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ON ASYMPTOTIC EXPANSION AND CLT OF LINEAR EIGENVALUE STATISTICS FOR SAMPLE COVARIANCE MATRICES WHEN $N/M \rightarrow 0$ ¹⁾

В работе изучается перенормированная вещественная выборочная ковариационная матрица $H = X^T X / \sqrt{MN} - \sqrt{M/N}$, где $N/M \rightarrow 0$ при $N, M \rightarrow \infty$. Мы всегда предполагаем, что $M = M(N)$. Здесь $X = [X_{jk}]_{M \times N}$ есть вещественная случайная $M \times N$ -матрица с независимыми одинаково распределенными элементами и мы предполагаем, что $\mathbf{E}|X_{11}|^{5+\delta} < \infty$ при некотором малом положительном δ . Рассматривается преобразование Стилтеса $m_N(z) = N^{-1} \text{Tr}(H - z)^{-1}$ и линейная статистика собственных значений матрицы H . Основное внимание уделяется асимптотическому разложению функции $\mathbf{E}\{m_N(z)\}$. Затем для некоторой хорошей тестовой функции устанавливается центральная предельная теорема. Мы показываем, что дисперсия предельного нормального распределения такая же, как в случае вещественной матрицы Вигнера с гауссовскими элементами.

Ключевые слова и фразы: выборочная ковариационная матрица, преобразование Стилтеса, асимптотическое разложение, линейная статистика собственных значений.

1. Introduction. As an important branch of the random matrix theory, the study towards sample covariance matrix traces back to the work of Hsu [8] and Wishart [17]. The modern formulation of a large dimensional sample covariance matrix always indicates the matrix in the form of $S_N = (1/M)X^T X$, where $X = [X_{jk}]_{M \times N}$ is an $M \times N$ real random matrix with mean zero and variance σ^2 i.i.d. entries and $N/M \rightarrow y \in (0, \infty)$ as $N, M := M(N)$ tend to infinity. So usually the sample size M and parameter number N are assumed to be with the same order. For a symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_N$, we denote its empirical spectral distribution by $F_N(x) = (1/N) \sum_{i=1}^N \mathbf{1}_{\{\lambda_i \leq x\}}$. And the limit of $F_N(x)$ as $N \rightarrow \infty$ is often referred to as limiting spectral distribution. In the case of S_N , it is well known that Marchenko and Pastur [13] first found: as N tends to

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infinity $F_N(x)$ almost surely converges to the so-called Marchenko–Pastur (MP) law with the density function

$$\rho_y(x) = \frac{1}{2\pi xy\sigma^2} \sqrt{(b-x)(x-a)} \mathbf{1}_{\{a \leq x \leq b\}},$$

and there is a point mass $1-1/y$ at the origin if $y > 1$, where $a = \sigma^2(1-\sqrt{y})^2$ and $b = \sigma^2(1+\sqrt{y})^2$.

While in modern statistics, the case of $N/M \rightarrow 0$ as both N and M tend to infinity is also very common, see [7] for example. In this paper, we will focus on the theoretical aspect of such a particular case of the sample covariance matrix. It is easy to see when $N/M \rightarrow y = 0$, the interval $[a, b]$ will shrink to the point σ^2 , so every eigenvalue of S will tend to σ^2 then. If we centralize S by subtracting $\sigma^2 I_N$ and multiplying it with $\sqrt{M/N}$, the range of the eigenvalues will be enlarged to be order 1 typically. In fact, under the assumption $N/M \rightarrow 0$ as $N \rightarrow \infty$, it was understood long time ago that the matrix

$$H_N := \sqrt{\frac{M}{N}} \left(\frac{1}{M} X^T X - \sigma^2 I_N \right) \quad (1.1)$$

behaves similarly to a Wigner matrix of dimension N on many spectral properties.

A real Wigner matrix can be defined as a real symmetric random matrix W_N with mean zero, finite variance i.i.d. diagonal entries and mean zero, variance ω^2/N i.i.d. above-diagonal entries, and all entries on or above diagonal are independent. As a cornerstone in the random matrix theory, the so-called semicircle law as the limiting spectral distribution of W_N was first raised by Wigner [16], with the density given by

$$\rho_{sc}(x) = \frac{1}{2\pi\omega^2} \sqrt{4\omega^2 - x^2} \mathbf{1}_{\{|x| \leq 2\omega\}}. \quad (1.2)$$

It is not difficult to see, when we fix N and only let M tend to infinity, that H_N will tend to a Wigner matrix with Gaussian entries under the effect of the central limit theorem. Thus if we let both N and M tend to infinity with $N/M \rightarrow 0$, it is natural to ask whether $F_N(x)$ of H_N tend to the semicircle law as well. A rigorous proof was given by Bai and Yin [5] through a moment method: under the assumption of the existence of 4-th moment of X_{jk} , $F_N(x)$ converges almost surely to the semicircle law (1.2) with $\omega = \sigma^2$. So a question naturally arises, how close are the spectral properties of the Wigner matrix W_N and the renormalized sample covariance matrix H_N when we set $\omega = \sigma^2$? However, such a question makes no sense if we do not specify some particular spectrum statistics to compare. In this paper, we will study some representative objects for H_N , like Stieltjes transform of $F_N(x)$ and

linear eigenvalue statistics of H_N . Then compare them to the corresponding results known for the Wigner matrix W_N .

The Stieltjes transform of $F_N(x)$ is defined by

$$m_N(z) = \int \frac{1}{x - z} dF_N(x)$$

for any $z = E + i\eta$ with $E \in \mathbf{R}$ and $\eta > 0$. We just consider fixed z , and specify $F_N(x)$ as the empirical spectral distribution of H_N throughout the paper. If we introduce the Green function $G_N(z) = (H_N - z)^{-1}$, we also have

$$m_N(z) = \frac{1}{N} \operatorname{Tr} G_N(z) = \frac{1}{N} \sum_{j=1}^N G_{jj}.$$

Here G_{jk} denotes the (j, k) entry of $G_N(z)$. Both as a powerful tool and as a relevant spectral statistic, the Stieltjes transform of the empirical spectral distribution has shown to be particularly important in the random matrix theory. As is well known, the convergence of probability measure sequence is equivalent to the convergence of its Stieltjes transform sequence towards the corresponding transform of the limiting measure (for example, see [4]). So under the assumption $\mathbf{E}\{X_{11}^4\} < \infty$, one has when $N \rightarrow \infty$, $m_N(z)$ almost surely converges to the Stieltjes transform $f(z)$ of the semicircle law given by

$$f(z) = \frac{-z + \sqrt{z^2 - 4\sigma^4}}{2\sigma^2}. \quad (1.3)$$

Here the square root is specified to be the one with positive imaginary part. What is more, due to the basic fact $|m_N(z)| \leq \eta^{-1}$, the a.s. convergence of $m_N(z)$ also implies

$$f_N(z) := \mathbf{E}\{m_N(z)\} = f(z) + o(1). \quad (1.4)$$

In this paper, we will take a step further to calculate the leading order term of the remainder. A corresponding work for the Wigner matrices has been taken in [10] and extended by some subsequent articles to some other matrix ensembles, see [1], [11]. Such a topic is often referred to as the asymptotic expansion of $f_N(z)$ in the literature. For ease of the expression and calculation, we just consider the normalized case of $\sigma^2 = 1$ hereafter, so the semicircle law in (1.2) will be a standard one, the general case is just analogous. We denote the α -th moment of X_{11} by ω_α and α -th cumulant by κ_α , particularly there are $\kappa_2 = \sigma^2 = 1$ and $\kappa_4 = \omega_4 - 3$ in our setting. Our first main theorem is on the asymptotic expansion of $f_N(z)$.

Theorem 1.1. Consider the matrix model H_N defined in (1.1) with $\sigma^2 = 1$. And $X = [X_{jk}]_{M \times N}$ be an $M \times N$ real random matrix with mean zero and variance 1 i.i.d. entries. We assume $M = M(N)$ and $N/M \rightarrow 0$ as $N \rightarrow \infty$. If $\mathbf{E}|X_{11}|^{5+\delta} < \infty$ with some small positive δ , then the following asymptotic expansion holds for any fixed z with $\eta > 0$:

$$f_N(z) = f(z) \left\{ 1 - \sqrt{\frac{N}{M}} \frac{f^3(z)}{1 - f^2(z)} + \frac{1}{N} \left(\frac{f^2(z)}{(1 - f^2(z))^2} + \kappa_4 \frac{f^2(z)}{1 - f^2(z)} \right) \right\} + o\left(\sqrt{\frac{N}{M}} \vee \frac{1}{N}\right).$$

Remark 1.1. By comparing with the corresponding result for the Wigner matrix in [10], we can find a different first order remainder with order $\sqrt{N/M}$ rather than $1/N$ appears when $M = o(N^3)$. Moreover, we do only require $\eta > 0$ in our result, rather than $\eta \geq 2\omega$ for Wigner matrix case in [10].

Equalities (1.3) and (1.4) show the convergence of $m_N(z)$ a.s. and in expectation respectively, as usual, we shall consider the fluctuation of $m_N(z)$ as a further job. In this paper, we will deal with the more general linear eigenvalue statistic

$$\mathcal{L}_N[\varphi] = \sum_{i=1}^N \varphi(\lambda_i)$$

of H_N with the test function φ satisfying some smooth conditions. Note that $Nm_N(z)$ is a linear eigenvalue statistic with $\varphi(x) = 1/(x - z)$. The topic of CLT for linear eigenvalue statistics is really classical and attractive in the Random Matrix Theory, there are a vast of related articles for different matrix ensembles, for examples, see [2], [3], [9], [12], [14], [15]. However, for the large dimensional sample covariance matrices, there is no corresponding result on the case of $N/M \rightarrow 0$. So as a complement, we will discuss it in this paper. For convenience, we follow the notation in [14] to denote $\xi^\circ = \xi - \mathbf{E}\{\xi\}$ for any random variable ξ below. And introduce the norm

$$\|\varphi\|_s^2 = \int (1 + 2|k|)^{2s} |\widehat{\varphi}(k)|^2 dk, \quad \widehat{\varphi}(k) = \frac{1}{2\pi} \int e^{ikx} \varphi(x) dx.$$

Then we can state our second result as follows.

