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З а м е ч а н и е 2. Для одномерной линейной диффузии

$$dX_t = -\theta X_t dt + dw(t)$$

асимптотическая нормальность оценки максимального правдоподобия при случайной нормировке $(\int_0^T X_t^2 dt)^{1/2}$ в нестационарном случае ($\theta < 0$) была получена в работе [7], и там же было показано, что указанная нормировка стремится к бесконечности п.н. при $T \rightarrow \infty$.

З а м е ч а н и е 3. Легко понять, что основной результат работы сохраняет силу и при фиксированных начальных условиях, т.е. если $\xi(0) = (x_0, y_0)$, так как это повлияет только на предельное распределение случайной величины $\eta(T) = I_1(T)/I_2(T)$, явный вид которого не играет никакой роли для асимптотической нормальности оценок.

СПИСОК ЛИТЕРАТУРЫ

1. Колмогоров А. Н., Арато М., Синай Я. Г. Об оценке параметров комплексного стационарного гауссовского марковского процесса. — Докл. АН СССР, 1962, т. 146, № 4, с. 747–750.
2. Арато М. Линейные стохастические системы с постоянными коэффициентами. М.: Наука, 1989, 304 с.
3. Липцер Р. Ш., Ширяев А. Н. Статистика случайных процессов. М.: Наука, 1974, 696 с.
4. Startsev A. N. A new approach to estimation of an autoregressive parameter. — Probability Theory and Mathematical Statistics (Kiev, 1991). River Edge: World Scientific Publ., 1992, p. 377–381.
5. Старцев А. Н. Об одном новом подходе к оценке параметра сноса процесса линейной диффузии. — Узб. матем. журн., 1995, № 4, с. 72–76.
6. Durbin J. Estimation of parameters in time-series regression models. — J. Roy. Statist. Soc. Ser. B, 1960, v. 22, p. 139–153.
7. Feigin P. D. Maximum likelihood estimation for continuous-time stochastic processes. — Adv. Appl. Probab., 1976, v. 8, № 4, с. 712–736.

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CERTAIN PERIODICALLY CORRELATED MULTI-COMPONENT LOCALLY STATIONARY PROCESSES¹⁾

Определяя $X^{ls}(t)$ как случайную смесь двух стационарных процессов, где зависящие от времени случайные веса имеют экспоненциально выпуклую ковариацию, мы показываем, что этот процесс имеет многокомпонентную локально стационарную ковариационную функцию в

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смысле Сильвермана. Мы также вводим $X^p(t)$ как некоторый периодически коррелированный процесс с непрерывным временем, чья ковариационная функция порождается ковариационной функцией в дискретном времени, определяя некоторую простую случайную меру на действительной прямой. Мы также накладываем условие бипериодической корреляции этого периодически коррелированного процесса с процессом $X^{ls}(t)$. Доказано существование такой случайной меры. Затем мы вводим $X(t) = X^{ls}(t) + X^p(t)$ как некоторый периодически коррелированный многокомпонентный локально стационарный процесс и характеризуем ковариационную структуру и меняющиеся во времени спектральное представление таких процессов.

Ключевые слова и фразы: периодическая коррелированность, спектральное представление, многокомпонентные локально стационарные процессы, экспоненциально выпуклая ковариация.

1. Introduction. Recently a large amount of work has been devoted to time series analysis, with the focus placed on locally stationary (LS) and periodically correlated (PC) processes which give plausible description of real world. Silverman (see [18]) presented a definition of LS process to model systems which behave as a function of time. He presented a relation between the covariance and the spectral density of LS processes which constitutes a natural generalization of the Wiener–Khinchin relation. A general class of LS processes whose spectral structure varies smoothly over time is introduced by Dahlhaus [4]. A computationally efficient state space method for estimating, predicting and making statistical inferences about these nonstationary models is proposed in [15]. Exponentially convex stochastic processes, their covariance functions, and spectral representation have been studied by Loeve [12] and others [6].

