in \mathbf{Z}\}.$$

Furthermore,

$$\langle \Psi_{j,b}, \Psi_{l,u} \rangle_{L^2(D)} = \delta_{j,l} \delta(b-u), \quad b, u \in L_j \cap \mathbf{Z}^d, \quad j, l \in \mathbf{Z},$$

where $\int_D \delta(b-u)h(u)du = h(b)$. Therefore, for every function $f \in L^2(D)$,

$$f(x) = \sum_j \int_{L_j} f_{j,b} \Psi_{j,b}(x) db,$$

and

$$X(f) = \sum_j \int_{L_j} f_{j,b} X(\Psi_{j,b}) db = \sum_j \int_{L_j} f_{j,b} \eta(g_j(b)) db,$$

where $g_j(b) = \mathcal{T}_X(\Psi_{j,b})$.

In the case where the RKHS of X is isomorphic to a fractional Sobolev space $H^{\alpha_0}(D)$, we have that operators $\mathcal{T}_X: \overline{H}^{-\alpha_0}(D) \rightarrow L^2(D)$ and $\mathcal{T}_X^*: L^2(D) \rightarrow H^{\alpha_0}(D)$ define isomorphisms. The isonormal process η is then defined, as before, on $\mathcal{T}_X(\overline{H}^{-\alpha_0}(D)) = L^2(D)$. The covariance function B_X provides the following inner product:

$$\langle f, g \rangle_{B_X} = B_X(f, g) = \langle \mathcal{T}_X(f), \mathcal{T}_X(g) \rangle_{L^2(D)} \quad \forall f, g \in \overline{H}^{-\alpha_0}(D),$$

on the space $\overline{H}^{-\alpha_0}(D)$, which generates an equivalent norm to the original one associated with the fractional Sobolev space $\overline{H}^{-\alpha_0}(D)$. Thus,

$$[\mathcal{H}(X)]^* \equiv (\overline{H}^{-\alpha_0}(D), \langle \cdot, \cdot \rangle_{B_X}) \simeq (\overline{H}^{-\alpha_0}(D), \langle \cdot, \cdot \rangle_{\overline{H}^{-\alpha_0}(D)}),$$

where $\simeq$ and $\equiv$, respectively, represent the isomorphic and the isometrical equivalences. Here, as before, $[\mathcal{H}(X)]^*$ denotes the dual of the RKHS space $\mathcal{H}(X)$ of X . Therefore, the generalized random field X can also be identified with an isonormal process on the Hilbert space $H = [\mathcal{H}(X)]^* \equiv (\overline{H}^{-\alpha_0}(D), \langle \cdot, \cdot \rangle_{B_X})$.

5. Asymptotic normality. This section is devoted to the derivation of our main result, Theorem 5.1, on the asymptotic Gaussian distribution of the minimum contrast estimator given in (3.11). The key previous results are Lemmas 4.1 and 4.3. Specifically, Lemma 4.3 allows the application of Corollary 4.1, obtained from Lemma 4.1, in the derivation of the proof of Theorem 5.1.

Different methodologies have been used for obtaining the asymptotic normality of minimum contrast estimators defined in the spectral domain. For instance, we could mention the limit result derived in [14] for nonstationary Gaussian processes with stationary increments, with convergence rate $\sqrt{N}$, being N the number of observations. Similarly, in [20], for the case of continuous observations on an interval $[0, T]$, the asymptotic normality of the minimum contrast parameter estimator defined for random processes and fields is obtained with convergence rate $T^{d/2}$, where d denotes the spatial dimension, in particular, $d = 1$ for the random process case (see also [4]).

In our result, the rate of convergence obtained is of order j^{-1} , due to the fact that the integral wavelet functionals involved in the proof of the main central limit result are defined from the derivative of the contrast process, which involves the partial derivative with respect to the parameter α of the asymptotic variance of the wavelet random field. This partial derivative is directly proportional to j . The divergence order of the asymptotic variance of these wavelet functionals is then of order j^2 . Note that parameter j defines the exponent that determines the dyadic potential increasing rate of the translation parameter domain, with respect to which our asymptotic approach is defined.

Theorem 5.1. *Under conditions A.1–A.4 and B.1–B.2, as $j \rightarrow \infty$,*

$$\kappa(j)(\tilde{\alpha}_j - \alpha_0) \longrightarrow N(0, \Gamma(\alpha_0))$$

in distribution, where $\kappa(j) = 1/j$, and

$$\Gamma(\alpha_0) = \nu(\alpha_0)^{-2} \left[\sum_{k=1}^{\infty} \frac{C_k^2}{k!} \sigma_k^2(\alpha_0) \right], \quad (5.1)$$

$$\nu(\alpha_0) = \lim_{j \rightarrow \infty} \int_{L_j} \omega(b) \frac{\partial^2}{\partial \alpha^2} [\ln B_X^\alpha(\Psi_{j,b}, \Psi_{j,b})](\alpha_0) db, \quad (5.2)$$

with

$$\begin{aligned} \sigma_k^2(\alpha_0) = 4 \left[\ln \frac{1}{2} \right]^2 & \left[\int_{\mathbf{R}^d \setminus \mathcal{B}_R^d(0)} \frac{\|x\|^{k(2\alpha_0-3d)}}{[\tilde{L}_d(\alpha_0)]^k} \omega * \omega(x) dx \right. \\ & \left. + \frac{1}{[2\alpha_0(2\alpha_0+d)]^k} \int_{\mathcal{B}_R^d(0)} \frac{[\xi_v(x)]^k}{[\tilde{L}_d(\alpha_0)]^k} \omega * \omega(x) dx \right], \end{aligned} \quad (5.3)$$

for $k \geq 1$, where $\mathcal{B}_R^d(0)$ denotes the ball of center zero and radius R , with $R \gg 1$ and ξ_v and $\tilde{L}_d(\alpha_0)$ are given by (3.6) and (3.7), respectively. Here

$$\omega * \omega(x) = \int_{D(x)} \omega(b) \omega(b-x) db \quad (5.4)$$

with $D(x) = \{b \in \mathbf{R}_+^d, b_i > x_i, i = 1, \dots, d\}$, for $x = (x_1, \dots, x_d)$ and $b = (b_1, \dots, b_d)$.

P r o o f o f T h e o r e m 5.1. The proof of Theorem 5.1 has two parts. The first part follows from Lemma 4.3 and Corollary 4.1, and it provides the convergence to the Gaussian distribution of a finite-dimensional sum of integral Hermite functionals of the normalized wavelet random field. The second part (see Lemma 5.1) shows the convergence to the Gaussian distribution of the infinite-dimensional integral Hermite series defining the partial derivative of the contrast process with respect to parameter α .