Theorem 1.2. Consider the matrix model H_N defined in (1.1) with $\sigma^2 = 1$. And $X = [X_{jk}]_{M \times N}$ be an $M \times N$ real random matrix with mean zero and variance 1 i.i.d. entries. We assume $M = M(N)$ and $N/M \rightarrow 0$ as $N \rightarrow \infty$. If $\mathbf{E}|X_{11}|^5 < \infty$, then for any real test function φ satisfying $\|\varphi\|_{3/2+\varepsilon} < \infty$ with any $\varepsilon > 0$, we have $\mathcal{L}_N^\circ[\varphi]$ converges weakly to the

centered Gaussian distribution with variance

$$V[\varphi] = \frac{1}{2\pi^2} \int_{-2}^2 \int_{-2}^2 \left(\frac{\varphi(\lambda_1) - \varphi(\lambda_2)}{\lambda_1 - \lambda_2} \right)^2 \frac{4 - \lambda_1 \lambda_2}{\sqrt{4 - \lambda_1^2} \sqrt{4 - \lambda_2^2}} d\lambda_1 d\lambda_2 \\ + \frac{\kappa_4}{4\pi^2} \left(\int_{-2}^2 \frac{\varphi(\mu)\mu}{\sqrt{4 - \mu^2}} d\mu \right)^2. \quad (1.5)$$

R e m a r k 1.2. Again we can compare it to the corresponding result of the Wigner matrices in [14], the variance coincides with the counterpart for a Gaussian Wigner matrix (Not GOE!) with variance $\omega_4 - 1$ diagonal entries. This can be explained as a result of the central limit theorem and asymptotic independence effect for the matrix entries. We require φ to be real in the above theorem, so when we deal with $m_N(z)$, we need to work with $\operatorname{Re} m_N(z)$ and $\operatorname{Im} m_N(z)$ as in [10].

Our article is organized as follows. We will provide some basic facts and tools in Section 2, and as a warm-up, we use them to revisit the semicircle law for Gaussian case (i.e., X_{11} is Gaussian) in an average sense. In Section 3 we present the proof of Theorem 1.1, which is based on the preliminaries provided in Section 2 together with some extra lemmas, whose proofs are postponed to Section 4. In Section 5, we present the proof of Theorem 1.2 in short and leave all technical details in the Appendix. Throughout the paper, we will use C and C_i ($i = 1, 2, 3, 4$) to denote positive constants which may be different from line to line, sometimes may depend on η .

2. Preliminaries and Gaussian case. In this section we present some basic tools and facts required in the sequel, and use them to revisit the semicircle law proved in [5] in an average sense. For convenience, when there is no confusion, we will get rid of the subscript N in the notation of matrices and z as a variable in the notation $G_N(z)$ below. If we set $Y = [Y_{jk}]_{M \times N} := (MN)^{-1/4} X$, we have $H = Y^T Y - \sqrt{M/N} I_N$. To do the job we state the following two lemmas without detailed but routine proofs.

Lemma 2.1. *For any real-valued random variable ξ with $\mathbf{E}\{|\xi|^{p+2}\} < \infty$ and complex-valued function $g(t)$ with continuous and bounded $p+1$ derivatives, we have*

$$\mathbf{E}\{\xi g(\xi)\} = \sum_{a=0}^p \frac{\kappa_{a+1}}{a!} \mathbf{E}\{g^{(a)}(\xi)\} + \varepsilon, \quad (2.1)$$

where κ_a is the a -th cumulant of ξ , and

$$|\varepsilon| \leq C \sup_t |g^{(p+1)}(t)| \mathbf{E}\{|\xi|^{p+2}\},$$

where the positive constant C depends on p .

R e m a r k 2.1. The proof of Lemma 2.1 can be found in a lot of references, for example, see [10] for details.

When ξ is centered Gaussian, (2.1) reduces to be

$$\mathbf{E} \{ \xi g(\xi) \} = \mathbf{E} \xi^2 \cdot \mathbf{E} \{ g'(\xi) \}. \quad (2.2)$$

Our second lemma is on the derivatives $D_{jk}\{\cdot\}$ with respect to the matrix entry Y_{jk} , which will be used frequently in Sections 2 and 3.

Lemma 2.2. *For any $\alpha, j \in \{1, 2, \dots, M\}$ and $\beta, k \in \{1, 2, \dots, N\}$, we have*

- (i) $D_{jk}G_{\alpha\beta} = -(YG)_{j\alpha}G_{\beta k} - (YG)_{j\beta}G_{\alpha k},$
- (ii) $D_{jk}(YG)_{\alpha\beta} = \delta_{\alpha j}G_{\beta k} - G_{\beta k}(YGY^T)_{j\alpha} - (YG)_{j\beta}(YG)_{\alpha k},$
- (iii) $D_{jk}[G_{kk}(YGY^T)_{jj}] = 2G_{kk}(YG)_{jk} - 4G_{kk}(YG)_{jk}(YGY^T)_{jj},$

where $\delta_{\alpha j}$ in (ii) is the Kronecker delta function.

R e m a r k 2.2.. The proof of Lemma 2.2 is based on the following resolvent identity for real symmetric matrices A and B :

$$(A + B - z)^{-1} - (A - z)^{-1} = -(A + B - z)^{-1}B(A - z)^{-1},$$

which implies

$$\frac{d}{d\varepsilon} (A + \varepsilon B - z)^{-1} \Big|_{\varepsilon=0} = -(A - z)^{-1}B(A - z)^{-1}. \quad (2.3)$$

Now we set $A = H$ and $A + \varepsilon B = (Y + \varepsilon E(j, k))^T(Y + \varepsilon E(j, k))$. Here $E(j, k)$ represents the $M \times N$ matrix with (j, k) -th entry to be 1 and others 0. Then by (2.3) it is not difficult to get (i) of Lemma 2.2. And (ii) and (iii) can be proved by the chain rule. We omit the details here, in fact it is quite similar with the counterparts in [10] and [12].

As a warm-up for the main task in Section 3, we use the above two lemmas to prove $f_N(z)$ tends to $f(z)$ (as $N \rightarrow \infty$) for the Gaussian case below. By the basic relation between a matrix and its Green function

$$G = -z^{-1} + z^{-1}GH,$$

we have

$$\begin{aligned} f_N(z) &= -z^{-1} + z^{-1} \frac{1}{N} \mathbf{E} \{ \text{Tr } GH \} \\ &= -z^{-1} - z^{-1} \sqrt{\frac{M}{N}} f_N(z) + z^{-1} \frac{1}{N} \sum_{j,k} \mathbf{E} \{ Y_{jk}(YG)_{jk} \}. \end{aligned} \quad (2.4)$$

Then we can use the relation (2.2) to (2.4), which yields

$$\begin{aligned}
 f_N(z) &= -z^{-1} - z^{-1} \sqrt{\frac{M}{N}} f_N(z) \\
 &\quad + z^{-1} \frac{1}{N} \frac{1}{\sqrt{MN}} \sum_{j,k} \mathbf{E} \left\{ G_{kk} - ((YG)_{jk})^2 - G_{kk}(YGY^T)_{jj} \right\} \\
 &= -z^{-1} - \frac{z^{-1}}{N} \frac{1}{\sqrt{MN}} \mathbf{E} \operatorname{Tr} YG^2Y^T - \frac{z^{-1}}{\sqrt{MN}} \mathbf{E} \{m_N(z) \operatorname{Tr} YGY^T\} \\
 &= -z^{-1} - z^{-1} \frac{N+1}{\sqrt{MN}} f_N(z) - z^{-1} \left(1 + \sqrt{\frac{N}{M}} z \right) \mathbf{E} \{m_N^2(z)\} \\
 &\quad - z^{-1} \frac{1}{N^2} \left(1 + \sqrt{\frac{N}{M}} z \right) \mathbf{E} \operatorname{Tr} G^2, \tag{2.5}
 \end{aligned}$$

where we have used the fact that

$$\begin{aligned}
 \operatorname{Tr} YGY^T &= \operatorname{Tr} \left(I_N + \left(\sqrt{\frac{M}{N}} + z \right) G \right), \\
 \operatorname{Tr} YG^2Y^T &= \operatorname{Tr} \left(G + \left(\sqrt{\frac{M}{N}} + z \right) G^2 \right).
 \end{aligned}$$

Note that one has the trivial bound $\|G\| \leq \eta^{-1}$ for the matrix norm of G , (2.5) implies

$$f_N(z) = -z^{-1} - z^{-1} \mathbf{E} \{m_N^2(z)\} + O\left(\frac{1}{N} \vee \sqrt{\frac{N}{M}}\right).$$

To estimate $\mathbf{E} \{m_N^2(z)\}$, we need to derive a bound for $\mathbf{D}\{m_N(z)\}$. By using the Poincaré inequality for Gaussian matrix entries as in [12, Proposition 2.4], it is not difficult to get

$$\mathbf{D}\{m_N(z)\} = O\left(\frac{1}{N^2}\right).$$

Consequently one has

$$f_N(z) = -z^{-1} - z^{-1} f_N^2(z) + O\left(\frac{1}{N} \vee \sqrt{\frac{N}{M}}\right). \tag{2.6}$$

On the other hand, it is well known that $f(z)$ satisfies the equation

$$f(z) = -z^{-1} - z^{-1} f^2(z). \tag{2.7}$$

By a routine comparison issue on (2.6) and (2.7) we have $f_N(z) \rightarrow f(z)$ as $N \rightarrow \infty$ (see [4, Section 2] for example), which implies the convergence of the expected empirical spectral distribution to the semicircle law.

However, for Theorem 1.1, we need to take a step further to calculate the leading order term of the remainder $f_N(z) - f(z)$ precisely for the general distribution case. So at first we need to use (2.1) instead of (2.2). As a result, more involved estimate towards the derivatives is required. What is more, the Poincaré inequality is no longer valid, we need to estimate $\mathbf{D}\{m_N(z)\}$ by a martingale difference method as in [14].

3. Asymptotic expansion for $f_N(z)$. To prove Theorem 1.1, we begin with the basic idea raised in [10], use (2.1) to (2.4) and estimate the derivatives. However, the standard process will be too involved for our matrix model H_N . It is shown in [10] that for the Wigner matrix, every term in the expansion formula can be factorized into the entries of $G_N(z)$ which can be bounded by η^{-1} . Owing to such a trivial bound, many estimates hold obviously, especially for the remainder term. And the approach of iterating the expansion process also works well. However as we will see, for our case, the derivatives are in more complicated forms. There are no trivial bound for some factors of the terms (see (3.9)), as well, iterating the expansion process will bring new types of factors. And to bound the remainder, we need to use a truncation technic at first.