Gladyshev [8] introduced PC sequences and showed that they all are harmonizable in the sense of Loève [12]. He provided the structure of the covariance function and an interesting spectral representation. At the most basic level, the connection of the PC structure to a group of shift and unitary operators was mentioned as a motivation for the development of spectral representations that are much like those of Gladyshev. One of the classical results of unitary operator theory is the spectral theorem, the notion of spectral measures and the time varying spectral representation of a PC process [9]. The contributions to the analysis of PC processes in the literature are [5], [9] and [16], where various properties of PC processes are presented. Harmonic series representation, coherent and component method for mean and covariance function estimation by using linear filtration of PC processes are studied in [10] and [11]. A class of bilinear processes with periodic time-varying coefficients of periodic ARMA and periodic GARCH models has been applied in [2].

We consider a certain nonstationary process $\{X(t), t \in \mathbf{R}^+\}$, say periodically correlated multi-component locally stationary (PC-LS) process. For introducing the structure of this process we consider a partition of the positive real line. This paper can be considered as an extension of previous works in three stages. Firstly by presenting spectral representation of certain continuous PC process. This is established by introducing some random measure on subsets of some proper partition of real line, where the increments of the process on such partition provide a discrete time PC process. We specify the covariance structure of such random measure through the covariance structure of the discrete time PC process, and provide the spectral representation of such continuous time PC process. The intuition of this process comes from the aggregated traffic on various networks. Secondly we present a locally stationary process in the Silverman sense as a time varying random mixture of stationary processes, where the coefficients have exponentially convex covariances and are independent of the stationary ones. This consideration provides smooth

transition in the correlation structure of the process which is in effect of two or more stationary resources at the nodes of the mentioned partitions. This makes a convenient way for analyzing traffic systems like the mobile cellular networks. Thirdly we introduce the process as sum of such PC and LS processes which have bi-periodic correlations and have potential to model periodic and local variation in network traffics. The idea of introducing such process comes from the behavior of traffic flow in different networks, like telecommunication, internet or transportation. This process is determined by aggregation of the effects of two main components. The first component often has periodic behavior of the flow in time, $X^p(t)$, which shows over all usage of the network. We assume that $X^p(t)$ in turn can be approximated via a periodic sequence X_j^p , which represents the increment of the flow on some proper partition of time, say B_j , $j \in \mathbf{N}$. Type and amount of the usage of the network by different users at a time have some interaction which effects the flow and provides the other component of the process and can be approximated as a multi-component locally stationary process $X^{ls}(t)$. We assume that $X^{ls}(t)$ is a random mixture, with some exponentially convex random weight process, of two stationary processes at time points inside each B_j . We consider some fixed $T \in \mathbf{N}$ and a partition of the positive real line as $0 = s_0 < s_1 < \dots$, $B_j = (s_{j-1}, s_j]$ for $j \in \mathbf{N}$ and $|B_j| = |B_{j+kT}|$, $k \in \mathbf{N}$, then $S = \sum_{i=1}^T |B_i|$. Let $X(t) = X^{ls}(t) + X^p(t)$, $t \in \mathbf{R}^+$, represents a stochastic process, where $X^{ls}(t) = \sum_{j=1}^\infty X_j^{ls}(t) I_{B_j}(t)$ and $X_j^{ls}(t)$ is a multi-component LS process which is an exponentially convex mixture of two stationary processes. Also let $\{X_j^p\}$, $j \in \mathbf{N}$, be the discrete time PC process, M_j , $j \in \mathbf{N}$, the sequence of random measure on B_j , where $M_j(B_j) := X_j^p$, and $X_j^p(t) := M_j(s_{j-1}, t]$ for $t \in B_j$, $j \in \mathbf{N}$, with some special correlation. As the amount of the process on a set A , say traffic, we consider simple random measure $M(A) = \sum_{j=1}^m M_{i_j}(A \cap B_{i_j})$, where $A = \bigcup_{j=1}^m (A \cap B_{i_j})$. We prove the existence of the measures introducing a measurable map and a T -variate random measure. Then we study $X^p(t) = \sum_{j=1}^\infty \Delta_j^p(t) I_{B_j}(t)$, where $\Delta_j^p(t)$ is a linear combination of X_{j-1}^p and $X_j^p(t)$, which describes elimination of the effect of traffic on B_{j-1} as a decreasing fraction of X_{j-1}^p and aggregate of the process on B_j as $X_j^p(t)$, and study its spectral representation.