First part: Convergence of the finite-dimensional Hermite sum. We now proceed to develop the first part of the proof of Theorem 5.1.

For j sufficiently large, the following identity holds:

$$\left[\frac{\partial}{\partial \alpha} U_j \right](\tilde{\alpha}_j) - \left[\frac{\partial}{\partial \alpha} U_j \right](\alpha_0) = \left[\frac{\partial^2}{\partial \alpha^2} U_j \right](\alpha_j^*)(\tilde{\alpha}_j - \alpha_0), \quad (5.5)$$

for α_j^* in the interval $(\alpha_0, \tilde{\alpha}_j)$. Equivalently,

$$-\left[\frac{\partial}{\partial \alpha} U_j \right](\alpha_0) = \left[\frac{\partial^2}{\partial \alpha^2} U_j \right](\alpha_j^*)(\tilde{\alpha}_j - \alpha_0), \quad \alpha_j^* \in (\alpha_0, \tilde{\alpha}_j).$$

Furthermore, from Lemmas 3.2 and 3.3,

$$-\left[\frac{\partial^2}{\partial \alpha^2} U_j \right](\alpha_j^*) \longrightarrow \nu(\alpha_0) \quad \text{in probability,}$$

as $j \rightarrow \infty$, where $\nu(\alpha_0)$ is defined as in equation (5.2). Therefore, from Slutskii's lemma, in order to prove that $\kappa(j)(\tilde{\alpha}_j - \alpha_0) \longrightarrow N(0, \Gamma(\alpha_0))$, it is

sufficient to show that

$$\begin{aligned} \kappa(j) \left[\frac{\partial}{\partial \alpha} U_j \right] (\alpha_0) &= \kappa(j) \int_{L_j} \frac{S(j, b, b) - \mathbf{E}_{\alpha_0}(S(j, b, b))}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^2} \\ &\times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) \omega(b) db \longrightarrow N(0, G(\alpha_0)), \end{aligned} \quad (5.6)$$

in distribution, where

$$G(\alpha_0) = \nu(\alpha_0)^2 \Gamma(\alpha_0) = \sum_{k=1}^{\infty} \frac{C_k^2}{k!} \sigma_k^2(\alpha_0)$$

with $\sigma_k^2(\alpha_0)$ being defined as in equation (5.3), for each $k \geq 1$.

For the derivation of limit (5.6), consider the process $Z_j(\alpha_0)$ given by

$$Z_j(\alpha_0) = \int_{L_j} \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) \frac{S(j, b, b)}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^2} \omega(b) db,$$

with expected value

$$\mathbf{E}(Z_j(\alpha_0)) = \int_{L_j} \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \omega(b) db.$$

For the application of Lemma 4.1, process $Z_j(\alpha_0)$ is rewritten as

$$\begin{aligned} Z_j(\alpha_0) &= \int_{L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \\ &\times F\left(\frac{X(\Psi_{j,b})}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}}\right) db, \end{aligned}$$

in terms of the functional

$$F\left(\frac{X(\Psi_{j,u})}{[B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{1/2}}\right) = \frac{|X(\Psi_{j,u})|^2}{B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})} = \frac{S(j, u, u)}{[B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]}$$

of the normalized wavelet random field $\{X_W^N(j, b), b \in L_j, j \in \mathbf{Z}\}$, given by

$$X_W^N(j, u) = \frac{X(\Psi_{j,u})}{[B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{1/2}}, \quad u \in L_j.$$

The Hermite expansion of F then leads to the following series expansion of process $Z_j(\alpha_0)$:

$$Z_j(\alpha_0) = \sum_{k=0}^{\infty} \frac{C_k}{k!} \eta_k^{\alpha_0}(X_W^N)(j),$$

where

$$\begin{aligned} \eta_k^{\alpha_0}(X_W^N)(j) &= \int_{L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) \\ &\times [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} H_k(X_W^N(j, b)) db. \end{aligned} \quad (5.7)$$

For $k \geq 1$, considering the isometry between the k th Wiener chaos associated with the generalized random field X , and the k th symmetric tensor product $H^{\odot k}$ of H , defined by the multiple Wiener–Itô stochastic integral I_k^X with respect to the process X , the following identity then holds:

$$\begin{aligned} \eta_k^{\alpha_0}(X_W^N)(j) &= I_k^X(f_j^{\alpha_0}) \\ &= \int_{D^k} \left[\int_{L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \right. \\ &\quad \times \left. \prod_{l=1}^k \frac{\Psi_{j,b}(x_l)}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} db \right] \prod_{l=1}^k \tilde{X}(x_l) dx_l \\ &= \int_{D^k} \left[\int_{L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \right. \\ &\quad \times \left. \prod_{l=1}^k \mathcal{T}_X \left(\frac{\Psi_{j,b}}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} \right) (x_l) db \right] \prod_{l=1}^k \eta(dx_l), \quad (5.8) \end{aligned}$$

where operator $\mathcal{T}_X$ and isonormal process η are defined as in Lemma 4.3. That is, here, $\eta(dx)$ denotes the Wiener measure. Therefore, the application of Lemma 4.1 will be performed in terms of the sequence of kernels $\{f_j^{\alpha_0} \in H^{\odot k}, j \geq 1\}$, defined from the identity

$$\begin{aligned} f_j^{\alpha_0}(x_1, \dots, x_k) &= \int_{L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \\ &\quad \times \prod_{l=1}^k \mathcal{T}_X \left(\frac{\Psi_{j,b}}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} \right) (x_l) db, \quad (5.9) \end{aligned}$$

for each $k \geq 1$. Here, since we are interested in the asymptotic results, when $j \rightarrow \infty$, without loss of generality, parameter j will move in the positive integers as in Lemma 4.1. From now on, we incorporate the scaling $\kappa(j)$ in the definition of the functional sequence $\{f_j^{\alpha_0}, j \geq 1\}$, i.e., we consider the sequence $\{\tilde{f}_j^{\alpha_0} = \kappa(j)f_j^{\alpha_0}, j \geq 1\}$.