Now we truncate X_{jk} at $\tau := (MN)^{1/4-t}$ with some small positive $t \leq \delta/100$ (say). Then we denote the empirical spectral distribution of the matrix defined in (1.1) with truncated entries $X_{jk}^\tau := X_{jk} \mathbf{1}_{\{|X_{jk}| \leq \tau\}}$ by $F_N^\tau(x)$. Further we denote the Stieltjes transform of $F_N^\tau(x)$ by $m_N^\tau(z)$ and set $f_N^\tau(z) = \mathbf{E} m_N^\tau(z)$. By taking into account that $|m_N(z)|, |m_N^\tau(z)| \leq \eta^{-1}$ and

$$\sum_{j,k} \mathbf{P}\{|X_{jk}| \geq \tau\} \leq CMN \cdot \tau^{-(5+\delta)} \leq C(MN)^{-1/4-t} = o\left(\sqrt{\frac{N}{M}} \vee \frac{1}{N}\right),$$

we have

$$|f_N(z) - f_N^\tau(z)| = o\left(\sqrt{\frac{N}{M}} \vee \frac{1}{N}\right).$$

After the truncation, all of the first 5 moments of X_{jk} will be modified. For example,

$$\begin{aligned} \left| \mathbf{E} \{X_{jk} \mathbf{1}_{\{|X_{jk}| \leq \tau\}}\} \right| &= \left| \mathbf{E} \{X_{jk} \mathbf{1}_{\{|X_{jk}| \geq \tau\}}\} \right| \leq \tau^{-(4+\delta)} \mathbf{E} |X_{jk}|^{5+\delta} \\ &\leq C(MN)^{-1-t}. \end{aligned}$$

Similarly we can calculate the modification of the α -th moments of X_{jk} under the truncation for all $\alpha = 2, 3, 4, 5$. So without loss of generality, we can always assume $X_{jk}, (j, k) \in \{1, \dots, M\} \times \{1, \dots, N\}$, are i.i.d. and

$$\begin{aligned} |X_{jk}| &\leq (MN)^{1/4-t}, & |\mathbf{E} \{X_{jk}\}| &\leq C(MN)^{-1-t}, \\ \mathbf{E} \{X_{jk}^2\} &= 1 + C(MN)^{-3/4-t}, & \mathbf{E} \{X_{jk}^\alpha\} &= \omega_\alpha + o(1), \quad \alpha = 3, 4, 5, \\ \mathbf{E} \{|X_{jk}|^{5+\delta}\} &\leq C. \end{aligned} \tag{3.1}$$

And for simplicity of the notation, we still use $f_N(z)$ to denote $f_N^\tau(z)$ below. Moreover, we need to present two lemmas before the rigorous proof of Theorem 1.1. The first lemma is on the estimate of $\mathbf{D}\{m_N(z)\}$, as we have seen in Section 2, it is a necessary ingredient for the final result. The second lemma is on an estimate towards every diagonal entry of the Green function, which can help to overcome the difficulties we have mentioned above. Their proofs will be postponed to Section 4.

Lemma 3.1. *Under the assumptions of $\omega_1 = 0$ and $\omega_4 < \infty$, we have*

$$\mathbf{D}\{m_N(z)\} = O(N^{-2})$$

for any fixed z with $\eta > 0$.

To present the second lemma, we need to introduce the matrix

$$\tilde{G} = [\tilde{G}_{jk}]_{M \times M} := \left(YY^T - \sqrt{\frac{M}{N}} - z \right)^{-1}.$$

Lemma 3.2. *Under the assumptions of Lemma 3.1, for any fixed z with $\eta > 0$ we have the following two estimates:*

(i) for any $k \in \{1, 2, \dots, N\}$

$$\mathbf{E} \left| G_{kk} + \frac{1}{z + f_N(z)} \right|^2 \leq C_1 \frac{1}{N} + C_2 \frac{N}{M}; \quad (3.2)$$

(ii) for any $j \in \{1, 2, \dots, M\}$

$$\mathbf{E} \left| \tilde{G}_{jj} + \frac{1}{\sqrt{M/N} + z + f_N(z)} \right|^2 \leq C_1 \left(\frac{N}{M} \right)^3 + C_2 \frac{N}{M^2}. \quad (3.3)$$

Proof of Theorem 1.1. Use (2.1) to (2.4), we have the following expansion for the general distribution case:

$$\begin{aligned} f_N(z) &= -z^{-1} - z^{-1} \sqrt{\frac{M}{N}} f_N(z) \\ &+ z^{-1} \frac{1}{N} \sum_{a=0}^3 \frac{1}{(MN)^{(a+1)/4}} \sum_{j,k} \frac{\kappa_{a+1}}{a!} \mathbf{E} D_{jk}^a(YG)_{jk} + R_N, \end{aligned} \quad (3.4)$$

where $D_{\alpha\beta}^a\{\cdot\}$ represents the a -th derivative with respect to $Y_{\alpha\beta}$ and

$$R_N \leq C \frac{1}{N} \frac{1}{(MN)^{5/4}} \sum_{j,k} \sup_{jk} \mathbf{E} |D_{jk}^4(YG)_{jk}|.$$

Observe that in (3.4), the $a = 0$ term is

$$z^{-1} \frac{1}{N} \frac{1}{(MN)^{1/4}} \mathbf{E} \{X_{11}\} \sum_{jk} \mathbf{E} (YG)_{jk} = o\left(\sqrt{\frac{M}{N}} \vee \frac{1}{N}\right)$$

by taking into account (3.1) and the estimation (3.13) proved below. So we will focus on the estimations of $a \geq 1$ terms and R_N in the sequel. Here $\sup_{jk}$ means the supremum is taken with respect to Y_{jk} and E_{jk} represents the conditional expectation with respect to Y_{jk} . By using Lemma 2.2 repeatedly we have

$$\begin{aligned} D_{jk}(YG)_{jk} &= G_{kk} - [(YG)_{jk}]^2 - G_{kk}(YGY^T)_{jj}, \\ D_{jk}^2(YG)_{jk} &= -6G_{kk}(YG)_{jk} + 6G_{kk}(YG)_{jk}(YGY^T)_{jj} + 2[(YG)_{jk}]^3, \end{aligned} \quad (3.5)$$

$$\begin{aligned} D_{jk}^3(YG)_{jk} &= -6(G_{kk})^2 + 36G_{kk}[(YG)_{jk}]^2 \\ &\quad + 12(G_{kk})^2(YGY^T)_{jj} - 36G_{kk}[(YG)_{jk}]^2(YGY^T)_{jj} \\ &\quad - 6(G_{kk})^2[(YGY^T)_{jj}]^2 - 6[(YG)_{jk}]^4, \end{aligned} \quad (3.6)$$

$$\begin{aligned} D_{jk}^4(YG)_{jk} &= 120[G_{kk}]^2(YG)_{jk} - 240G_{kk}[(YG)_{jk}]^3 \\ &\quad - 240[G_{kk}]^2(YG)_{jk}(YGY^T)_{jj} + 240G_{kk}[(YG)_{jk}]^3(YGY^T)_{jj} \\ &\quad + 120[G_{kk}]^2(YG)_{jk}[(YGY^T)_{jj}]^2 + 24[(YG)_{jk}]^5 \end{aligned}$$

for any $(j, k) \in \{1, \dots, M\} \times \{1, \dots, N\}$. Observe that for any integer $a \geq 1$, $D_{jk}^a(YG)_{jk}$ is a linear combination of terms in the form of

$$F_{jk}(a_1, a_2, a_3) := [G_{kk}]^{a_1} [(YG)_{jk}]^{a_2} [(YGY^T)_{jj}]^{a_3} \quad (3.7)$$

with nonnegative integers a_1, a_2, a_3 satisfying $a_1 + a_2 + a_3 \leq a + 1$. Moreover, by Lemma 2.2, it is easy to see one $(YGY^T)_{jj}$ must appears together with one G_{kk} , thus we always have $a_1 \geq a_3$. By observing that

$$YGY^T = \tilde{G}YY^T = I_M + \left(\sqrt{\frac{M}{N}} + z \right) \tilde{G} \quad (3.8)$$

and the trivial fact $\|\tilde{G}\| \leq \eta^{-1}$ we have

$$|(YGY^T)_{jj}| \leq C \sqrt{\frac{M}{N}}.$$

Similarly we have the following estimate:

$$|(YG)_{jk}| \leq \left(\sum_k |(YG)_{jk}|^2 \right)^{1/2} = |(YGG^*Y^T)_{jj}|^{1/2} \leq C \left(\frac{M}{N} \right)^{1/4}. \quad (3.9)$$

But these two bounds are too bad, thus we need to figure out the higher order term R_N more carefully. Fortunately, when we take expectations in (3.4), we can use the following estimate:

$$\mathbf{E} \left| (YGY^T)_{jj} - \frac{f_N(z)}{\sqrt{M/N} + z + f_N(z)} \right|^2 \leq C_1 \left(\frac{N}{M} \right)^2 + C_2 \frac{1}{M}, \quad (3.10)$$

which is a direct consequence of (3.3) and (3.8).

To deal with R_N , we observe that

$$\begin{aligned}
 |F_{jk}(a_1, a_2, a_3)| &\leq C |(YG)_{jk}|^{a_2} |(YGY^T)_{jj}|^{a_3} \\
 &= C \left| \sum_{\alpha} Y_{j\alpha} G_{\alpha k} \right|^{a_2} \left| \sum_{\beta, \gamma} Y_{j\beta} Y_{j\gamma} G_{\beta\gamma} \right|^{a_3} \\
 &\leq \left(\left(\sum_{\alpha} Y_{j\alpha}^2 \right) \left(\sum_{\alpha} |G_{\alpha j}|^2 \right) \right)^{a_2/2} \\
 &\quad \times \left(\left(\sum_{\beta, \gamma} Y_{j\beta}^2 Y_{j\gamma}^2 \right) \left(\sum_{\beta, \gamma} |G_{\beta\gamma}|^2 \right) \right)^{a_3/2} \\
 &\leq CN^{a_3/2} \left(\sum_{\alpha} Y_{j\alpha}^2 \right)^{(a_2+2a_3)/2}. \tag{3.11}
 \end{aligned}$$

In the second inequality we used Cauchy–Schwarz inequality and in the last step we used the fact that

$$\begin{aligned}
 \sum_{\alpha} |G_{\alpha j}|^2 &= [G^*G]_{jj} \leq C, \quad \sum_{\beta, \gamma} |G_{\beta\gamma}|^2 = \text{Tr } G^*G = CN, \\
 \sum_{\beta, \gamma} Y_{j\beta}^2 Y_{j\gamma}^2 &= \left(\sum_{\alpha} Y_{j\alpha}^2 \right)^2.
 \end{aligned}$$

So by (3.11) we have

$$\begin{aligned}
 \sup_{j,k} \mathbf{E}_{jk} |F_{jk}(a_1, a_2, a_3)| &\leq CN^{a_3/2} \left(\sup_{j,k} |Y_{jk}|^{a_2+2a_3} + \mathbf{E} \left(\sum_{\alpha \neq k} Y_{j\alpha}^2 \right)^{(a_2+2a_3)/2} \right) \\
 &\leq CN^{a_3/2} \cdot (MN)^{-t(a_2+2a_3)} + CN^{a_3/2} \cdot N^{(a_2+2a_3)/2} (MN)^{-(a_2+2a_3)/4}.
 \end{aligned}$$

Then it is not difficult to see $|R_N| = o(\sqrt{N/M} \vee (1/N))$ by taking into account that $a_2 + 2a_3 \leq 5$.

