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A NOTE ON A LAW OF ITERATED LOGARITHM FOR BOUNDED N -DEMIMARTINGALES

Понятие N -полумартингала является обобщением понятия отрицательной ассоциированности. Для такого рода случайных величин имеются разнообразные результаты. В настоящей статье доказан закон повторного логарифма для ограниченных N -полумартингалов.

Ключевые слова и фразы: случайно взвешенные суммы, сильная сходимость, отрицательно ассоциированные случайные величины, N -полумартингал.

1. Introduction. Let $\{S_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition 1. A sequence of L_1 random variables $\{S_n, n \geq 1\}$ is called an N -demimartingale if for all $i = 1, 2, \dots$

$$\mathbf{E}[(S_{i+1} - S_i)f(S_1, \dots, S_i)] \leq 0,$$

for all componentwise nondecreasing functions f provided the expectation is defined. Moreover, if f is assumed to be nonnegative, the sequence $\{S_n, n \geq 1\}$ is called an N -demisupermartingale.

The notion of an N -demimartingale was first introduced by Christofides in [1] as generalization of the concept of negative association which was introduced by Joag-Dev and Proschan in [6]. Over the recent years, various results has been proved. These results are mainly different kinds of inequalities (see, e.g., [1]–[4], [8], [9]), but also strong laws of large numbers [9], and some results concerning complete convergence of N -demimartingales [4].

Definition 2. A finite collection of random variables $X_1, \dots, X_n$ is said to be negatively associated if

$$\text{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0,$$

for any disjoint subsets A and B of $\{1, \dots, n\}$ and for any two componentwise nondecreasing functions f, g on $\mathbf{R}^A$ and $\mathbf{R}^B$, respectively, provided that the covariance is defined. An infinite collection is negatively associated if for every finite subcollection is negatively associated.

One can easily show that a sequence of partial sums of zero mean negatively associated random variables is an N -demimartingale. The class of N -demimartingale includes as special case martingales with respect to natural choice of σ -algebras. Furthermore, it can be shown that a supermartingale is an N -demisupermartingale. From the definition of Brownian motion we know that the increments of Brownian motion are independent. The law of iterated logarithm for Brownian motion can be found in [7]. The motivation of this study was a question whether the result for Brownian motion can be extended, under some additional conditions, to the case of N -demimartingales. In this paper, we prove the law of iterated logarithm for N -demimartingale. As an application of main results we obtain the law of iterated logarithm for randomly weighted partial sums of zero mean negatively associated random variables and the counterpart of the Hardy–Littlewood law of iterated logarithm for empirical distributions of negatively associated random variables.

In this work, we set $\lg x = \ln(\max\{x, e\})$ and $\lg_2 x = \lg(\lg x)$ and we denote by C a generic positive number that is not necessarily the same at each appearance.

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2. Results.

Theorem 1. Let $\{S_n, n \in \mathbf{N}\}$ (with $S_0 \equiv 0$) be an N -demimartingale such that for positive numbers $c_i, i = 1, 2, \dots$,

$$|S_i - S_{i-1}| \leq c_i < \infty.$$

Suppose that $\Lambda = \sum_{i=1}^{\infty} c_i^2 < \infty$, $\mathbf{E} S_n = 0$ and $\mathbf{E} S_n^2 = \sigma^2 > 0$ for all $n \in \mathbf{N}$. Then

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{n \lg_2 n}} \leq 2\sqrt{2}\Lambda\sigma \quad \text{a.s.}$$

In the proof of Theorem 1 we use the following exponential inequality for N -demimartingales, which was shown in the proof of Theorem 6 in [2].