In Section 2, we study the general framework and preliminaries of the harmonizable representation of stationary and PC processes and also, we present exponentially convex processes. The main result as the covariance structure and spectral representation of such continuous time multi-component PC-LS processes are given in Section 3.

2. Preliminaries. We review the harmonizable representations of PC processes based on unitary operators and also the definition of exponentially convex processes in this section.

2.1. Spectral representation of PC processes. We review harmonizable representation of PC processes by unitary operators. For a comprehensive review of these one can refer to Hurd and Miamee [9]. Every wide sense stationary process $X(t)$ has harmonizable representation

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ(\lambda), \quad (2.1)$$

where $Z(\lambda)$ has orthogonal increments.

A unitary operator on a Hilbert space $\mathcal{H}$ is a linear operator from $\mathcal{H}$ to $\mathcal{H}$ which preserves inner product as $\langle Ux, Uy \rangle = \langle x, y \rangle = \text{cov}(x, y)$ for every $x, y \in \mathcal{H}$.

Theorem 2.1. *For any unitary operator U on a Hilbert space $\mathcal{H}$, there exists a unique spectral measure Q on the Borel subsets of $[0, 2\pi)$ such that $U = \int_0^{2\pi} e^{i\lambda} Q(d\lambda)$, and $U^t = \int_0^{2\pi} e^{i\lambda t} Q(d\lambda)$ for any integer t .*

Existence of unitary operator for a PC sequence characterizes its spectral representation.

Proposition 2.1. *A second order stochastic sequence X_j is PC with period T if and only if for every $j \in \mathbf{Z}$, there exist a unitary operator $U = V^T$ and a periodic sequence*

(process) P_j with period T taking values in $\mathcal{H}X = \overline{\text{sp}}\{X_j, j \in \mathbf{Z}\}$ for which $X_j = V^j P_j$, where $V = \int_0^{2\pi} e^{i\lambda/T} Q(d\lambda)$ and Q is the spectral measure defined in Theorem 2.1, so $X_j = \int_0^{2\pi} e^{i\lambda j/T} Q(d\lambda) P_j$.

2.2. Exponentially convex process. We give a brief description of exponentially convex process and its covariance function, for more details see [6], [7].

Definition 2.1. The covariance function of a second order zero mean process $\{Z(t), t \in \mathbf{R}\}$ with finite variance, $\langle Z(t_i), \overline{Z(t_j)} \rangle = \psi(t_i + t_j)$, is called exponentially convex if and only if $\sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} \psi(t_i + t_j) \geq 0$, for all finite sets of complex coefficients $a_1, \dots, a_n$ and points $t_1, \dots, t_n \in \mathbf{R}$.

A stochastic process with such covariance function is called exponentially convex. The result of Berg, Christensen, and Ressel [1] implies that a continuous function is exponentially convex if and only if it is the Laplace transform of a nonnegative finite measure.

3. Main results: A continuous time PC-LS model. We introduce a new class of certain nonstationary processes, say multi-component periodically correlated locally stationary (PC-LS) processes, as

$$X(t) = X^{ls}(t) + X^p(t), \quad t > 0, \quad (3.1)$$

where $X^p(\cdot)$ is a continuous time PC process and $X^{ls}(\cdot)$ is a multi-component LS process which have bi-periodic correlation. We introduce their special structures in Subsections 3.1 and 3.2, respectively. Their covariance and cross covariance structures are studied in Subsection 3.3. Finally, the spectral representation of the process is characterized in Subsection 3.4.