Now we can turn our attention to (3.5) and (3.6). Firstly, we shall provide some coarse estimate on $\sum_{j,k} |(YG)_{jk}|^{a_2}$. When $a_2 \geq 2$ we have

$$\begin{aligned}
 \left| \sum_{j,k} [(YG)_{jk}]^{a_2} \right| &\leq \sum_{j,k} [(G^*Y^T)_{kj} (YG)_{jk}]^{a_2/2} \leq \sum_k [(G^*Y^T YG)_{kk}]^{a_2/2} \\
 &= \sum_k \left[G_{kk} + \left(\sqrt{\frac{M}{N}} + \bar{z} \right) (G^*G)_{kk} \right]^{a_2/2} \\
 &= O(M^{a_2/4} N^{1-a_2/4}). \tag{3.12}
 \end{aligned}$$

When $a_2 = 1$, we can use the elementary inequality

$$\left| \sum_{j,k} (YG)_{jk} \right| \leq \left(MN \sum_{j,k} |(YG)_{jk}|^2 \right)^{1/2} = O((MN)^{3/4}). \tag{3.13}$$

And for this time we only need to bound the expectations, so (3.10) can be used. By Lemma 3.2, (3.10), (3.12), and (3.13), we can discard all the terms except for $-6F_{jk}(1, 1, 0)$, $-6F_{jk}(1, 0, 0)$ from (3.5), (3.6), respectively. For $F_{jk}(1, 1, 0)$, according to Lemma 3.2 and (3.13), it suffices to estimate $\sum_{j,k} \mathbf{E}\{(YG)_{jk}\}$ instead. To get a lower bound than the coarse one (3.13), we need to iterate the expansion process, the main term of $\sum_{j,k} \mathbf{E}\{(YG)_{jk}\}$ is enough for us. We do it as follows:

$$\begin{aligned} \mathbf{E} \sum_{j,k} (YG)_{jk} &= \sum_{j,\alpha,k} \mathbf{E}(Y_{j\alpha} G_{\alpha k}) \\ &= \sum_{j,\alpha,k} \left[\frac{1}{\sqrt{MN}} \mathbf{E}(D_{j\alpha} G_{\alpha k}) + \frac{\kappa_3}{2} \frac{1}{(MN)^{3/4}} \mathbf{E}(D_{j\alpha}^2 G_{\alpha k}) \right] + \varepsilon_N, \end{aligned} \quad (3.14)$$

where

$$|\varepsilon_N| \leq C \frac{\omega_4}{MN} \sum_{j,\alpha,k} \sup_{j\alpha} \mathbf{E}_{j\alpha} |D_{j\alpha}^3 G_{\alpha k}|.$$

By using Lemma 2.2 again we can get

$$D_{j\alpha} G_{\alpha k} = -(YG)_{j\alpha} G_{\alpha k} - (YG)_{jk} G_{\alpha\alpha}, \quad (3.15)$$

$$\begin{aligned} D_{j\alpha}^2 G_{\alpha k} &= -2G_{\alpha\alpha} G_{\alpha k} + 2G_{\alpha k} [(YG)_{j\alpha}]^2 \\ &\quad + 2G_{\alpha\alpha} G_{\alpha k} (YGY^T)_{jj} + 4G_{\alpha\alpha} (YG)_{j\alpha} (YG)_{jk}, \end{aligned} \quad (3.16)$$

$$\begin{aligned} D_{j\alpha}^3 G_{\alpha k} &= 18G_{\alpha\alpha} G_{\alpha k} (YG)_{j\alpha} + 6G_{\alpha\alpha}^2 (YG)_{jk} \\ &\quad - 18G_{\alpha\alpha} G_{\alpha k} (YG)_{j\alpha} (YGY^T)_{jj} - 6G_{\alpha k} [(YG)_{j\alpha}]^3 \\ &\quad - 18G_{\alpha\alpha} (YG)_{jk} [(YG)_{j\alpha}]^2 - 6[G_{\alpha\alpha}]^2 (YG)_{jk} (YGY^T)_{jj}. \end{aligned}$$

Inserting (3.15) and (3.16) into (3.14) one finds that

$$\begin{aligned} \mathbf{E} \sum_{j,k} (YG)_{jk} &= -\frac{1}{\sqrt{MN}} \sum_{j,k} \mathbf{E} [(YG^2)_{jk} + (YG)_{jk} \text{Tr } G] \\ &\quad + \frac{\kappa_3}{(MN)^{3/4}} \sum_{\alpha,k} \mathbf{E} [-MG_{\alpha\alpha} G_{\alpha k} + G_{\alpha k} (GY^T YG)_{\alpha\alpha} \\ &\quad \times G_{\alpha\alpha} G_{\alpha k} \text{Tr } (YGY^T) \\ &\quad + 2(GY^T YG)_{\alpha k} G_{\alpha\alpha}] + \varepsilon_N. \end{aligned}$$

Similarly to (3.13) we can provide

$$\mathbf{E} \sum_{j,k} (YG^2)_{jk} = O((MN)^{3/4}).$$

The other terms in the expansion are easier to estimate by a similar calculation as we have done above. Moreover, similar to the case of R_N it

is not difficult to check that ε_N has no contribution to the main term of $\sum_{j,k} \mathbf{E} \{(YG)_{jk}\}$. So we can get a bound as

$$\mathbf{E} \sum_{j,k} (YG)_{jk} = O(M^{1/4} N^{5/4}).$$

Therefore, under the estimates above, we arrive at

$$\begin{aligned} f_N(z) &= -z^{-1} - z^{-1} \frac{1}{N} \mathbf{E} \sum_{j,k} [(YG)_{jk}]^2 \\ &\quad - z^{-1} \frac{1}{N} \mathbf{E} \sum_{j,k} G_{kk} (YGY^T)_{jj} - z^{-1} \kappa_4 \frac{1}{N^2} \mathbf{E} \sum_k G_{kk}^2 + o\left(\frac{1}{N}\right). \end{aligned}$$

By (2.5) and Lemma 3.1, we can rewrite it as

$$\begin{aligned} f_N(z) &= -z^{-1} - z^{-1} \sqrt{\frac{N}{M}} f_N(z) - z^{-1} \left(1 + \sqrt{\frac{N}{M}} z\right) f_N^2(z) \\ &\quad - z^{-1} \frac{1}{N^2} \mathbf{E} \{\text{Tr} G^2(z)\} - z^{-1} \kappa_4 \frac{1}{N^2} \mathbf{E} \sum_k G_{kk}^2 + o\left(\frac{1}{N}\right), \end{aligned}$$

which implies

$$\left| f_N(z) + \frac{1}{z + f_N(z)} \right| = O\left(\frac{1}{N} \vee \sqrt{\frac{N}{M}}\right). \quad (3.17)$$

Furthermore, we know both $f_N(z)$ and $f(z)$ are analytic function of z . Then by (1.4) and the Cauchy formula, we also have

$$\frac{d}{dz} f_N(z) = \frac{d}{dz} f(z) + o(1).$$

Now we note that

$$\frac{1}{N} \mathbf{E} \text{Tr} G^2(z) = \frac{d}{dz} f_N(z) = \frac{d}{dz} f(z) + o(1) = -\frac{f(z)}{z + 2f(z)} + o(1).$$

Together with (3.2) we finally have

$$\begin{aligned} f_N(z) &= f(z) - \sqrt{\frac{N}{M}} \frac{f(z) + z f^2(z)}{z + 2f(z)} \\ &\quad + \frac{1}{N} \left[\frac{f(z)}{(z + 2f(z))^2} - \frac{\kappa_4 f^2(z)}{z + 2f(z)} \right] + O\left(\frac{N}{M}\right) + o\left(\frac{1}{N}\right). \end{aligned}$$

This concludes the proof by inserting the basic relation $z + 2f(z) = f(z) - 1/f(z)$. Theorem 1.1 is proved.

4. Variance estimates. In this section, we will state the proofs of Lemmas 3.1 and 3.2. In fact, we can use the asymptotic expansion method again to estimate $\mathbf{D}\{m_N(z)\}$, but it is really more complicated than the Wigner matrix case. We will use a martingale difference method used in [14] very recently. For convenience we introduce the notation γ_N to replace $\text{Tr } G$ below.

P r o o f o f L e m m a 3.1. If we denote by $\mathbf{E}_{\leq k}\{\cdot\}$ and $\mathbf{E}_k\{\cdot\}$ the expectation with respect to the random variables of the first k columns and k -th column of Y_N , respectively, by the classical martingale method in [6], one has

$$\begin{aligned}\mathbf{D}\{\gamma_N\} &= \sum_{k=1}^N \mathbf{E} \left\{ |\mathbf{E}_{\leq k-1}\{\gamma_N\} - \mathbf{E}_{\leq k}\{\gamma_N\}|^2 \right\} \\ &= \sum_{k=1}^N \mathbf{E} \left\{ |\mathbf{E}_{\leq k-1}\{\gamma_N - \mathbf{E}_k\{\gamma_N\}\}|^2 \right\} \\ &\leq \sum_{k=1}^N \mathbf{E} \left\{ |\gamma_N - \mathbf{E}_k\{\gamma_N\}|^2 \right\}.\end{aligned}\quad (4.1)$$

Now we let y_k be the k -th column of Y_N and B_k the $M \times (N-1)$ matrix consisting of the other $N-1$ columns of Y_N , then we have

$$H = \begin{pmatrix} y_1 \cdot y_1 - \sqrt{M/N} & (B_1^T y_1)^T \\ B_1^T y_1 & B_1^T B_1 - \sqrt{M/N} I_{N-1} \end{pmatrix}.$$

With the notations

$$H^{(k)} = B_k^T B_k - \sqrt{\frac{M}{N}} I_{N-1}$$

and $G^{(k)} = (H^{(k)} - z)^{-1}$, we have

$$\begin{aligned}\text{Tr } G - \text{Tr } G^{(1)} &= \frac{1 + y_1 \cdot B_1 B_1^T (B_1 B_1^T - \sqrt{M/N} - z)^{-2} y_1}{y_1 \cdot y_1 - \sqrt{M/N} - z - y_1 \cdot B_1 B_1^T (B_1 B_1^T - \sqrt{M/N} - z)^{-1} y_1} \\ &=: \frac{1+U}{V}\end{aligned}\quad (4.2)$$

and $G_{11} = V^{-1}$. Here we used the basic identity

$$B_1 \left(B_1^T B_1 - \sqrt{\frac{M}{N}} - z \right)^{-n} B_1^T = B_1 B_1^T \left(B_1 B_1^T - \sqrt{\frac{M}{N}} - z \right)^{-n}, \quad n = 1, 2.$$