Lemma 1. Let $\{S_n, n \in \mathbf{N}\}$ (with $S_0 \equiv 0$) be an N -demimartingale with $\mathbf{E} S_1 \leq 0$ and assume that

$$|S_i - S_{i-1}| \leq c_i < \infty,$$

where $c_1, c_2, \dots$ are positive numbers. Then for any $t > 0$ and every $\varepsilon > 0$

$$\begin{aligned} \mathbf{P}(S_n > n\varepsilon) &\leq \exp\left(-tn\varepsilon + \frac{t^2 \sum_{i=1}^n c_i}{2}\right), \\ \mathbf{P}(|S_n| > n\varepsilon) &\leq 2 \exp\left(-tn\varepsilon + \frac{t^2 \sum_{i=1}^n c_i}{2}\right). \end{aligned}$$

Proof of Theorem 1. Let $Y_n = S_n/\sigma, n = 1, 2, \dots$. In order to prove the theorem we first show that for arbitrarily small $\varepsilon > 0$, with probability 1 only at most finitely many events $Y_n > 2(1 + \varepsilon)\sqrt{2\Lambda n \lg_2 n}$ occur.

Assume that λ is equal to $\sqrt{1 + \varepsilon}$, set $n_k = [\lambda^k]$, where $[\cdot]$ is the integer part, and let k_0 be so large that for $k > k_0$

$$n_{k+1} > n_k + 1, \quad 1 < \frac{n_{k+1}}{n_k} < \lambda^2, \quad 1 < \frac{n_{k+1} \lg_2 n_{k+1}}{n_k \lg_2 n_k} < \lambda^3. \quad (1)$$

Let $\{Y_{n_k}, k \in \mathbf{N}\}$ be a subsequence of $\{Y_n, n \in \mathbf{N}\}$. Applying Lemma 1 with $t = \sqrt{(2 \lg_2 n_k)/(\Lambda n_k)}$, we obtain

$$\begin{aligned} \mathbf{P}\left(|Y_{n_k}| > 2(1 + \varepsilon)\sqrt{2\Lambda n_k \lg_2 n_k}\right) &\leq 2 \exp\left(-2(1 + \varepsilon) \lg_2 n_k + \frac{\lg_2 n_k \sum_{i=1}^{n_k} c_i^2}{n_k \Lambda}\right) \\ &\leq 2 \exp\left(-2(1 + \varepsilon) \lg_2 n_k + \frac{\lg_2 n_k}{n_k}\right) \\ &\leq 2 \exp(-2(1 + \varepsilon) \lg_2 n_k + 1) \leq \frac{C}{k^{2(1 + \varepsilon)}}. \end{aligned}$$

Thus, the series $\sum_{k=1}^{\infty} \mathbf{P}(|Y_{n_k}| > 2(1 + \varepsilon)\sqrt{2\Lambda n_k \lg_2 n_k})$ is convergent for every $\varepsilon > 0$ and according to the Borel–Cantelli lemma

$$\limsup_{n \rightarrow \infty} \frac{Y_{n_k}}{\sqrt{2n_k \lg_2 n_k}} \leq 2\Lambda \quad \text{a.s.}$$

In order to prove that the whole sequence behaves properly, let us denote by $A_k(\varepsilon)$ the event

$$\max_{n_k \leq j \leq n_{k+1}} |Y_j| > 2(1 + \varepsilon)\sqrt{2\Lambda n_k \lg_2 n_k}.$$

By the Borel–Cantelli lemma, with probability 1, at most finitely many of the events $A_k(\varepsilon)$ will occur, once we show that

$$\sum_{k=1}^{\infty} \mathbf{P}(A_k(\varepsilon)) < \infty$$

for arbitrarily small $\varepsilon > 0$.

Now, by (1), [3, Corollary 8], and the Cauchy–Schwarz inequality, for $k > k_0$ we have

$$\begin{aligned} \mathbf{P}(A_k(\varepsilon)) &\leq \mathbf{P}\left(\max_{1 \leq j \leq n_{k+1}} |Y_j| > 2(1+\varepsilon)\sqrt{2\Lambda n_k \lg_2 n_k}\right) \\ &\leq \mathbf{P}\left(\max_{1 \leq j \leq n_{k+1}} |Y_j| > 2\lambda^{-3/2}(1+\varepsilon)\sqrt{2\Lambda n_{k+1} \lg_2 n_{k+1}}\right) \\ &\leq \frac{C(\mathbf{E}|Y_{n_{k+1}}|^2)^{1/2}}{\sqrt{n_{k+1} \lg_2 n_{k+1}}} \left(\mathbf{P}\left(|Y_{n_{k+1}}| > (1+\varepsilon)^{1/4}\sqrt{2\Lambda n_{k+1} \lg_2 n_{k+1}}\right)\right)^{1/2}. \end{aligned}$$