3.1. Continuous PC process. Let $\{X_j^p, j \in \mathbf{N}\}$ be a positive second order discrete time PC process with period T and M_j a random measure on Borel field of subsets of B_j , where $M_j(B_j) := X_j^p$, and for $A, B \subset B_j$, $D \subset B_k$, we define $\mathbf{E}[M_j(A)] = (|A|/|B_j|) \mathbf{E}[X_j^p]$ and covariance functions

$$\gamma_{j,j}(A, B) = \frac{\lambda|A||B| + (1-\lambda)a_j|A \cap B|}{a_j^2} \gamma_{jj}^p, \quad \gamma_{j,k}(A, D) := \frac{|A||D|}{a_j a_k} \gamma_{jk}^p, \quad (3.2)$$

where $\gamma_{j,j}(A, B) = \langle M_j(A), M_j(B) \rangle$, $\gamma_{j,k}(A, D) = \langle M_j(A), M_k(D) \rangle$, $\gamma_{jk}^p = \langle X_j^p, X_k^p \rangle$, $\gamma_{jj}^p = \mathbf{D}(X_j^p)$, $0 \leq \lambda \leq 1$, and $a_j = |B_j|$.

If $B = B_k$, then for $j \neq k$, $\langle M_j(A), M_k(B_k) \rangle = (|A|/a_j) \gamma_{jk}^p$. This inner product is well defined. Let $X_j^p(t) := M_j(s_{j-1}, t]$, $t \in B_j$, $j \in \mathbf{N}$. So for $t, u \in B_j$, $t \leq u$, and $v \in B_k$

$$\langle X_j^p(t), X_j^p(u) \rangle = \frac{\lambda a_j^t a_j^u + (1-\lambda)a_j a_j^t}{a_j^2} \gamma_{jj}^p, \quad \langle X_j^p(t), X_k^p(v) \rangle = \frac{a_j^t a_k^v}{a_j a_k} \gamma_{jk}^p, \quad (3.3)$$

where $a_j^t = t - s_{j-1}$. Thus by defining

$$N_j(t - s_{j-1}) := M_j(s_{j-1}, t] = X_j^p(t), \quad (3.4)$$

we find that $N_j(y)$ is a discrete time PC process with period T with respect to j for fixed $0 < y \leq a_j$ and $X_j^p(t)$ is a bi-periodic process with period T in j and period $S = \sum_{i=1}^T |B_i|$ in t , that is $\langle X_j^p(t), X_k^p(u) \rangle = \langle X_{j+T}^p(t+S), X_{k+T}^p(u+S) \rangle$. Also for $t > 0$

$$X^p(t) = \sum_{j=1}^{\infty} \Delta_j^p(t) I_{B_j}(t), \quad \Delta_j^p(t) = \frac{a_j - a_j^t}{a_j} X_{j-1}^p + X_j^p(t), \quad (3.5)$$

is a continuous time PC process with respect to t with period S . Using a method similar to one described by Soltani and Parvardeh [17], we prove the existence of such random measures by the following.

Let $(D, \mathcal{D})$ be a measurable space and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbf{P})$ the Hilbert space of real random variables on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with finite second moment. A mapping $\Phi: \mathcal{D} \rightarrow \mathcal{L}^2(\Omega, \mathcal{F}, P)$ is a second order random measure on $\mathcal{D}$ if $\Phi(\emptyset) = 0$ and