To estimate (4.1), we only need to deal with the first term

$$\mathbf{E} \left\{ |\gamma_N - \mathbf{E}_1\{\gamma_N\}|^2 \right\} = \mathbf{E} \left\{ \left| \frac{1+U}{V} - \mathbf{E}_1 \left\{ \frac{1+U}{V} \right\} \right|^2 \right\}, \quad (4.3)$$

since the others are analogous. Thus it suffices to estimate $\mathbf{E}\{|UV^{-1} - \mathbf{E}_1\{UV^{-1}\}|^2\}$ and $\mathbf{E}\{|V^{-1} - \mathbf{E}_1\{V^{-1}\}|^2\}$. We will only present the estimate for the first one below. Clearly we have

$$\begin{aligned} \mathbf{E}_1\left\{|UV^{-1} - \mathbf{E}_1\{UV^{-1}\}|^2\right\} &\leq \mathbf{E}_1\left\{\left|UV^{-1} - \frac{\mathbf{E}_1\{U\}}{\mathbf{E}_1\{V\}}\right|^2\right\} \\ &= \mathbf{E}_1\left\{\left|\frac{U - \mathbf{E}_1\{U\}}{\mathbf{E}_1\{V\}} - \frac{V - \mathbf{E}_1\{V\}}{\mathbf{E}_1\{V\}} \frac{U}{V}\right|^2\right\}. \end{aligned} \quad (4.4)$$

By observing that $|\mathbf{E}_1\{V\}|^{-1} \leq \eta^{-1}$ and

$$\left|\frac{U}{V}\right| \leq \left|\frac{U}{\text{Im } V}\right| = \frac{y_1 \cdot B_1 B_1^T |B_1 B_1^T - \sqrt{M/N} - z|^{-2} y_1}{\eta + \eta y_1 \cdot B_1 B_1^T |B_1 B_1^T - \sqrt{M/N} - z|^{-2} y_1} \leq \eta^{-1},$$

it suffices to provide the following estimates:

$$\mathbf{E}_1\{|U - \mathbf{E}_1\{U\}|^2\}, \quad \mathbf{E}_1\{|V - \mathbf{E}_1\{V\}|^2\} = O\left(\frac{1}{N}\right). \quad (4.5)$$

For simplicity we introduce

$$\begin{aligned} M^{[1]} &= [M_{jk}^{[1]}]_{M \times M} := B_1 B_1^T \left(B_1 B_1^T - \sqrt{\frac{M}{N}} - z \right)^{-1}, \\ M^{[2]} &= [M_{jk}^{[2]}]_{M \times M} := B_1 B_1^T \left(B_1 B_1^T - \sqrt{\frac{M}{N}} - z \right)^{-2}. \end{aligned}$$

Thus we have

$$\begin{aligned} U - \mathbf{E}_1\{U\} &= \sum_{i \neq j} M_{ij}^{[2]} Y_{i1} Y_{j1} + \sum_i M_{ii}^{[2]} (Y_{i1}^2 - \mathbf{E}_1\{Y_{i1}^2\}), \\ V - \mathbf{E}_1\{V\} &= y_1 \cdot y_1 - \sqrt{\frac{M}{N}} - \sum_{i \neq j} M_{ij}^{[1]} Y_{i1} Y_{j1} - \sum_i M_{ii}^{[1]} (Y_{i1}^2 - \mathbf{E}_1\{Y_{i1}^2\}). \end{aligned}$$

It is not difficult to get

$$\mathbf{E}_1\{|U - \mathbf{E}_1\{U\}|^2\} \leq C \frac{1}{MN} \sum_{i,j} |M_{ij}^{[2]}|^2 = C \frac{1}{MN} \text{Tr } |M^{[2]}|^2, \quad (4.6)$$

$$\mathbf{E}_1\{|V - \mathbf{E}_1\{V\}|^2\} \leq C_1 \frac{1}{MN} \text{Tr } |M^{[1]}|^2 + C_2 \frac{1}{N}. \quad (4.7)$$

Now we denote eigenvalues of $H^{(k)}$ by $\mu_1^{(k)}, \dots, \mu_{N-1}^{(k)}$. Note that $\mu_\alpha^{(k)}$ ($\alpha = 1, \dots, N-1$) are also the eigenvalues of the matrix

$$\check{H}^{(k)} := B_k B_k^T - \sqrt{\frac{M}{N}} I_M,$$

which has a $(M - N + 1)$ -multiple eigenvalue $-\sqrt{M/N}$. So it follows

$$\mathrm{Tr} |M^{[n]}|^2 = \sum_{\alpha=1}^{N-1} \frac{(\mu_{\alpha}^{(1)} + \sqrt{M/N})^2}{|\mu_{\alpha}^{(1)} - z|^{2n}} = O(M), \quad n = 1, 2,$$

which implies (4.5) by taking into account (4.6) and (4.7). Lemma 3.1 is proved.

P r o o f o f T h e o r e m 3.2. We will only prove for G_{11} and $\tilde{G}_{11}$, the others are analogous. As we have shown above

$$G_{11} = V^{-1} = \left(y_1 \cdot y_1 - \sqrt{\frac{M}{N}} - z - y_1 \cdot M^{[1]} y_1 \right)^{-1}. \quad (4.8)$$

We denote the unit eigenvector of $\check{H}^{(1)}$ corresponding to the eigenvalue $\mu_{\alpha}^{(1)}$ by $v_{\alpha}^{(1)} = (v_{\alpha}^{(1)}(1), \dots, v_{\alpha}^{(1)}(M))$ ($\alpha = 1, \dots, N-1$) and set

$$\xi_{\alpha}^{(1)} = \sqrt{MN} |y_1 \cdot v_{\alpha}^{(1)}|^2.$$

It is easy to check $\mathbf{E}_1\{\xi_{\alpha}^{(1)}\} = 1$. Then by (4.8), we have

$$\begin{aligned} G_{11} &= \left(y_1 \cdot y_1 - \sqrt{\frac{M}{N}} - z - \frac{1}{\sqrt{MN}} \sum_{\alpha=1}^{N-1} \frac{(\mu_{\alpha}^{(1)} + \sqrt{M/N}) \xi_{\alpha}^{(1)}}{\mu_{\alpha}^{(1)} - z} \right)^{-1} \\ &=: (-z - m_N(z) + r_1)^{-1}, \end{aligned}$$

where

$$\begin{aligned} r_1 &= \left(y_1 \cdot y_1 - \sqrt{\frac{M}{N}} \right) + \left(m_N(z) - \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{\sqrt{N/M} \mu_{\alpha}^{(1)} + 1}{\mu_{\alpha}^{(1)} - z} \right) \\ &\quad - \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{(\sqrt{N/M} \mu_{\alpha}^{(1)} + 1)(\xi_{\alpha}^{(1)} - 1)}{\mu_{\alpha}^{(1)} - z} \\ &= \left(m_N(z) - \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{\sqrt{N/M} \mu_{\alpha}^{(1)} + 1}{\mu_{\alpha}^{(1)} - z} \right) + V - \mathbf{E}_1\{V\}. \end{aligned}$$

As a consequence we have

$$\begin{aligned} \left| G_{11} + \frac{1}{z + f_N(z)} \right| &= \left| \frac{f_N(z) - m_N(z) + r_1}{(-z - m_N(z) + r_1)(z + f_N(z))} \right| \\ &\leq C |(f_N(z) - m_N(z)) + r_1|. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\mathbf{E} \left\{ \left| G_{11} + \frac{1}{z + f_N(z)} \right|^2 \right\} \\ &\leq C \left(\mathbf{D}\{m_N(z)\} + \mathbf{E} \left\{ \left| m_N(z) - \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{\sqrt{N/M} \mu_{\alpha}^{(1)} + 1}{\mu_{\alpha}^{(1)} - z} \right|^2 \right\} \right. \\ &\quad \left. + \mathbf{E} \{ |V - \mathbf{E}_1\{V\}|^2 \} \right) \end{aligned} \quad (4.9)$$

Note the trivial bound

$$\sqrt{\frac{N}{M}} \left| \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{\mu_{\alpha}^{(1)}}{\mu_{\alpha}^{(1)} - z} \right| = O\left(\sqrt{\frac{N}{M}}\right), \quad (4.10)$$

and the fact that

$$\begin{aligned} \left| m_N(z) - \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{1}{\mu_{\alpha}^{(1)} - z} \right| &= \left| \int \frac{dF_N(x)}{x - z} - \left(1 - \frac{1}{N}\right) \int \frac{dF_N^{(1)}(x)}{x - z} \right| \\ &= \frac{1}{N} \left| \int \frac{NF_N(x) - (N-1)F_N^{(1)}(x)}{(x - z)^2} dx \right| \\ &\leq \frac{1}{N} \int \frac{dx}{|x - z|^2} = \frac{\pi}{N\eta}, \end{aligned} \quad (4.11)$$

where $F_N^{(1)}(x)$ is the empirical spectral distribution of $H^{(1)}$. In the inequality above, we use the well-known interlacing property between eigenvalues of an Hermitian matrix and its submatrix as $\lambda_1 \leq \mu_1^{(1)} \leq \lambda_2 \leq \dots \leq \mu_{N-1}^{(1)} \leq \lambda_N$.

Combining Lemma 3.1, (4.5), (4.10), and (4.11) we conclude the proof of (i) of Lemma 3.2.

It is similar to get (3.3) for $\tilde{G}_{11}$, we now turn to $\tilde{H} = YY^T - \sqrt{M/N} I_M$. We denote the k -th row of Y_N by $\tilde{y}_k$ and the $(M-1) \times N$ matrix consisting of the other $M-1$ rows of Y_N by D_k . And further introduce the matrix

$$\tilde{H}^{(k)} = [\tilde{H}_{jk}^{(k)}]_{N \times N} =: D_k^T D_k - \sqrt{\frac{M}{N}} I_N$$

with its eigenvalues $\tilde{\mu}_{\alpha}^{(k)}$, $\alpha = 1, \dots, N$. Then we have the following representation

$$\tilde{H} = \begin{pmatrix} \tilde{y}_1 \cdot \tilde{y}_1 - \sqrt{M/N} & \tilde{y}_1 D_1^T \\ D_1 \tilde{y}_1^T & D_1 D_1^T - \sqrt{M/N} I_{M-1} \end{pmatrix}.$$

If we use the $\tilde{E}_1\{\cdot\}$ to denote the expectation with respect to $\tilde{y}_1$, and set

$$\tilde{V} = \tilde{y}_1 D_1^T D_1 \left(D_1^T D_1 - \sqrt{\frac{M}{N}} - z \right)^{-1} \tilde{y}_1^T,$$

we have

$$\tilde{G}_{11} = \frac{1}{\tilde{y}_1 \cdot \tilde{y}_1 - \sqrt{M/N} - z - \tilde{V}} =: \frac{1}{-\sqrt{M/N} - z - m_N(z) + \tilde{r}_1},$$

where

$$\begin{aligned} \tilde{r}_1 &= \tilde{y}_1 \cdot \tilde{y}_1 + m_N(z) + \tilde{V} \\ &= \tilde{y}_1 \cdot \tilde{y}_1 + m_N(z) - \frac{1}{N} \sum_{\alpha=1}^N \frac{\sqrt{N/M} \tilde{\mu}_{\alpha}^{(1)} + 1}{\tilde{\mu}_{\alpha}^{(1)} + z} + \tilde{V} - \tilde{E}_1\{\tilde{V}\}. \end{aligned}$$