By the assumptions, $\mathbf{E}|Y_{n_{k+1}}|^2 = 1$, and for sufficiently large k , we have $n_{k+1} \geq \lg_2 n_{k+1}$. Hence,

$$\begin{aligned} \mathbf{P}(A_k(\varepsilon)) &\leq \frac{C}{\lg_2 n_{k+1}} \left(\mathbf{P}\left(|Y_{n_{k+1}}| > (1+\varepsilon)^{1/4}\sqrt{2\Lambda n_{k+1} \lg_2 n_{k+1}}\right)\right)^{1/2} \\ &\leq \frac{C}{\lg k} \left(\mathbf{P}\left(|Y_{n_{k+1}}| > (1+\varepsilon)^{1/4}\sqrt{2\Lambda n_{k+1} \lg_2 n_{k+1}}\right)\right)^{1/2}. \end{aligned}$$

By Lemma 1 and (1) we get

$$\begin{aligned} \mathbf{P}(A_k(\varepsilon)) &\leq \frac{C}{\lg k} \exp\left(-\frac{t(1+\varepsilon)^{1/4}\sqrt{2\Lambda n_{k+1} \lg_2 n_{k+1}}}{2} + \frac{t^2 \sum_{i=1}^{n_{k+1}} c_i^2}{4}\right) \\ &\leq \frac{C}{\lg k} \exp\left(-\frac{t(1+\varepsilon)^{1/4}\sqrt{2\Lambda n_k \lg_2 n_k}}{2} + \frac{t^2 \Lambda}{4}\right). \end{aligned} \quad (2)$$

When

$$t = (1+\varepsilon)^{3/4} \left(\sqrt{\frac{2\lg_2 n_k}{\Lambda n_k}} + \frac{\lg_2 k}{\sqrt{2\Lambda n_k \lg_2 n_k}}\right),$$

we obtain

$$\begin{aligned} \frac{t^2 \Lambda}{4} &= \frac{(1+\varepsilon)^{3/2}(2\lg_2 n_k + \lg_2 k)^2}{8n_k \lg_2 n_k} \leq \frac{(1+\varepsilon)^{3/2}(2\lg_2 n_k + \lg_2 k)^2}{8(\lg_2 n_k)^2} \\ &= \frac{(1+\varepsilon)^{3/2}}{2} \left(\frac{2\lg_2 n_k + \lg_2 k}{2\lg_2 n_k}\right)^2 \leq \frac{(1+\varepsilon)^{3/2}}{2} \left(1 + \frac{\lg_2 k}{2\lg_2 n_k}\right)^2 \\ &\leq 2(1+\varepsilon)^{3/2}, \end{aligned} \quad (3)$$

since $\lg k \geq \lg_2 k$. Hence by (3), we obtain in (2) that

$$\begin{aligned} \mathbf{P}(A_k(\varepsilon)) &\leq \frac{C}{\lg k} \exp\left(-(1+\varepsilon)\lg(k \lg \lambda) - \frac{1+\varepsilon}{2} \lg_2 k\right) \\ &= \frac{C}{k^{1+\varepsilon}(\lg k)^{(3+\varepsilon)/2}}. \end{aligned}$$

Thus, the series $\sum_{k=1}^{\infty} \mathbf{P}(A_k(\varepsilon))$ is convergent for every $\varepsilon > 0$ and according to the Borel–Cantelli lemma,

$$\limsup_{n \rightarrow \infty} \frac{Y_n}{\sqrt{2n \lg_2 n}} \leq 2\Lambda \quad \text{a.s.}$$

Because the collection $\{-S_n: n \in \mathbf{N}\}$ is also N -demimartingale, using similar arguments one can show that

$$\liminf_{n \rightarrow \infty} \frac{Y_n}{\sqrt{2n \lg_2 n}} \geq -2\Lambda \quad \text{a.s.}$$

This proves the theorem.