$\mathbf{E} |\Phi(\bigcup_{i=1}^{\infty} A_i) - \sum_{i=1}^n \Phi(A_i)|^2 \rightarrow 0$ as $n \rightarrow \infty$, for disjoint sets $A_1, A_2, \dots \in \mathcal{D}$. We introduce simple random measure $M(A)$ for $A \in \mathcal{D}$ by $M(A) = M_1(A \cap B_1) + M_2(A \cap B_2) + \dots + M_T(A \cap B_T)$, where B_j is the support of M_j , $j = 1, \dots, T$, and $B_i \cap B_j = \emptyset$ for $i \neq j$. For the random measure M , $\mathbf{E} |M(dx)|^2 = \gamma_{j,j}(dx)$ for $x \in B_j$. Also for $x \in B_j$ and $y \in B_i$ we have $\mathbf{E} (M(dx)M(dy)) = \mathbf{E} (M_j(dx), M_i(dy)) = \gamma_{j,k}(dx, dy)$. To show the existence of such random measures, let $\Psi = (\Psi_1, \dots, \Psi_T)$ be a T -variate random measure on $(X, \mathcal{X})$, where $X = B_1 \times \dots \times B_T$ and $\mathcal{X}$ is the corresponding class of finite disjoint union of measurable rectangles. In this case for $A \in \mathcal{D}$, $A = \bigcup_{j=1}^m (A \cap B_{i_j})$ and $\{i_1, i_2, \dots, i_m\}$ are m distinct integer of $\{1, 2, \dots, T\}$. Let $X^* = B_{i_1} \times \dots \times B_{i_m}$, and let $\mathcal{X}^*$ the class of finite disjoint rectangles $F^* = (F \cap B_{i_1}) \times \dots \times (F \cap B_{i_m})$, where $F \subset D^* = \bigcup_{j=1}^m B_{i_j}$, and $S: \mathcal{X}^* \rightarrow D^*$ a measurable map such that $S(A^*) = \bigcup_{i=1}^m \{x_i: \underline{x} = (x_1, \dots, x_m) \in A^*\}$ for $A^* \in \mathcal{X}^*$. Then $S^{-1}(F) = \{\underline{x} = (x_1, \dots, x_m): x_j \in F \cap B_{i_j}, j = 1, \dots, m\}$ for $F \subset D^*$. Also let $M_{i_j}(F) = \Psi_{i_j} S^{-1}(F)$, and let $\Psi^* = (\Psi_{i_1}, \dots, \Psi_{i_m})$ be an m -variate random measure on $(X^*, \mathcal{X}^*)$ with covariance matrix $\mu = [\mu_{j,k}]$, then for the resulting M and for $1 \leq j, k \leq m$

$$\begin{aligned} \mu_{j,k} \left(S^{-1}(F) \right) &= \mathbf{E} \left(\Psi_{i_j} (S^{-1}(F)) \Psi_{i_k} (S^{-1}(F)) \right) = \gamma_{i_j, i_k} (F \cap B_{i_j}, F \cap B_{i_k}), \\ \mu_{j,j} \left(S^{-1}(F) \right) &= \mathbf{E} \left(\Psi_{i_j} (S^{-1}(F)) \Psi_{i_j} (S^{-1}(F)) \right) = \gamma_{i_j, i_j} (F \cap B_{i_j}, F \cap B_{i_j}). \end{aligned}$$

Interestingly $\nu(A \times C) = \sum_{j,k} \gamma_{j,k} (A \cap B_j, C \cap B_k)$ defines a product measure on $(D^* \times D^*, \mathcal{Y})$, where $\mathcal{Y}$ is the class of finite disjoint union of corresponding measurable rectangles. Also for $E_{j,k} \in \mathcal{Y}$, where $E_{j,k} \subset B_{i_j} \times B_{i_k}$, $\nu(E_{j,k}) = \mu_{j,k} \{\underline{x}: \underline{x} \in X^*, (x_j, x_k) \in E_{j,k}\}$. Thus for $E \in \mathcal{Y}$ and $E_{j,k} = E \cap (B_{i_j} \times B_{i_k})$

$$\begin{aligned} \nu(E) &= \sum_{j,k} \nu(E_{j,k}) = \sum_{l=-m+1}^{m-1} \sum_{j=\max\{1-l, 1\}}^{\min\{m-l, m\}} \nu(E_{j, l+j}) \\ &= \sum_{j=1}^m \nu(E_{j,j}) + \sum_{l=1}^{m-1} \left\{ \sum_{j=1}^{m-l} \nu(E_{j, l+j}) + \sum_{j=m-l+1}^{\min\{2m-l, m\}} \nu(E_{j, l+j-m}) \right\}. \end{aligned}$$