Now we denote the set of event

$$\Omega_{\circ} := \left\{ |f_N(z) - m_N(z) + \tilde{r}_1(z)| \geq \frac{1}{2} \sqrt{\frac{M}{N}} \right\},$$

and observe that on $\Omega_{\circ}^c$

$$\left| \left(\sqrt{\frac{M}{N}} + z + m_N(z) - \tilde{r}_1(z) \right) \left(\sqrt{\frac{M}{N}} + z + f_N(z) \right) \right| \geq C \frac{M}{N}.$$

Then we have

$$\begin{aligned} & \mathbf{E} \left| \tilde{G}_{11} + \frac{1}{\sqrt{M/N} + z + f_N(z)} \right|^2 \\ &= \mathbf{E} \left| \frac{f_N(z) - m_N(z) + \tilde{r}_1(z)}{(\sqrt{M/N} + z + m_N(z) - \tilde{r}_1(z))(\sqrt{M/N} + z + f_N(z))} \right|^2 \\ &\leq C \left[\left(\frac{N}{M} \right)^2 + \frac{N}{M} \mathbf{P}(\Omega_{\circ}) \right] \cdot \mathbf{E} |f_N(z) - m_N(z) + \tilde{r}_1(z)|^2, \quad (4.12) \end{aligned}$$

where we used the fact that

$$\left| \left(\sqrt{\frac{M}{N}} + z + m_N(z) - \tilde{r}_1(z) \right) \left(\sqrt{\frac{M}{N}} + z + f_N(z) \right) \right| \geq C \sqrt{\frac{M}{N}} \quad (4.13)$$

holds on the full set Ω . To see this, we denote the unit eigenvector of $\tilde{H}^{(1)}$ corresponding to the eigenvalue $\tilde{\mu}_{\alpha}^{(1)}$ by $\tilde{v}_{\alpha}^{(1)} = (\tilde{v}_{\alpha}^{(1)}(1), \dots, \tilde{v}_{\alpha}^{(1)}(N))$ ($\alpha = 1, \dots, N$) and set

$$\tilde{\xi}_{\alpha}^{(1)} = \sqrt{MN} |\tilde{y}_1 \cdot \tilde{v}_{\alpha}^{(1)}|^2.$$

Then we have

$$\begin{aligned} & \sqrt{\frac{M}{N}} + z + m_N(z) - \tilde{r}_1(z) \\ &= \sqrt{\frac{M}{N}} + z - \tilde{y}_1 \left[I - \left(\tilde{H}^{(1)} + \sqrt{\frac{M}{N}} \right) (\tilde{H}^{(1)} - z)^{-1} \right] \tilde{y}_1^T \\ &= \sqrt{\frac{M}{N}} + z + \frac{1}{\sqrt{MN}} \sum_{\alpha=1}^N \left(\frac{\sqrt{M/N} + \tilde{\mu}_{\alpha}^{(1)}}{\tilde{\mu}_{\alpha}^{(1)} - z} - 1 \right) \tilde{\xi}_{\alpha}^{(1)} \\ &= \left(\sqrt{\frac{M}{N}} + z \right) \left(1 + \frac{1}{\sqrt{MN}} \sum_{\alpha=1}^N \frac{1}{\tilde{\mu}_{\alpha}^{(1)} - z} \tilde{\xi}_{\alpha}^{(1)} \right) \\ &=: \left(\sqrt{\frac{M}{N}} + z \right) (1 + S). \quad (4.14) \end{aligned}$$

Taking into account that $\tilde{\mu}_\alpha^{(1)} \geq -\sqrt{M/N}$ and $\tilde{\xi}_\alpha^{(1)} \geq 0$, we have

$$\operatorname{Re} S \geq -C \sqrt{\frac{M}{N}} \operatorname{Im} S, \quad \operatorname{Im} S \geq 0,$$

which implies

$$|1 + S| \geq C \sqrt{\frac{N}{M}}. \quad (4.15)$$

Then (4.13) is a direct consequence of (4.14) and (4.15).

Now we proceed to the estimate of (4.12). By observing

$$\mathbf{P}(\Omega_o) \leq \frac{4N}{M} \mathbf{E} |f_N(z) - m_N(z) + \tilde{r}_1(z)|^2,$$

it suffices to provide

$$\mathbf{E} |f_N(z) - m_N(z) + \tilde{r}_1(z)|^2 \leq C_1 \frac{N}{M} + C_2 \frac{1}{N},$$

which can be derived by a similar estimate process as that of (4.9). Thus we complete the proof. Lemma 3.2 is proved.

5. CLT for linear eigenvalue statistics. To prove Theorem 1.2, we will follow the recent article [14] by Shcherbina. Since there are only some technical differences, we will state the main body of the proof in this section, and leave all the technical details to the Appendix.

P r o o f o f T h e o r e m 1.2. Similar to [14], as we will see, ω_8 is needed in our proof. So first we need to truncate the random variable X_{jk} at $(MN)^{1/4}$, and then re-centralize it. To use the truncated matrix, it is necessary to show its linear eigenvalue statistics have the same limit distribution as the original one at first. We denote the truncated matrix by

$$\hat{Y} = [\hat{Y}_{ij}]_{M \times N}, \quad \hat{Y}_{ij} = Y_{ij} \mathbf{1}_{\{|Y_{ij}| \leq 1\}}, \quad \check{Y} = [\check{Y}_{ij}]_{M \times N}, \quad \check{Y}_{ij} = \hat{Y}_{ij} - \mathbf{E} \hat{Y}_{ij}.$$

And further introduce

$$\begin{aligned} \hat{H}_N &= \hat{Y}^T \hat{Y} - \sqrt{\frac{M}{N}} I_N, & \widehat{\mathcal{L}}_N[\varphi] &= \operatorname{Tr} \varphi(\hat{H}), \\ \check{H}_N &= \check{Y}^T \check{Y} - \sqrt{\frac{M}{N}} I_N, & \check{\mathcal{L}}_N[\varphi] &= \operatorname{Tr} \varphi(\check{H}), \\ \hat{H}_N(t) &= [\hat{Y} + t(Y - \hat{Y})]^T [\hat{Y} + t(Y - \hat{Y})] - \sqrt{\frac{M}{N}} I_N. \end{aligned}$$

We denote the eigenvalues and corresponding eigenvectors of $\widehat{H}_N(t)$ by $\lambda_i(t)$ and $v_i(t)$ ($i = 1, \dots, N$) below. It suffices to prove that

$$\begin{aligned} \mathbf{E} \left\{ |e^{ix\mathcal{L}_N^\circ[\varphi]} - e^{ix\widehat{\mathcal{L}}_N^\circ[\varphi]}| \right\} &\leq 2\mathbf{P}\{\widehat{Y} \neq Y\} \\ &\quad + |x| \left| \mathbf{E} \{ \mathcal{L}_N[\varphi] \} - \mathbf{E} \{ \widehat{\mathcal{L}}_N[\varphi] \} \right| \rightarrow 0, \end{aligned} \quad (5.1)$$

$$\mathbf{E} \left\{ |e^{ix\widehat{\mathcal{L}}_N^\circ[\varphi]} - e^{ix\check{\mathcal{L}}_N^\circ[\varphi]}| \right\} \leq 2|x| \mathbf{E} \left\{ |\widehat{L}_N[\varphi] - \check{\mathcal{L}}_N[\varphi]| \right\} \rightarrow 0 \quad (5.2)$$

as N tends to infinity. The proof of (5.2) is similar to (5.1), we only give the proof of (5.1) below. $\mathbf{P}\{\widehat{Y} \neq Y\}$ is obviously $o(1)$ by the truncation. Also, we have

$$\begin{aligned} \mathbf{E} \left\{ |\mathcal{L}_N[\varphi] - \widehat{\mathcal{L}}_N[\varphi]| \right\} &= \int_0^1 dt \mathbf{E} \left\{ \sum_i |\varphi'(\lambda_i(t)) \lambda_i'(t)| \right\} \\ &\leq \|\varphi'\|_\infty \int_0^1 dt \mathbf{E} \left\{ \sum_i v_i^T(t) \widehat{H}'(t) v_i(t) \right\} \\ &\leq \|\varphi'\|_\infty \int_0^1 dt \mathbf{E} \left\{ \text{Tr}[(Y - \widehat{Y})^T \widehat{Y} + \widehat{Y}^T (Y - \widehat{Y}) + 2t(Y - \widehat{Y})^T (Y - \widehat{Y})] \right\} \\ &\leq \|\varphi'\|_\infty \int_0^1 dt \mathbf{E} \left\{ \sum_{i,j} |[(Y - \widehat{Y})^T \widehat{Y} + \widehat{Y}^T (Y - \widehat{Y}) + 2t(Y - \widehat{Y})^T (Y - \widehat{Y})]_{ij}| \right\} \\ &\leq 2\|\varphi'\|_\infty \int_0^1 dt \sum_{i,j,k} \left(\mathbf{E} \left\{ |(Y_{ki} - \widehat{Y}_{ki})| \cdot |\widehat{Y}_{kj}| \right\} + t \mathbf{E} \left\{ |(Y_{ki} - \widehat{Y}_{ki})(Y_{kj} - \widehat{Y}_{kj})| \right\} \right) \\ &\leq 2\|\varphi'\|_\infty \int_0^1 dt \left\{ \sum_{i,k} \left[\mathbf{E} \{ |Y_{ki} - \widehat{Y}_{ki}| \cdot |\widehat{Y}_{ki}| \} + t \mathbf{E} \{ |Y_{ki} - \widehat{Y}_{ki}|^2 \} \right] \right. \\ &\quad \left. + \sum_{i \neq j} \sum_k \left[\mathbf{E} |Y_{ki} - \widehat{Y}_{ki}| \cdot \mathbf{E} |\widehat{Y}_{kj}| + t \mathbf{E} |Y_{ki} - \widehat{Y}_{ki}| \cdot \mathbf{E} |Y_{kj} - \widehat{Y}_{kj}| \right] \right\}. \end{aligned} \quad (5.3)$$

By the truncation, we have

$$\begin{aligned} \mathbf{E} |\widehat{Y}_{kj}| &\leq C(MN)^{-1/4}, \quad \mathbf{E} |Y_{ki} - \widehat{Y}_{ki}| \leq C(MN)^{-5/4}, \\ \mathbf{E} |Y_{ki} - \widehat{Y}_{ki}|^2 &\leq C(MN)^{-5/4}. \end{aligned}$$

Then it is easy to check (5.3) tends to 0 as N tends to infinity, so (5.1) holds.

So without loss of generality, we may and do assume in the sequel

$$\mathbf{E} \{X_{jk}\} = 0, \quad \mathbf{E} \{X_{jk}^2\} = 1 + o(1), \quad \mathbf{E} \{X_{jk}^4\} = \omega_4 + o(1), \quad (5.4)$$

$$\mathbf{E} \{X_{jk}^6\} \leq C(MN)^{1/4}, \quad \mathbf{E} \{X_{jk}^8\} \leq C(MN)^{3/4}. \quad (5.5)$$

Relying on (5.4) and (5.5) we have the following lemma which collects all estimates we need to derive the central limit theorem for $\mathcal{L}_N[\varphi]$.