Before applying Lemma 4.1, we compute, for $k \geq 1$,

$$\lim_{j \rightarrow \infty} k! \|\tilde{f}_j^{\alpha_0}\|_{H^{\otimes k}}^2 = \lim_{j \rightarrow \infty} \mathbf{E} (I_k^X(\tilde{f}_j^{\alpha_0})^2).$$

Thus, for $k \geq 1$,

$$\begin{aligned} \mathbf{E} (I_k^X(\tilde{f}_j^{\alpha_0})^2) &= k! \|\tilde{f}_j^{\alpha_0}\|_{H^{\otimes k}}^2 \\ &= \kappa^2(j) k! \int_{D^k \times D^k} \left[\int_{L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \right. \\ &\quad \times \left. \prod_{l=1}^k \frac{\Psi_{j,b}(x_l)}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} db \right] \end{aligned}$$

$$\begin{aligned}
& \times \left[\int_{L_j} \omega(u) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,u}, \Psi_{j,u}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{-1} \right. \\
& \quad \left. \times \prod_{l=1}^k \frac{\Psi_{j,u}(y_l)}{[B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{1/2}} du \right] \prod_{l=1}^k B_{\tilde{X}}(x_l, y_l) dx_l dy_l \\
& = \kappa^2(j) k! \int_{L_j \times L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \\
& \quad \times \omega(u) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,u}, \Psi_{j,u}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{-1} \\
& \quad \times \frac{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^k}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{k/2} [B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{k/2}} du db. \quad (5.10)
\end{aligned}$$

For j sufficiently large, from Lemma 3.1, equation (5.10) can be rewritten as

$$\begin{aligned}
& \mathbf{E} (I_k^X(\tilde{f}_j^{\alpha_0})^2) \\
& = \kappa^2(j) k! \int_{L_j \times L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} 2^{-2j\alpha} C_\Psi(\alpha, b, j) \right] (\alpha_0) 2^{2j\alpha_0} [C_\Psi(\alpha_0, b, j)]^{-1} \\
& \quad \times \omega(u) \left[\frac{\partial}{\partial \alpha} 2^{-2j\alpha} C_\Psi(\alpha, u, j) \right] (\alpha_0) 2^{2j\alpha_0} [C_\Psi(\alpha_0, u, j)]^{-1} \\
& \quad \times \frac{2^{-2jk\alpha_0} [C_\Psi(\alpha_0, b, u, j)]^k}{2^{-j\alpha_0 k} [C_\Psi(\alpha_0, b, j)]^{k/2} 2^{-j\alpha_0 k} [C_\Psi(\alpha_0, u, j)]^{k/2}} du db \\
& = \kappa^2(j) k! 2^{4j\alpha_0} \int_{L_j \times L_j} \omega(b) \omega(u) \left[\frac{\partial}{\partial \alpha} 2^{-2j\alpha} C_\Psi(\alpha, b, j) \right] (\alpha_0) [C_\Psi(\alpha_0, b, j)]^{-1} \\
& \quad \times [C_\Psi(\alpha_0, u, j)]^{-1} \left[\frac{\partial}{\partial \alpha} 2^{-2j\alpha} C_\Psi(\alpha, u, j) \right] (\alpha_0) \\
& \quad \times \frac{[C_\Psi(\alpha_0, b, u, j)]^k}{[C_\Psi(\alpha_0, b, j)]^{k/2} [C_\Psi(\alpha_0, u, j)]^{k/2}} du db, \quad (5.11)
\end{aligned}$$

where $C_\Psi(\alpha_0, b, u, j)$ and $C_\Psi(\alpha_0, b, j)$ are defined as in Lemma 3.1. Moreover, from this Lemma, for $j \rightarrow \infty$,

$$\begin{aligned}
& \left[\frac{\partial}{\partial \alpha} 2^{-2j\alpha} C_\Psi(\alpha, \infty) \right] (\alpha_0) \simeq 2j 2^{-2j\alpha_0} \ln(1/2) \tilde{L}_d(\alpha_0) + 2^{-2j\alpha_0} \left[\frac{\partial}{\partial \alpha} \tilde{L}_d \right] (\alpha_0) \\
& = 2j 2^{-2j\alpha_0} \ln \frac{1}{2} \tilde{L}_d(\alpha_0) \\
& \quad + 2^{-2j\alpha_0} \frac{2[\ln 2] \alpha_0 (2\alpha_0 + d) 2^{-(2\alpha_0 + d - 2)} - (4\alpha_0 + d) [1 - 2^{-(2\alpha_0 + d - 2)}]}{[\alpha_0 (2\alpha_0 + d)]^2}. \quad (5.12)
\end{aligned}$$

Thus, as $j \rightarrow \infty$, the following limit expression is obtained for equation (5.11):

$$\mathbf{E} (I_k^X(\tilde{f}_j^{\alpha_0})^2) \rightarrow \left[2 \ln \frac{1}{2} \right]^2 k! \int_{\mathbf{R}_+^d \times \mathbf{R}_+^d} \omega(b) \omega(u) \frac{[C_\Psi(\alpha_0, b - u, \infty)]^k}{[\tilde{L}_d(\alpha_0)]^k} du db$$

$$\begin{aligned}
&= \left[2 \ln \frac{1}{2} \right]^2 k! \int_{\mathbf{R}^d} \frac{[C_{\Psi}(\alpha_0, x, \infty)]^k}{[\tilde{L}_d(\alpha_0)]^k} \omega * \omega(x) dx \\
&\sim 4 \left[\ln \frac{1}{2} \right]^2 k! \left[\int_{\mathbf{R}^d \setminus \mathcal{B}_R^d(0)} \frac{\|x\|^{k(2\alpha_0-3d)}}{[\tilde{L}_d(\alpha_0)]^k} \omega * \omega(x) dx \right. \\
&\quad \left. + \frac{1}{[2\alpha_0(2\alpha_0+d)]^k} \int_{\mathcal{B}_R^d(0)} \frac{[\xi_v(x)]^k}{[\tilde{L}_d(\alpha_0)]^k} \omega * \omega(x) dx \right] \\
&= k! \sigma_k^2(\alpha_0), \tag{5.13}
\end{aligned}$$

for $\omega \in L^1(\mathbf{R}_+^d)$, and with $\omega * \omega(x)$, $x \in \mathbf{R}^d$, being defined as in equation (5.4). Note that

$$\begin{aligned}
\sigma_1^2(\alpha_0) &= 4 \left[\ln \frac{1}{2} \right]^2 \left[\int_{\mathbf{R}^d \setminus \mathcal{B}_R^d(0)} \frac{\|x\|^{2\alpha_0-3d}}{[\tilde{L}_d(\alpha_0)]} \omega * \omega(x) dx \right. \\
&\quad \left. + \frac{1}{[2\alpha_0(2\alpha_0+d)]} \int_{\mathcal{B}_R^d(0)} \frac{[\xi_v(x)]}{[\tilde{L}_d(\alpha_0)]} \omega * \omega(x) dx \right] \tag{5.14}
\end{aligned}$$