Now define $\mu_0, \dots, \mu_{m-1}$ on D^* by

$$\begin{aligned} \mu_0(A) &= \sum_{j=1}^m \mu_{j,j} \left(S^{-1}(A) \right), \\ \mu_l(A) &= \sum_{j=m-l+1}^{\min\{2m-l, m\}} \mu_{j, j+l-m} \left(S^{-1}(A) \right) + \sum_{j=1}^{m-l} \mu_{j, j+l} \left(S^{-1}(A) \right) \quad \text{for } l = 1, \dots, m-1; \end{aligned}$$

then for $E \in \mathcal{Y}$ we have $\nu(E) = \sum_{l=0}^{m-1} \mu_l \{x \in D^*; (x, y) \in E \text{ for some } y\}$. Therefore the product measure ν is specified by m set functions $\mu_0, \dots, \mu_{m-1}$ on D^* that are determined by $\mu = [\mu_{j,k}]$, the covariance matrix of m -variate random vector $M = (M_1, \dots, M_m)$. The covariance matrix μ also can be specified from $\mu_0, \dots, \mu_{m-1}$, namely, for $G \in \mathcal{X}^*$

$$\begin{aligned} \mu_{j,k}(G) &= \mu_{m+k-j} \left(\{x_k: \underline{x} \in G\} \cup \{x_j: \underline{x} \in G\} \right), \quad k-j < 0, \\ \mu_{j,k}(G) &= \mu_{k-j} \left(\{x_k: \underline{x} \in G\} \cup \{x_j: \underline{x} \in G\} \right), \quad k-j \geq 0. \end{aligned}$$

Therefore ν and μ uniquely specify each other.

3.2. Multi-component LS process. To provide the smoothing transition and LS behavior of the process between inside each partition we introduce a random mixture of two or more stationary processes or resources at nodes of such partition with exponentially convex coefficients which provides an LS processes in the Silverman sense. Let $\{Y_j^s(t), t \in$

$B_j \cup B_{j+1}\}$, $Y_j^s(t) = \int_{-\infty}^{\infty} e^{it\lambda} \eta_j(d\lambda)$, $j \in \mathbf{N}$, be a sequence of independent stationary processes, $\eta_j(\cdot) \stackrel{d}{=} \eta_{j+T}(\cdot)$ and $\{Y_j^s(t)\} \stackrel{d}{=} \{Y_{j+kT}^s(t+kS)\}$ for $k \in \mathbf{N}$. We define

$$X^{ls}(t) = \sum_{j=1}^{\infty} X_j^{ls}(t) I_{B_j}(t), \quad X_j^{ls}(t) = U^{j-1}(t) Y_{j-1}^s(t) + U^j(t) Y_j^s(t), \quad (3.6)$$

for $t \in B_j$, $j = 2, \dots, T$, and $X_1^{ls}(t) = U^1(t) Y_1^s(t)$ for $t \in B_1$, where $\{U^j(t), j \in \mathbf{N}\}$ is a random weight process with exponentially convex covariance and are independent of the process $Y_j^s(\cdot)$. We call $X_j^{ls}(\cdot)$ a multi-component LS process motivated from its covariance function, which is represented in Lemma 3.1.

3.3. Covariance function of PC-LS process. We present some lemmas in this section which we use to provide the covariance function of the process $X(t)$ in Theorem 3.1.