Lemma 5.1. *Under the assumptions (5.4), (5.5) and our basic assumption $N/M \rightarrow 0$ as $N \rightarrow \infty$, we have for any fixed z with $\eta > 0$ and z_1, z_2 such that $\text{Im } z_{1,2} > 0$, $1 > \delta > 0$ the following estimates hold:*

$$\mathbf{D}\{\gamma_N\} \leq CN^{-1} \sum_{i=1}^N \text{Big}(\mathbf{E}\{|G_{ii}(z)|^{1+\delta}\}) (\eta^{-3-\delta} + \eta^{-3+\delta}), \quad (5.6)$$

$$\mathbf{E}\{(V^\circ)^3\}, \quad \mathbf{E}\{(U^\circ)^3\}, \quad \mathbf{E}\{|V^\circ|^4\}, \quad \mathbf{E}\{|U^\circ|^4\} = O(N^{-3/2}), \quad (5.7)$$

$$\begin{aligned} N \mathbf{E}_1\{V^\circ(z_1)V^\circ(z_2)\} &= \omega_4 - 1 - \frac{\omega_4 - 1}{M} \text{Tr}[M^{[1]}(z_1) + M^{[1]}(z_2)] \\ &+ \frac{\omega_4 - 3}{M} \sum_i M_{ii}^{[1]}(z_1)M_{ii}^{[1]}(z_2) + \frac{2}{M} \text{Tr}[M^{[1]}(z_1)M^{[1]}(z_2)] \\ &+ \frac{1}{M} \left(\sqrt{\frac{M}{N}} + z_1\right) \left(\sqrt{\frac{M}{N}} + z_2\right) \gamma_N^{\circ(1)}(z_1) \gamma_N^{\circ(1)}(z_2), \end{aligned} \quad (5.8)$$

$$\begin{aligned} \mathbf{D}\{N \mathbf{E}_1\{V^\circ(z_1)V^\circ(z_2)\}\}, \quad \mathbf{D}\{N \mathbf{E}_1\{V^\circ(z_1)U^\circ(z_2)\}\} &= O\left(\frac{N}{M} \vee \frac{1}{N}\right), \\ \mathbf{E}\{|\gamma_N^{\circ(1)}|^4\} &\leq CN^{1/2}\eta^{-12} + o(N^{1/2}), \quad \mathbf{E}\{|\gamma_N^{\circ(1)} - \gamma_N^\circ|^4\} = O(N^{-3/2}), \end{aligned} \quad (5.9)$$

$$\left| \frac{\mathbf{E}\{\gamma_N^{(1)}\}}{N} - f(z) \right|, \quad |\mathbf{E}^{-1}\{V\} + f(z)| = o(1). \quad (5.10)$$

We postpone the proof of Lemma 5.1 to the Appendix. Here we need to mention that the constants in (5.6) or (5.9) are independent of η .

By [14, Proposition 1 and 3] and (5.6) we only need to prove Theorem 1.2 for the functions $\varphi = \varphi_\eta$ which are the convolution of some φ_0 with the Poisson kernel $P_\eta = \eta/(\pi(x^2 + \eta^2))$. And φ_0 is restricted to satisfy $\int |\varphi_0(\lambda)| d\lambda \leq C$ with some positive constant C . For such test function φ we have

$$\mathcal{L}_N^\circ[\varphi] = \frac{1}{\pi} \int \varphi_0(\mu) \text{Im } \gamma_N^\circ(z_\mu) d\mu, \quad z_\mu = \mu + i\eta.$$

Following the notation of [14], we set

$$Z_N(x) = \mathbf{E}\{e^{ix\mathcal{L}_N^\circ[\varphi]}\}, \quad e(x) = e^{ix\mathcal{L}_N^\circ[\varphi]}, \quad e_1(x) = e^{ix(\mathcal{L}_N^{(1)}[\varphi])^\circ}, \quad (5.11)$$

where $\mathcal{L}_N^{(1)}[\varphi]$ stands for the corresponding linear eigenvalue statistics of $H^{(1)}$. Observe that Theorem 1.2 can be proved by providing that

$$\frac{d}{dx} Z_N(x) = -xV[\varphi_0, \eta]Z_N(x) + o(1). \quad (5.12)$$

To do this, we introduce

$$Y_N(z, x) := \mathbf{E}\{\text{Tr } G(z)e^\circ(x)\} = \sum_k \mathbf{E}\{G_{kk}(z)e^\circ(x)\}. \quad (5.13)$$

Thus we have

$$\frac{d}{dx} Z_N(x) = \frac{1}{2\pi} \int \varphi_0(\mu) (Y(z_\mu, x) - Y(\bar{z}_\mu, x)) d\mu.$$

We only need to deal with the first term in the summation (5.13), the others are analogous, so we can deal with $N \mathbf{E} \{V^{-1}e^\circ(x)\} := T_1 + T_2$ instead, where

$$T_1 = -N \mathbf{E} \{(V^{-1})^\circ e_1(x)\}, \quad T_2 = -N \mathbf{E} \{(V^{-1})^\circ (e(x) - e_1(x))\}.$$

Since the calculations towards T_1 and T_2 are similar to those in the case studied in [14], we will not state the tedious process. Indeed, by inserting the estimate in Lemma 5.1, we easily obtain

$$T_1 = f^2(z) Y_N(z, x) + O\left(\frac{1}{\sqrt{N}} \vee \sqrt{\frac{N}{M}}\right),$$

and

$$T_2 = ix Z_N(x) \int d\mu \varphi_0(\mu) \frac{D(z, z_\mu) - D(z, \bar{z}_\mu)}{2i\pi} + o(1),$$

where

$$\begin{aligned} D(z, z_\mu) := & 2f^2(z)f^2(z_\mu)(1 + f'(z_\mu)) \left(\frac{f(z) - f(z_\mu)}{z - z_\mu} + \frac{\omega_4 - 1}{2} \right) \\ & + 2f^2(z)f(z_\mu) \frac{d}{dz_\mu} \left(\frac{f(z) - f(z_\mu)}{z - z_\mu} \right). \end{aligned} \quad (5.14)$$

So we have

$$Y_N(z, x) = f^2(z) Y_N(z, x) + ix Z_N(x) \int d\mu \varphi_0(\mu) \frac{D(z, z_\mu) - D(z, \bar{z}_\mu)}{2i\pi(1 - f^2(z))} + o(1).$$

Thus the variance in (5.12) can be represented by

$$\begin{aligned} V[\varphi_0, \eta] = & \frac{1}{4\pi^2} \int \int \varphi_0(\mu_1) \varphi_0(\mu_2) \left(C(z_{\mu_1}, \bar{z}_{\mu_2}) + C(\bar{z}_{\mu_1}, z_{\mu_2}) \right. \\ & \left. - C(z_{\mu_1}, z_{\mu_2}) - C(\bar{z}_{\mu_1}, \bar{z}_{\mu_2}) \right) d\mu_1 d\mu_2, \end{aligned}$$

where

$$C(z, z_\mu) = \frac{D(z, z_\mu)}{1 - f^2(z)}.$$

We can get (1.5) by comparing (5.14) to (2.40) of [14], then we complete the proof. Theorem 1.2 is proved.

6. Appendix.

P r o o f o f L e m m a 5.1. We begin with (5.6), we will use (4.1) and (4.3), and so we need to estimate (4.4) more carefully. According to (4.6) and (4.7), we have

$$\begin{aligned} \mathbf{E}_1 \left\{ \left| \frac{V - \mathbf{E}_1\{V\}}{\mathbf{E}_1\{V\}} \right|^2 \right\} &\leq C_1 \frac{1}{MN} \frac{\text{Tr } |M^{[1]}|^2}{|z + (1/\sqrt{MN}) \text{Tr } M^{[1]}|^2} \\ &\quad + C_2 \frac{1}{N} \frac{1}{|\mathbf{E}_1\{V\}|^2} \end{aligned} \quad (6.1)$$

and

$$\mathbf{E}_1 \left\{ \left| \frac{U - \mathbf{E}_1\{U\}}{\mathbf{E}_1\{V\}} \right|^2 \right\} = C \frac{1}{MN} \frac{\text{Tr } |M^{[2]}|^2}{|z + (1/\sqrt{MN}) \text{Tr } M^{[1]}|^2}.$$

For (6.1), if we set $N^{(1)} = B_1^T B_1$, we have

$$\text{Tr } M^{[1]} = \text{Tr } G^{(1)} N^{(1)}, \quad \text{Tr } |M^{[1]}|^2 = \text{Tr } G^{(1)} N^{(1)} G^{(1)*} N^{(1)}.$$

So

$$\begin{aligned} \frac{1}{MN} \text{Tr } |M^{[1]}|^2 &= \frac{1}{MN} \text{Tr } G^{(1)} N^{(1)} G^{(1)*} N^{(1)} \\ &\leq \frac{1}{MN} [\text{Tr } G^{(1)} N^{(1)} G^{(1)*}]^{1-\delta} [\text{Tr } G^{(1)} (N^{(1)})^{(1+\delta)/\delta} G^{(1)*}]^\delta \\ &\leq \frac{1}{MN} [\text{Tr } G^{(1)} N^{(1)} G^{(1)*}]^{1-\delta} [\text{Tr } (N^{(1)})^{(1+\delta)/\delta}]^\delta \eta^{-2\delta} \\ &\leq \frac{1}{N} \left[\frac{1}{\sqrt{MN}} \text{Tr } G^{(1)} N^{(1)} G^{(1)*} \right]^{1-\delta} \left[\frac{1}{N} \left(\frac{N}{M} \right)^{(1+\delta)/2\delta} \text{Tr } (N^{(1)})^{(1+\delta)/\delta} \right]^\delta \eta^{-2\delta} \\ &\leq \frac{1}{N} \left[\frac{1}{\sqrt{MN}} \text{Tr } G^{(1)} N^{(1)} G^{(1)*} \right]^{1-\delta} \left[\frac{1}{N} \text{Tr} \left(\frac{1}{M} X^{(1)T} X^{(1)} \right)^{(1+\delta)/\delta} \right]^\delta \eta^{-2\delta}. \end{aligned}$$

Furthermore we have

$$\text{Im } \text{Tr } G^{(1)} N^{(1)} = \eta \text{Tr } G^{(1)} N^{(1)} G^{(1)*},$$

which implies

$$\begin{aligned} \mathbf{E} \left\{ \frac{1}{MN} \frac{\text{Tr } |M^{[1]}|^2}{|z + (1/\sqrt{MN}) \text{Tr } M^{[1]}|^2} \right\} \\ &\leq \frac{1}{N} \eta^{-1-\delta} \mathbf{E} \left\{ \frac{[(1/N) \text{Tr } (1/M) X^{(1)T} X^{(1)})^{(1+\delta)/\delta}]^\delta}{|\mathbf{E}_1\{V\}|^{1+\delta}} \right\} \\ &\leq \frac{1}{N} \eta^{-1-\delta} \mathbf{E} \left\{ \left[\frac{1}{N} \text{Tr} \left(\frac{1}{M} X^{(1)T} X^{(1)} \right)^{(1+\delta)/\delta} \right]^\delta \mathbf{E}_1\{|G_{11}|^{1+\delta}\} \right\}. \end{aligned} \quad (6.2)$$

Here we have used Jensen's inequality $|\mathbf{E}_1\{V\}|^{-1} \leq \mathbf{E}_1\{|V|^{-1}\}$.