is clearly finite from the asymptotic order of function $\|x\|^{2\alpha_0-3d}$ and the continuity of function ξ_v , given in equation (3.6), as well as from the absolute integrability of function ω . Moreover, for $k \geq 2$, from Lemma 3.1, applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\lim_{j \rightarrow \infty} \mathbf{E} (I_k^X(\tilde{f}_j^{\alpha_0})^2) &\leq 4 \left[\ln \frac{1}{2} \right]^2 k! \left[\int_{\mathbf{R}_+^d \times \mathbf{R}_+^d} \omega(b) \omega(u) db du \right]^{1/2} \\
&\quad \times \left[\int_{\mathbf{R}_+^d \times \mathbf{R}_+^d} \omega(b) \omega(u) \frac{[C_{\Psi}(\alpha_0, b, u, \infty)]^{2k}}{[\tilde{L}_d(\alpha_0)]^{2k}} db du \right]^{1/2} \\
&\leq 4 \left[\ln \frac{1}{2} \right]^2 k! \|\omega\|_{L^1(\mathbf{R}_+^d)} \left[\int_{\mathbf{R}_+^d \times \mathbf{R}_+^d} \omega(b) \omega(u) \frac{[C_{\Psi}(\alpha_0, b, u, \infty)]^2}{[\tilde{L}_d(\alpha_0)]^2} db du \right]^{1/2} \\
&\sim 4 \left[\ln \frac{1}{2} \right]^2 k! \|\omega\|_{L^1(\mathbf{R}_+^d)} \left[\int_{\mathbf{R}^d \setminus \mathcal{B}_R^d(0)} \frac{\|x\|^{4\alpha_0-6d}}{[\tilde{L}_d(\alpha_0)]^2} \omega * \omega(x) dx \right. \\
&\quad \left. + \frac{1}{[2\alpha_0(2\alpha_0+d)]^2} \int_{\mathcal{B}_R^d(0)} \frac{[\xi_v(x)]^2}{[\tilde{L}_d(\alpha_0)]^2} \omega * \omega(x) dx \right]^{1/2}, \tag{5.15}
\end{aligned}$$

which again is finite in view of the asymptotic order of function $\|x\|^{4\alpha_0-6d}$, and the continuity of function $[\xi_v]^2$, as well as from the absolute integrability of the weight function ω .

We now proceed to check condition (ii) of Lemma 4.1. Specifically, for each $k \geq 2$, the contraction of order p , for $p = 1, \dots, k-1$, is given by

$$\tilde{f}_j^{\alpha_0} \otimes_p \tilde{f}_j^{\alpha_0}(x_1, \dots, x_{2k-2p}) = \int_{D^p \times D^p} \tilde{f}_j^{\alpha_0}(x_1, \dots, x_{k-p}, y_1, \dots, y_p)$$

$$\begin{aligned}
& \times \tilde{f}_j^{\alpha_0}(x_{k-p+1}, \dots, x_{2k-2p}, z_1, \dots, z_p) \prod_{l=1}^p dy_l dz_l \\
& = \kappa^2(j) \int_{L_j \times L_j} \omega(b) \omega(u) \\
& \quad \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,u}, \Psi_{j,u}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{-1} \\
& \quad \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \\
& \quad \times \prod_{l=1}^{k-p} \mathcal{T}_X \left(\frac{\Psi_{j,u}}{[B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{1/2}} \right) (x_l) \\
& \quad \times \prod_{l=k-p+1}^{2k-2p} \mathcal{T}_X \left(\frac{\Psi_{j,b}}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} \right) (x_l) \\
& \quad \times \left[\prod_{l=1}^p \int_{D^p \times D^p} \frac{\Psi_{j,u}(y_l)}{[B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{1/2}} \right. \\
& \quad \quad \left. \times \frac{\Psi_{j,b}(z_l)}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} B_{\tilde{X}}(y_l, z_l) dy_l dz_l \right] du db
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
& = \kappa^2(j) \int_{L_j \times L_j} \omega(b) \omega(u) \\
& \quad \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,u}, \Psi_{j,u}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{-1} \\
& \quad \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \\
& \quad \times \prod_{l=1}^{k-p} \mathcal{T}_X \left(\frac{\Psi_{j,u}}{[B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{1/2}} \right) (x_l) \\
& \quad \times \prod_{l=k-p+1}^{2k-2p} \mathcal{T}_X \left(\frac{\Psi_{j,b}}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} \right) (x_l)
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
& \times \frac{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,u})]^p}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b}) B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{p/2}} db du.
\end{aligned} \tag{5.18}$$

Its norm in the space $H^{\otimes 2k-2p}$ is then computed as follows:

$$\begin{aligned}
& \|\tilde{f}_j^{\alpha_0} \otimes_p \tilde{f}_j^{\alpha_0}\|_{H^{\otimes 2k-2p}} = \int_{D^{2k-2p}} \kappa^4(j) \int_{L_j \times L_j} \int_{L_j \times L_j} \omega(b_1) \omega(u_1) \omega(b_2) \omega(u_2) \\
& \quad \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b_1}, \Psi_{j,b_1}) \right] (\alpha_0) [B_X^\alpha(\Psi_{j,b_1}, \Psi_{j,b_1})]^{-1} \\
& \quad \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,u_1}, \Psi_{j,u_1}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,u_1}, \Psi_{j,u_1})]^{-1} \\
& \quad \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b_2}, \Psi_{j,b_2}) \right] (\alpha_0) [B_X^\alpha(\Psi_{j,b_2}, \Psi_{j,b_2})]^{-1}
\end{aligned}$$