In case when $\{S_n, n \in \mathbf{N}\}$ is an N -demimartingale with uniformly bounded differences, i.e., when $|S_i - S_{i-1}| \leq \alpha < \infty$ for $i = 1, 2, \dots$, we can prove the following proposition.

Proposition 1. *Let $\{S_n, n \in \mathbf{N}\}$ (with $S_0 \equiv 0$) be an N -demimartingale with uniformly bounded differences, i.e., for $i = 1, 2, \dots$*

$$|S_i - S_{i-1}| \leq \alpha < \infty.$$

Suppose that $\mathbf{E} S_n = 0$ for all $n \in \mathbf{N}$. Then

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{2n \lg n}} \leq \alpha \quad a.s.$$

P r o o f. Let $a_n = \alpha(1 + \varepsilon)^{1/2} \sqrt{2n \lg n}$, where $\varepsilon > 0$ is arbitrarily small. Then, by Theorem 6 in [2] we obtain

$$\begin{aligned} \mathbf{P}(S_n > a_n) &\leq \exp\left(-\frac{2(1 + \varepsilon)\alpha^2 n \lg n}{2\alpha^2 n}\right) \\ &\leq \exp(-(1 + \varepsilon) \lg n) = n^{-(1 + \varepsilon)}. \end{aligned}$$

Thus, for every $\varepsilon > 0$ the right-hand side converges. Hence, by the Borel–Cantelli lemma it follows that with probability 1 only finitely many events $A_n(\varepsilon) = \{S_n > a_n\}$ occur, and

$$\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \lg n}} \leq \alpha \quad a.s.$$

Because the collection $\{-S_n: n \in \mathbf{N}\}$ is also an N -demartingale, using similar arguments one can show that

$$\liminf_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \lg n}} \geq -\alpha \quad a.s.$$

Proposition 1 is proved.

3. Applications. Given that the randomly weighted partial sums of zero mean negatively associated random variables form an N -demimartingale (see [3, Proposition 9]), we have the following corollary.

Corollary 1. *Let $\{X_i, i \in \mathbf{N}\}$ (with $X_0 \equiv 0$) be zero mean sequence of negatively associated random variables with common standard deviation σ and such that $|X_i| \leq c_i$ for all $i \geq 1$. Let $\{Y_i: i \in \mathbf{N}\}$ be a sequence of nonnegative, independent, identically distributed, and uniformly bounded random variables such that Y_i is independent of X_i for all $i \geq 1$. Suppose that $\Lambda = \sum_{i=1}^{\infty} c_i^2 < \infty$ and $\mathbf{E}|Y_1|^2 < \infty$. Then*

$$\limsup_{n \rightarrow \infty} \left| \frac{\sum_{i=1}^n Y_i X_i}{\sqrt{n \lg_2 n}} \right| \leq 2\sqrt{2M\Lambda} \sigma (\mathbf{E}|Y_1|^2)^{1/2} \quad a.s.$$

P r o o f. From Proposition 9 in [3], it follows that sequence of randomly weighted partial sums $S_n = \sum_{i=1}^n Y_i X_i$ of zero mean negatively associated random variables is an N -demimartingale. By the assumptions,

$$|S_i - S_{i-1}| = |X_i Y_i| \leq M c_i \quad \text{for all } i \geq 1.$$

Thus by Theorem 1 we get the assertion.

Corollary 2. *Let $\{X_i, i \in \mathbf{N}\}$ be a sequence of negatively associated random variables with common distribution F . For a given $x \in \mathbf{R}$, put*

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x).$$

Then

$$\limsup_{n \rightarrow \infty} \left| \frac{F_n(x) - F(x)}{\sqrt{(2 \lg n)/n}} \right| \leq 1 \quad a.s.$$

P r o o f. By a fundamental property of negatively associated random variables, for a given $x \in \mathbf{R}$, sequence $\{I(X_n \leq x), n \in \mathbf{N}\}$ is also negatively associated. Furthermore, we have that

$$|S_i - S_{i-1}| = |I(X_i \leq x) - F(x)| \leq 1 \quad \text{for all } i \geq 1.$$

Thus, by Proposition 1 we get the assertion.

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