If we control (6.2) by using [4, Theorem 5.9] and by replacing y with N/M , we can get

$$\mathbf{E}_1 \left\{ \left| \frac{V - \mathbf{E}_1\{V\}}{\mathbf{E}_1\{V\}} \right|^2 \right\} \leq C \frac{1}{N} \mathbf{E} |G_{11}|^{1+\delta} (\eta^{-1-\delta} + \eta^{-1+\delta}).$$

Then by the fact that $\text{Im } V = -\eta U - \eta$, we also have

$$\begin{aligned} \mathbf{E} \left\{ \left| \frac{V - \mathbf{E}_1\{V\}}{\mathbf{E}_1\{V\}} \frac{U}{V} \right|^2 \right\} &\leq C \frac{1}{N} \left(\mathbf{E} \{|G_{11}|^{1+\delta}\} \right) (\eta^{-3-\delta} + \eta^{-3+\delta}) \\ \mathbf{E} \left\{ \left| \frac{U - \mathbf{E}_1\{U\}}{\mathbf{E}_1\{V\}} \right|^2 \right\} &\leq C \frac{1}{N} \left(\mathbf{E} \{|G_{11}|^{1+\delta}\} \right) (\eta^{-3-\delta} + \eta^{-3+\delta}). \end{aligned}$$

Next we turn to the proof of the first inequality of (5.9), which will be used in the proof of (5.7). We use the following inequality for martingale (see [6]):

$$\mathbf{E} \{ |\gamma_N^\circ|^4 \} \leq CN \sum_{k=1}^N \mathbf{E} \{ |\gamma_N - \mathbf{E}_k\{\gamma_N\}|^4 \}.$$

Similarly to (4.3), it suffices to check

$$\mathbf{E} \{ |U - \mathbf{E}_1\{U\}|^4 \} \leq CN^{-3/2} \eta^{-8} + o(N^{-3/2}), \quad (6.3)$$

$$\mathbf{E} \{ |V - \mathbf{E}_1\{V\}|^4 \} \leq CN^{-3/2} \eta^{-4} + o(N^{-3/2}). \quad (6.4)$$

We only present the proof of (6.4) below, (6.3) is analogous. Observing that $M_{ii}^{[1]} = (B_1 G^{(1)} B_1^T)_{ii}$, (3.9) and (3.10) are valid as well. We have

$$\sum_i \mathbf{E} \{ |M_{ii}^{[1]}|^4 \} \leq C \eta^{-2} \frac{M}{N} \sum_i \mathbf{E} \{ |M_{ii}^{[1]}|^2 \} = CM \eta^{-4},$$

which implies

$$\begin{aligned} &\mathbf{E} \{ |V - \mathbf{E}_1\{V\}|^4 \} \\ &\leq C \mathbf{E} \left\{ \mathbf{E}_1 \left[\left| \sum_{i \neq j} M_{ij}^{[1]} Y_{i1} Y_{j1} \right|^4 + \left| \sum_i M_{ii}^{[1]} (Y_{i1}^2 - \frac{1}{\sqrt{MN}}) \right|^4 \right. \right. \\ &\quad \left. \left. + \left| y_1 \cdot y_1 - \sqrt{\frac{M}{N}} \right|^4 \right] \right\} \\ &\leq C_1 \frac{1}{(MN)^2} \mathbf{E} \{ \text{Tr} |M^{[1]}|^4 \} + C_2 \frac{1}{(MN)^2} \sum_i \mathbf{E} \{ |M_{ii}^{[1]}|^4 \} \omega_8 \\ &\quad + C_3 \frac{1}{(MN)^2} \mathbf{E} \{ (\text{Tr} |M|^2)^2 \} + C_4 \frac{1}{MN^2} \omega_8 \\ &\leq CN^{-3/2} \eta^{-4} + o(N^{-3/2}). \end{aligned}$$

For (5.7), we only deal with $\mathbf{E}\{|V^\circ|^4\}$ below, the others are similar. We have

$$\begin{aligned} V^\circ &= V - \mathbf{E}_1\{V\} + \mathbf{E}_1\{V\} - \mathbf{E}\{V\} \\ &= V - \mathbf{E}_1\{V\} + \frac{1}{\sqrt{MN}} (\text{Tr } M^{[1]} - \mathbf{E} \text{Tr } M^{[1]}) \\ &= V - \mathbf{E}_1\{V\} + \left(\frac{1}{N} + \frac{1}{\sqrt{MN}} z_1 \right) \left(\gamma_N^{(1)} - \mathbf{E} \gamma_N^{(1)} \right). \end{aligned}$$

So use the first inequality of (5.9) to $\gamma_N^{(1)}$, together with (6.5) we can get (5.7).

For (5.8), note the following expansion:

$$\begin{aligned} N \mathbf{E}_1\{V^\circ(z_1)V^\circ(z_2)\} &= (\omega_4 - 1) - \frac{\omega_4 - 1}{M} \text{Tr}[M^{[1]}(z_1) + M^{[1]}(z_2)] \\ &\quad + \frac{\omega_4 - 3}{M} \sum_i M_{ii}^{[1]}(z_1)M_{ii}^{[1]}(z_2) + \frac{2}{M} \text{Tr}[M^{[1]}(z_1)M^{[1]}(z_2)] \\ &\quad + \frac{1}{M} \left[(\text{Tr } M^{[1]}(z_1) - \mathbf{E} \text{Tr } M^{[1]}(z_1))(\text{Tr } M^{[1]}(z_2) - \mathbf{E} \text{Tr } M^{[1]}(z_2)) \right] \\ &= (\omega_4 - 1) - \frac{\omega_4 - 1}{M} \text{Tr}[M^{[1]}(z_1) + M^{[1]}(z_2)] \\ &\quad + \frac{\omega_4 - 3}{M} \sum_i M_{ii}^{[1]}(z_1)M_{ii}^{[1]}(z_2) + \frac{2}{M} \text{Tr}[M^{[1]}(z_1)M^{[1]}(z_2)] \\ &\quad + \frac{1}{M} \left(\sqrt{\frac{M}{N}} + z_1 \right) \left(\sqrt{\frac{M}{N}} + z_2 \right) \gamma_N^{\circ(1)}(z_1) \gamma_N^{\circ(1)}(z_2). \end{aligned} \quad (6.5)$$

To deal with the first estimate in (5.9), we only need to take care of the variance of the third and fourth terms of (6.5). Using (3.10) to $M_{ii}^{[1]}$ again we can get

$$\mathbf{D}\{M_{ii}^{[1]}\} \leq C_1 \left(\frac{N}{M} \right)^2 + C_2 \frac{1}{M},$$

together with the trivial bound $|M_{ii}^{[1]}| \leq C\sqrt{M/N}$ we have

$$\begin{aligned} \mathbf{D}\left\{ \frac{1}{M} \sum_i M_{ii}^{[1]}(z_1)M_{ii}^{[1]}(z_2) \right\} &\leq \frac{1}{M} \sum_i \mathbf{D}\{M_{ii}^{[1]}(z_1)M_{ii}^{[1]}(z_2)\} \\ &\leq \frac{1}{M} \sum_i \mathbf{E}(M_{ii}^{[1]}(z_1)M_{ii}^{[1]\circ}(z_2) + M_{ii}^{[1]\circ}(z_1)\mathbf{E} M_{ii}^{[1]}(z_2))^2 \leq C_1 \frac{N}{M} + C_2 \frac{1}{N}. \end{aligned}$$

Also we have

$$\text{Tr } M^{[1]}(z_1)M^{[1]}(z_2) = \sum_{\alpha=1}^{N-1} \frac{(\mu_\alpha^{(1)} + \sqrt{M/N})^2}{(\mu_\alpha^{(1)} - z_1)(\mu_\alpha^{(1)} - z_2)}$$

$$\begin{aligned}
&= \left[\frac{M}{N} - z_1 z_2 + \left(\sqrt{\frac{M}{N}} + \frac{z_1 + z_2}{2} \right) (z_1 + z_2) \right] \sum_{\alpha=1}^{N-1} \frac{1}{(\mu_{\alpha}^{(1)} - z_1)(\mu_{\alpha}^{(1)} - z_2)} \\
&\quad + \left(\sqrt{\frac{M}{N}} + \frac{z_1 + z_2}{2} \right) \sum_{\alpha=1}^{N-1} \frac{2\mu_{\alpha}^{(1)} - z_1 - z_2}{(\mu_{\alpha}^{(1)} - z_1)(\mu_{\alpha}^{(1)} - z_2)} + N - 1 \\
&= \left[\frac{M}{N} - z_1 z_2 + \left(\sqrt{\frac{M}{N}} + \frac{z_1 + z_2}{2} \right) (z_1 + z_2) \right] \operatorname{Tr} \frac{G^{(1)}(z_1) - G^{(1)}(z_2)}{z_1 - z_2} \\
&\quad + \left(\sqrt{\frac{M}{N}} + \frac{z_1 + z_2}{2} \right) \operatorname{Tr}(G^{(1)}(z_1) + G^{(1)}(z_2)) + N - 1.
\end{aligned}$$

Using (5.6) to $\operatorname{Tr} G^{(1)} = \gamma_N^{(1)}$ we have

$$\mathbf{D} \left\{ \frac{1}{M} \operatorname{Tr} M^{[1]}(z_1) M^{[1]}(z_2) \right\} \leq \frac{1}{N^2}.$$

The estimate towards $\mathbf{D}\{N \mathbf{E}_1\{V^{\circ}(z_1)U^{\circ}(z_2)\}\}$ is straightforward by taking into account $(G^{(1)})^2 = dG^{(1)}/dz$.

The second estimate of (5.9) is a direct consequence of (4.2) and (5.7). The first part of (5.10) is just the consequence of (1.4) if we replace $f_N(z) = \gamma_N/N$ by $\gamma_N^{(1)}/N$, and second one follows directly from the fact

$$\mathbf{E}\{V\} = z + \frac{1}{N} \gamma_N^{(1)} + O\left(\sqrt{\frac{N}{M}}\right) = z + f(z) + o(1)$$

and $f(z) = -1/(z + f(z))$. So we complete the proof of Lemma 5.1.

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