Lemma 3.1. *Let B_j , $j \in \mathbf{N}$, be the introduced partition of positive real line. The covariance function of $X^{ls}(t)$, $t \in \mathbf{R}^+$, satisfies*

$$\gamma^{ls}(t, u) \equiv \langle X^{ls}(t), X^{ls}(u) \rangle = \sum_{m=1}^{\infty} \sum_{n=m-1}^{m+1} \gamma_{mn}^{ls}(t, u) I_{B_m}(t) I_{B_n}(u), \quad (3.7)$$

where

$$\begin{aligned} \gamma_{mm}^{ls}(t, u) &\equiv \langle X_m^{ls}(t), X_m^{ls}(u) \rangle = \psi_m(t+u) \gamma_m(t-u) + \psi_{m-1}(t+u) \gamma_{m-1}(t-u), \\ \gamma_{mn}^{ls}(t, u) &\equiv \langle X_m^{ls}(t), X_n^{ls}(u) \rangle = \psi_k(t+u) \gamma_k(t-u), \end{aligned}$$

$k = \min\{m, n\}$ and $|n-m| = 1$ in which $\psi_m(t+u) = \langle U^m(t), U^m(u) \rangle$, $\gamma_m(t-u) = \langle Y_m^s(t), Y_m^s(u) \rangle$ for $m \in \mathbf{N}$.

P r o o f. According to (3.6) and the fact that $U^j(\cdot)$ and $Y_j^s(\cdot)$ are independent processes, $\gamma_{mm}^{ls}(t, u)$ and $\gamma_{mn}^{ls}(t, u)$ for $|n-m| = 1$ are as presented by the lemma. Thus $X_j^{ls}(\cdot)$ is a multi-component LS process in the Silverman sense [18]. Also by (3.6)

$$\gamma^{ls}(t, u) = \sum_{m=1}^{\infty} \langle X_m^{ls}(t), X^{ls}(u) \rangle I_{B_m}(t) = \sum_{m=1}^{\infty} \sum_{n=m-1}^{m+1} \gamma_{mn}^{ls}(t, u) I_{B_m}(t) I_{B_n}(u).$$

Lemma 3.2. *Let B_j , $j \in \mathbf{N}$, be a partition of positive real line, and let $\gamma_{mn}^p = \langle X_m^p(t), X_n^p(u) \rangle$. The covariance function of $X^p(t)$ for $t \in B_m$ and $u \in B_n$, $t \leq u$, is*

$$\begin{aligned} \gamma^p(t, u) &\equiv \langle X^p(t), X^p(u) \rangle \\ &= \begin{cases} A_{t,m} \left(A_{u,m} \gamma_{m-1,m-1}^p + \frac{a_m^u}{a_m} \gamma_{m-1,m}^p \right) + A_{u,m} \frac{a_m^t}{a_m} \gamma_{m-1,m}^p \\ \quad + B_m(t, u) \gamma_{mm}^p, & n = m, \\ A_{t,m} \left(A_{u,n} \gamma_{m-1,n-1}^p + \frac{a_n^u}{a_n} \gamma_{m-1,n}^p \right) + A_{u,n} \frac{a_m^t}{a_m} \gamma_{m,n-1}^p \\ \quad + \frac{a_m^t a_n^u}{a_m a_n} \gamma_{mn}^p, & |m-n| \geq 1, \end{cases} \end{aligned} \quad (3.8)$$

where

$$A_{t,m} = \frac{1-a_m^t}{a_m}, \quad A_{u,n} = 1 - \frac{a_n^u}{a_n}, \quad B_m(t, u) = \frac{\lambda a_m^t a_m^u + (1-\lambda) a_m a_m^t}{a_m^2}.$$

P r o o f. By (3.5) the covariance function of $X^p(\cdot)$ for $t \in B_m$ and $u \in B_n$ is $\gamma^p(t, u) = \langle X_m^p(t), X_n^p(u) \rangle$ and by (3.3) we have the result.