$$\begin{aligned}
& \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,u_2}, \Psi_{j,u_2}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,u_2}, \Psi_{j,u_2})]^{-1} \\
& \times \frac{[B_X^{\alpha_0}(\Psi_{j,b_1}, \Psi_{j,u_1})]^p}{[B_X^{\alpha_0}(\Psi_{j,b_1}, \Psi_{j,b_1}) B_X^{\alpha_0}(\Psi_{j,u_1}, \Psi_{j,u_1})]^{p/2}} \\
& \times \frac{[B_X^{\alpha_0}(\Psi_{j,b_2}, \Psi_{j,u_2})]^p}{[B_X^{\alpha_0}(\Psi_{j,b_2}, \Psi_{j,b_2}) B_X^{\alpha_0}(\Psi_{j,u_2}, \Psi_{j,u_2})]^{p/2}} \\
& \times \prod_{l=1}^{k-p} \frac{\Psi_{j,u_1}(x_l)}{[B_X^{\alpha_0}(\Psi_{j,u_1}, \Psi_{j,u_1})]^{1/2}} \prod_{l=k-p+1}^{2k-2p} \frac{\Psi_{j,b_1}(x_l)}{[B_X^{\alpha_0}(\Psi_{j,b_1}, \Psi_{j,b_1})]^{1/2}} \\
& \times \prod_{l=1}^{k-p} \frac{\Psi_{j,u_2}(y_l)}{[B_X^{\alpha_0}(\Psi_{j,u_2}, \Psi_{j,u_2})]^{1/2}} \prod_{l=k-p+1}^{2k-2p} \frac{\Psi_{j,b_2}(y_l)}{[B_X^{\alpha_0}(\Psi_{j,b_2}, \Psi_{j,b_2})]^{1/2}} \\
& \times \prod_{l=1}^{2k-2p} B_{\bar{X}}(x_l, y_l) dx_l dy_l db_1 db_2 du_1 du_2 \\
& = \kappa^4(j) \int_{L_j \times L_j} \int_{L_j \times L_j} \omega(b_1) \omega(b_2) \omega(u_1) \omega(u_2) \\
& \times [B_X^{\alpha_0}(\Psi_{j,b_1}, \Psi_{j,b_1}) B_X^{\alpha_0}(\Psi_{j,b_2}, \Psi_{j,b_2})]^{-1} \\
& \times [B_X^{\alpha_0}(\Psi_{j,u_1}, \Psi_{j,u_1}) B_X^{\alpha_0}(\Psi_{j,u_2}, \Psi_{j,u_2})]^{-1} \\
& \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b_1}, \Psi_{j,b_1}) \right] (\alpha_0) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b_2}, \Psi_{j,b_2}) \right] (\alpha_0) \\
& \times \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,u_1}, \Psi_{j,u_1}) \right] (\alpha_0) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,u_2}, \Psi_{j,u_2}) \right] (\alpha_0) \\
& \times \frac{[B_X^{\alpha_0}(\Psi_{j,b_1}, \Psi_{j,u_1})]^p}{[B_X^{\alpha_0}(\Psi_{j,b_1}, \Psi_{j,b_1}) B_X^{\alpha_0}(\Psi_{j,u_1}, \Psi_{j,u_1})]^{p/2}} \\
& \times \frac{[B_X^{\alpha_0}(\Psi_{j,b_2}, \Psi_{j,u_2})]^p}{[B_X^{\alpha_0}(\Psi_{j,b_2}, \Psi_{j,b_2}) B_X^{\alpha_0}(\Psi_{j,u_2}, \Psi_{j,u_2})]^{p/2}} \\
& \times \left[\frac{[B_X^{\alpha_0}(\Psi_{j,b_1}, \Psi_{j,b_2})]^{k-p}}{[B_X^{\alpha_0}(\Psi_{j,b_1}, \Psi_{j,b_1}) B_X^{\alpha_0}(\Psi_{j,b_2}, \Psi_{j,b_2})]^{(k-p)/2}} \right] \\
& \times \left[\frac{[B_X^{\alpha_0}(\Psi_{j,u_1}, \Psi_{j,u_2})]^{k-p}}{[B_X^{\alpha_0}(\Psi_{j,u_1}, \Psi_{j,u_1}) B_X^{\alpha_0}(\Psi_{j,u_2}, \Psi_{j,u_2})]^{(k-p)/2}} \right] db_1 db_2 du_1 du_2. \quad (5.19)
\end{aligned}$$

Considering the case $k = 2$ and $p = 1$, for j sufficiently large, from Lemmas 3.1, the following approximation is considered:

$$\begin{aligned}
& \|\tilde{f}_j^{\alpha_0} \otimes_p \tilde{f}_j^{\alpha_0}\|_{H^{\otimes 2k-2p}} \\
& \sim \frac{16[\ln \frac{1}{2}]^4}{[\tilde{L}_d(\alpha_0)]^4} \int_{[0,2^j]^{4d}} \omega(b_1) \omega(b_2) \omega(u_1) \omega(u_2) C_{\Psi}(\alpha_0, \|b_1 - u_1\|, \infty) \\
& \quad \times C_{\Psi}(\alpha_0, \|b_2 - u_2\|, \infty) C_{\Psi}(\alpha_0, \|b_1 - b_2\|, \infty) \\
& \quad \times C_{\Psi}(\alpha_0, \|u_1 - u_2\|, \infty) db_1 db_2 du_1 du_2 \\
& = K \int_{[-2^j, 2^j]^d \times [-2^j, 2^j]^d \times [-2^{j+1}, 2^{j+1}]^d} \left[\int_{D_{b_1}(h-\tau, z, h, j)} \omega(b_1) \omega(b_1 - z) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \omega(b_1 - h) \omega(b_1 + \tau - h) db_1 \Big] \\
& \times C_{\Psi}(\alpha_0, \|z\|, \infty) C_{\Psi}(\alpha_0, \|\tau\|, \infty) C_{\Psi}(\alpha_0, \|h - z\|, \infty) \\
& \times C_{\Psi}(\alpha_0, \|h - \tau\|, \infty) dh d\tau dz, \tag{5.20}
\end{aligned}$$

where

$$K = \frac{16[\ln \frac{1}{2}]^4}{[\widetilde{L}_d(\alpha_0)]^4};$$

$$D_{b_1}(h - \tau, z, h, j) = \prod_{i=1}^d [h_i - \tau_i, h_i - \tau_i + 2^j] \cap \prod_{i=1}^d [z_i, z_i + 2^j] \cap \prod_{i=1}^d [h_i, h_i + 2^j].$$

Denoting by

$$I_{\omega}(z, \tau, h, j) = \int_{D_{b_1}(h - \tau, z, h, j)} \omega(b_1) \omega(b_1 - z) \omega(b_1 - h) \omega(b_1 + \tau - h) db_1,$$

equation (5.20) can be rewritten, and upper bounded, from Lemmas 3.1, as follows:

$$\begin{aligned}
& \|\widetilde{f}_j^{\alpha_0} \otimes_p \widetilde{f}_j^{\alpha_0}\|_{H^{\otimes 2k-2p}} \sim K \int_{[-2^{j+1}, 2^{j+1}]^d \times [-2^j, 2^j]^d \times [-2^j, 2^j]^d} C_{\Psi}(\alpha_0, \|z\|, \infty) \\
& \times C_{\Psi}(\alpha_0, \|\tau\|, \infty) C_{\Psi}(\alpha_0, \|h - z\|, \infty) C_{\Psi}(\alpha_0, \|h - \tau\|, \infty) \\
& \times I_{\omega}(z, \tau, h, j) d\tau dz dh \\
& \leq K \int_{[-2^{j+1}, 2^{j+1}]^d} \left[\int_{[-2^j, 2^j]^d \setminus B_R(0) \times [-2^j, 2^j]^d \setminus B_R(0)} R^{4\alpha_0 - 6d} \right. \\
& \quad \times C_{\Psi}(\alpha_0, \|h - z\|, \infty) C_{\Psi}(\alpha_0, \|h - \tau\|, \infty) I_{\omega}(z, \tau, h, j) d\tau dz \Big] dh \\
& + K \int_{[-2^{j+1}, 2^{j+1}]^d} \left[\int_{[-2^j, 2^j]^d \setminus B_R(0) \times B_R(0)} R^{2\alpha_0 - 3d} \left[\max_{x \in B_R(0)} \xi_v(x) \right] \right. \\
& \quad \times C_{\Psi}(\alpha_0, \|h - z\|, \infty) C_{\Psi}(\alpha_0, \|h - \tau\|, \infty) I_{\omega}(z, \tau, h, j) d\tau dz \Big] dh \\
& + K \int_{[-2^{j+1}, 2^{j+1}]^d} \left[\int_{B_R(0) \times [-2^j, 2^j]^d \setminus B_R(0)} R^{2\alpha_0 - 3d} \left[\max_{x \in B_R(0)} \xi_v(x) \right] \right. \\
& \quad \times C_{\Psi}(\alpha_0, \|h - z\|, \infty) C_{\Psi}(\alpha_0, \|h - \tau\|, \infty) I_{\omega}(z, \tau, h, j) d\tau dz \Big] dh \\
& + K \int_{[-2^{j+1}, 2^{j+1}]^d} \left[\int_{B_R(0) \times B_R(0)} \left[\max_{x \in B_R(0)} \xi_v(x) \right]^2 \right. \\
& \quad \times C_{\Psi}(\alpha_0, \|h - z\|, \infty) C_{\Psi}(\alpha_0, \|h - \tau\|, \infty) I_{\omega}(z, \tau, h, j) d\tau dz \Big] dh \\
& = K \int_{[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}}]^d \times [-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}}]^d} \int_{D_h(x, y, j)} R^{4\alpha_0 - 6d}
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
& \times C_{\Psi}(\alpha_0, \|x\|, \infty) C_{\Psi}(\alpha_0, \|y\|, \infty) I_{\omega}(h-x, h-y, h, j) dh dx dy \\
& + K \int_{D_y(j, R, d) \times [-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}}]^d} R^{2\alpha_0 - 3d} \left[\max_{x \in B_R(0)} \xi_v(x) \right] \\
& \quad \times \int_{D_h(x, y, j)} C_{\Psi}(\alpha_0, \|x\|, \infty) C_{\Psi}(\alpha_0, \|y\|, \infty) \\
& \quad \times I_{\omega}(h-x, h-y, h, j) dh dx dy \\
& + K \int_{[-2^{j+1} - \frac{R}{\sqrt{d}}, 2^{j+1} + \frac{R}{\sqrt{d}}]^d \times D_x(j, R, d)} R^{2\alpha_0 - 3d} \left[\max_{x \in B_R(0)} \xi_v(x) \right] \\
& \quad \times \int_{D_h(x, y, j)} C_{\Psi}(\alpha_0, \|x\|, \infty) C_{\Psi}(\alpha_0, \|y\|, \infty) \\
& \quad \times I_{\omega}(h-x, h-y, h, j) dh dx dy \\
& + K \int_{D_y(j, R, d) \times D_x(j, R, d)} \left[\max_{x \in B_R(0)} \xi_v(x) \right]^2 \\
& \quad \times \int_{D_h(x, y, j)} C_{\Psi}(\alpha_0, \|x\|, \infty) C_{\Psi}(\alpha_0, \|y\|, \infty) \\
& \quad \times I_{\omega}(h-x, h-y, h, j) dh dx dy \\
& = I_1(j) + I_2(j) + I_3(j) + I_4(j), \tag{5.22}
\end{aligned}$$

where $D_h(x, y, j)$ and $D_z(j, R, d)$, $z = x, y$, are given as in equation (3.2).

Applying again Lemma 3.1 to equation (5.22), in a similar way to equation (5.21), each one of the integrals $I_i(j)$, $i = 1, 2, 3, 4$, can be decomposed into four integrals $I_{l,m}(j)$; $l = 1, \dots, 4$; $m = 1, \dots, 4$, where similar upper bounds in terms of R and $[\max_{x \in B_R(0)} \xi_v(x)]$ can be considered. Specifically, we obtain

$$\|\tilde{f}_j^{\alpha_0} \otimes_p \tilde{f}_j^{\alpha_0}\|_{H^{\otimes 2k-2p}} \leq \sum_{l=1}^4 \sum_{m=1}^4 f_{l,m}(R) g_{l,m}(\xi_v) I_{l,m}(j),$$

where

$$\begin{aligned}
I_{l,m}(j) &= \int_{D_{l,m}(j) \times D_h(x, y, j) \times D_{b_1}(h-x, y, h, j)} \omega(b_1) \omega(b_1 - y) \\
& \quad \omega(b_1 - h) \omega(b_1 - (h - x)) db_1 dh dx dy, \tag{5.23}
\end{aligned}$$

with $D_{l,m}(j)$, $l, m = 1, \dots, 4$, and $D_{b_1}(h-x, y, h, j)$ being defined as in equation (3.2). Under condition B.2, $I_{l,m}(j) \rightarrow 0$, as $j \rightarrow \infty$, for $l, m = 1, \dots, 4$, and the Gaussian asymptotic distribution holds for $k = 2$. Also, under condition B.2, from Cauchy–Schwarz inequality applied to equation (5.19), after considering Lemma 3.1, we obtain, for $k > 2$, as $j \rightarrow \infty$, in a similar way to the case $k = 2$, and $p = 1$, that

$$\begin{aligned}
& \|\tilde{f}_j^{\alpha_0} \otimes_p \tilde{f}_j^{\alpha_0}\|_{H^{\otimes 2k-2p}} \\
& \sim K_1 \int_{L_j \times L_j} \int_{L_j \times L_j} \omega(b_1) \omega(b_2) \omega(u_1) \omega(u_2) \frac{[C_{\Psi}(\alpha_0, b_1, u_1, \infty)]^p}{[\widetilde{L_d}(\alpha_0)]^p}
\end{aligned}$$

$$\begin{aligned}
& \times \frac{[C_{\Psi}(\alpha_0, b_2, u_2, \infty)]^p}{[\widetilde{L}_d(\alpha_0)]^p} \frac{[C_{\Psi}(\alpha_0, b_1, b_2, \infty)]^{k-p}}{[\widetilde{L}_d(\alpha_0)]^{k-p}} \\
& \times \frac{[C_{\Psi}(\alpha_0, u_1, u_2, \infty)]^{k-p}}{[\widetilde{L}_d(\alpha_0)]^{k-p}} db_1 db_2 du_1 du_2 \\
& \leq K_1 \left[\int_{L_j \times L_j} \int_{L_j \times L_j} \omega(b_1) \omega(b_2) \omega(u_1) \omega(u_2) db_1 db_2 du_1 du_2 \right]^{1/2} \\
& \times \left[\int_{L_j \times L_j} \int_{L_j \times L_j} \omega(b_1) \omega(b_2) \omega(u_1) \omega(u_2) \frac{[C_{\Psi}(\alpha_0, b_1, u_1, \infty)]^{2p}}{[\widetilde{L}_d(\alpha_0)]^{2p}} \right. \\
& \times \frac{[C_{\Psi}(\alpha_0, b_2, u_2, \infty)]^{2p}}{[\widetilde{L}_d(\alpha_0)]^{2p}} \frac{[C_{\Psi}(\alpha_0, b_1, b_2, \infty)]^{2(k-p)}}{[\widetilde{L}_d(\alpha_0)]^{2(k-p)}} \\
& \times \left. \frac{[C_{\Psi}(\alpha_0, u_1, u_2, \infty)]^{2(k-p)}}{[\widetilde{L}_d(\alpha_0)]^{2(k-p)}} db_1 db_2 du_1 du_2 \right]^{1/2} \\
& \leq K_1 \left[\int_{L_j} \omega(b) db \right]^2 \left[\int_{L_j \times L_j} \int_{L_j \times L_j} \omega(b_1) \omega(b_2) \omega(u_1) \omega(u_2) \right. \\
& \times \frac{[C_{\Psi}(\alpha_0, b_1, u_1, \infty)]^2}{[\widetilde{L}_d(\alpha_0)]^2} \frac{[C_{\Psi}(\alpha_0, b_2, u_2, \infty)]^2}{[\widetilde{L}_d(\alpha_0)]^2} \\
& \times \left. \frac{[C_{\Psi}(\alpha_0, b_1, b_2, \infty)]^2}{[\widetilde{L}_d(\alpha_0)]^2} \frac{[C_{\Psi}(\alpha_0, u_1, u_2, \infty)]^2}{[\widetilde{L}_d(\alpha_0)]^2} db_1 db_2 du_1 du_2 \right]^{1/2} \\
& = K_2 \int_{[-2j, 2j]^d \times [-2j, 2j]^d \times [-2j+1, 2j+1]^d} \\
& \times \left[\int_{D_{b_1}(h-\tau, z, h, j)} \omega(b_1) \omega(b_1 - z) \omega(b_1 - h) \omega(b_1 + \tau - h) db_1 \right] \\
& \times [C_{\Psi}(\alpha_0, \|z\|, \infty) C_{\Psi}(\alpha_0, \|\tau\|, \infty) C_{\Psi}(\alpha_0, \|h - z\|, \infty) \\
& \times C_{\Psi}(\alpha_0, \|h - \tau\|, \infty)]^2 dh d\tau dz \\
& \leq K_2 \sum_{l=1}^4 \sum_{m=1}^4 \widetilde{f}_{l,m}(R) \widetilde{g}_{l,m}(\xi_v) I_{l,m}(j) \longrightarrow 0 \tag{5.24}
\end{aligned}$$

where $K_1 = 16[\ln(1/2)]^4$ and $K_2 = K_1 [\int_{L_j} \omega(b) db]^2 [\widetilde{L}_d(\alpha_0)]^{-8}$ and $I_{l,m}(j)$, $l, m = 1, \dots, 4$, are defined as in equation (5.23).

Thus, from Lemma 4.1, for every $k \geq 2$, the stochastic integral $I_k^X(\tilde{f}_j^{\alpha_0})$ converges in distribution to a Gaussian random variable with variance $k! \sigma_k^2(\alpha_0)$, as $j \rightarrow \infty$. For a finite n , the convergence of

$$S_n(j) = \sum_{k=1}^n \frac{C_k}{k!} I_k^X(\tilde{f}_j^{\alpha_0}),$$

as $j \rightarrow \infty$, to a Gaussian distribution with variance

$$\sum_{k=1}^n \frac{C_k^2}{k!} \sigma_k^2(\alpha_0),$$

then follows from Corollary 4.1.

Convergence of the global Hermite expansion. The following lemma allows to establish the asymptotic normality of the minimum contrast estimator from the result previously derived on the convergence to the Gaussian distribution of the finite-dimensional Hermite sum.

As in the first part of the proof of Theorem 5.1, for each $k \geq n$, the sequence of kernels $\{\tilde{f}_j^{\alpha_0} = \kappa(j)f_j^{\alpha_0}, j \geq 1\}$ is defined from the following identity:

$$f_j^{\alpha_0}(x_1, \dots, x_k) = \int_{L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \\ \times \prod_{l=1}^k \mathcal{I}_X \left(\frac{\Psi_{j,b}}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{1/2}} \right) (x_l) db. \quad (5.25)$$

Lemma 5.1. *Under the conditions assumed in Theorem 5.1, if for any $n \geq 1$, as $j \rightarrow \infty$,*

$$S_n(j) = \sum_{k=1}^n \frac{C_k}{k!} I_k^X(\tilde{f}_j^{\alpha_0})$$

converges to a normal distribution with variance

$$\sum_{k=1}^n \frac{C_k^2}{k!} \sigma_k^2(\alpha_0),$$

then, the weak-convergence, as $n \rightarrow \infty$ and $j \rightarrow \infty$, of

$$S_n(j) = \sum_{k=1}^n \frac{C_k}{k!} I_k^X(\tilde{f}_j^{\alpha_0})$$

to the normal distribution with variance

$$\sigma^2 = \sum_{k=1}^{\infty} \frac{C_k^2}{k!} \sigma_k^2(\alpha_0)$$

also holds.

P r o o f. The proof follows from the convergence in probability to zero, when $n \rightarrow \infty$, of the tail

$$\tilde{S}_n(j) = \sum_{k=n}^{\infty} \frac{C_k}{k!} I_k^X(\tilde{f}_j^{\alpha_0}),$$

uniformly in j , under conditions of Theorem 5.1. Specifically, for $n \geq 2$, applying Cauchy–Schwarz inequality, for j sufficiently large,

$$\mathbf{E}(\tilde{S}_n(j))^2 = \mathbf{E} \left(\sum_{k=n}^{\infty} \frac{C_k}{k!} I_k^X(\tilde{f}_j^{\alpha_0}) \right)^2$$

$$\begin{aligned}
&= \sum_{k=n}^{\infty} \frac{C_k^2}{k!} \kappa^2(j) \int_{L_j \times L_j} \omega(b) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,b}, \Psi_{j,b}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{-1} \\
&\quad \times \omega(u) \left[\frac{\partial}{\partial \alpha} B_X^\alpha(\Psi_{j,u}, \Psi_{j,u}) \right] (\alpha_0) [B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{-1} \\
&\quad \times \frac{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,u})]^k}{[B_X^{\alpha_0}(\Psi_{j,b}, \Psi_{j,b})]^{k/2} [B_X^{\alpha_0}(\Psi_{j,u}, \Psi_{j,u})]^{k/2}} du db \\
&\leq \sigma_2^2(\alpha_0) \sum_{k=n}^{\infty} \frac{C_k^2}{k!} = F(n) \xrightarrow{n \rightarrow \infty} 0,
\end{aligned} \tag{5.26}$$

since by Parseval identity $\sum_{k=1}^{\infty} (C_k^2/k!) < \infty$. The rest of the proof follows as in [19]. Specifically, we consider the following probabilities:

$$\mathbf{P}\{|\tilde{S}_n(j)| > \varepsilon\} \leq \frac{F(n)}{\varepsilon^2} \rightarrow 0, \quad n \rightarrow \infty.$$

Moreover, for $S(j) = S_n(j) + \tilde{S}_n(j)$,

$$\begin{aligned}
\mathbf{P}\{S(j) \leq x\} &= \mathbf{P}\{S(j) \leq x, |\tilde{S}_n(j)| \leq \varepsilon\} + \mathbf{P}\{S(j) \leq x, |\tilde{S}_n(j)| > \varepsilon\} \\
&\leq \mathbf{P}\{S_n(j) \leq x + \varepsilon\} + \frac{F(n)}{\varepsilon^2}.
\end{aligned}$$

Thus, for any $\varepsilon > 0$, and $n \geq 1$

$$\overline{\lim}_{j \rightarrow \infty} \mathbf{P}\{S(j) \leq x\} \leq \Phi_n(x + \varepsilon) + \frac{F(n)}{\varepsilon^2}, \tag{5.27}$$

where Φ_n is a cumulative distribution function of a Gaussian r.v. with zero mean and variance $\sigma_n^2(z) = \sum_{k=1}^n (C_k^2/k!) \sigma_k^2(\alpha_0)$.

On the other hand,

$$\begin{aligned}
\mathbf{P}\{S_n(j) \leq x - \varepsilon\} &\leq \mathbf{P}\{S_n(j) + \varepsilon \leq x, |\tilde{S}_n(j)| \leq \varepsilon\} \\
&\quad + \mathbf{P}\{S_n(j) + \varepsilon \leq x, |\tilde{S}_n(j)| > \varepsilon\} \\
&\leq \mathbf{P}\{S(j) \leq x\} + \frac{F(n)}{\varepsilon^2},
\end{aligned}$$

that is, for any $\varepsilon > 0$, and $n \geq 1$, as $j \rightarrow \infty$,

$$\underline{\lim} \mathbf{P}\{S(j) \leq x\} \geq \Phi_n(x - \varepsilon) - \frac{F(n)}{\varepsilon^2}. \tag{5.28}$$

If $n \rightarrow \infty$, we obtain from equations (5.27) and (5.28) that, for every $j \geq 1$,

$$\Phi_\infty(x - \varepsilon) \leq \underline{\lim} \mathbf{P}\{S(j) \leq x\} \leq \overline{\lim} \mathbf{P}\{S(j) \leq x\} \leq \Phi_\infty(x + \varepsilon),$$

where Φ_∞ is a cumulative distribution function of a Gaussian r.v. with zero mean and the variance

$$\sigma^2(z) = \sum_{k=1}^{\infty} \frac{C_k^2}{k!} \sigma_k^2(\alpha_0).$$

Thus, if $\varepsilon \rightarrow 0$, $\lim_{j \rightarrow \infty} \mathbf{P}\{S(j) \leq x\} = \Phi_\infty(x)$, $x \in \mathbf{R}$. Lemma 5.1 is proved.

6. Final comments. The asymptotic normality of the wavelet-based minimum contrast estimator proposed in [36] is obtained from the application of the central limit result derived in [29]. The main step of the proof consists of the identification of the wavelet random field with an isonormal process (see Lemma 4.3). This step allows us to establish the convergence to the Gaussian distribution from the application of Lemma 4.1. Thus, we adopt an alternative methodological line to the method of moments and diagram formulae, based on the convergence to zero of the norms of the self-contractions of the symmetric kernels involved in the construction of the sequence of multiple stochastic integrals under study. In our context, this sequence is formulated from the Hermite expansion of the integral weighted non-linear functional of the wavelet random field, defining the derivative of the contrast process. The increasing domain asymptotic approach in the wavelet domain is considered in the derivation of the mentioned limit results. This approach opens new research lines in the investigation of fixed-domain asymptotic properties of estimators in a parametric or semiparametric framework. The same idea can be applied to the nonparametric estimator defined in terms of the wavelet periodogram. The flexibility of the wavelet domain, due to the existence of a large variety of wavelet bases satisfying different regularity and moment conditions, allows the design of a vast class of contrast functions. In addition, the non-stationary Gaussian random field case can be treated in a suitable way.

The extension of the derived results to a non-Gaussian framework can be addressed, in the linear random field case, from the central limit results previously obtained for integral non-linear functionals and bilinear forms involving Appell polynomials (see [15], [38], where limit results for moving average sequences with long-range dependence are formulated, and [8], where an extension to the case of multidimensional indices of the functional analysis approach proposed by [7] is derived).

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