Let $a_j = |B_j|$, $a = \mathbf{E}[U^j(t)]$ for $t \in \mathbf{R}^+$ and $\theta_r(j, u) = \langle X_j^p(t), Y_r^s(u) \rangle$. We also define the cross covariance function of $X_j^p(t) = M_j(s_{j-1}, t]$ and $Y_r^s(u)$ for $t \in B_j$, $u \in B_r$ by

$$\langle X_j^p(t), Y_r^s(u) \rangle = \frac{a_j^t}{a_j} \theta_r(j, u), \quad \langle X_j^p(t), X_r^{ls}(u) \rangle = \frac{a a_j^t}{a_j} [\theta_{r-1}(j, u) + \theta_r(j, u)]. \quad (3.9)$$

R e m a r k 3.1. The cross covariance function of $X^{ls}(t)$ and $X^p(t)$ can be written as

$$\begin{aligned}\gamma^{p,ls}(t, u) &\equiv \langle X^p(t), X^{ls}(u) \rangle \\ &= a \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{a_m^t}{a_m} D_{m,n}(u) + A_{t,m} D_{m-1,n}(u) \right] I_{B_m}(t) I_{B_n}(u),\end{aligned}\quad (3.10)$$

where $D_{m,n}(u) = \theta_{n-1}(m, u) + \theta_n(m, u)$, $A_{t,m} = 1 - a_m^t/a_m$, and $\theta_r(m, u)$ is defined by (3.9).

Theorem 3.1. *The covariance function of the multi-component PC-LS process $X(t) = X^{ls}(t) + X^p(t)$, where $X^{ls}(t)$ and $X^p(t)$ are dependent and defined by (3.6) and (3.5), respectively, is $\gamma(t, u) = \text{cov}(X(t), X(u)) = \gamma^{ls}(t, u) + \gamma^p(t, u) + \gamma^{p,ls}(t, u)$, where $\gamma^{ls}(t, u)$, $\gamma^p(t, u)$, and $\gamma^{p,ls}(t, u)$ are as in Lemma 3.1, Lemma 3.2, and Remark 3.1, respectively.*

3.4. Spectral representation. In this section we obtain spectral representations of PC process $\{X^p(t), t \in \mathbf{R}^+\}$, and locally stationary process $\{X^{ls}(t), t \in \mathbf{R}^+\}$. Then by imposing bi-periodic property for the cross covariance function of $X^{ls}(t)$ and $X^p(t)$ their cross spectrum is characterized. Using these results the spectral representation of the multi-component PC-LS process $X(\cdot)$ and its spectral measure is provided by Theorem 3.2.

Let $\{X_j^p, j \in \mathbf{N}\}$ be a PC process with period T . By using the Gladyshev result for PC processes, it has been shown in [9] that for $j, k \in \mathbf{N}$

$$X_j^p = \int_0^{2\pi} e^{i\lambda j} d\vartheta(\lambda), \quad (3.11)$$

where $d\vartheta(\lambda) = \sum_{k=0}^{T-1} Q(T d\lambda - 2k\pi) \tilde{P}_k I_{\Delta_k}(\lambda)$, $\Delta_k = [2k\pi/T, 2(k+1)\pi/T]$, $\tilde{P}_k$ are Fourier coefficients of the periodic sequence P_j , $Q(\cdot)$ and P_j are defined by Proposition 2.1, that $P_j = \sum_{k=0}^{T-1} \tilde{P}_k e^{i2\pi k j/T}$. The support of the spectral density $\tilde{\theta}(d\lambda, d\omega) = \langle \vartheta(d\lambda), \vartheta(d\omega) \rangle$ is the intersection of the set $\Psi = \{(\lambda, \omega): \lambda = \omega - 2\pi k/T, k \in [-(T-1), T-1]\}$ and the square $[0, 2\pi) \times [0, 2\pi)$.

R e m a r k 3.2. Let $\{X_j^p\}$ be a sequence of PC process with period T , by Proposition 2.1 $X_j^p = \int_0^{2\pi} e^{i\lambda j/T} \xi(d\lambda, j)$, where $\xi(d\lambda, j) = Q(d\lambda) P_j$, $Q(d\lambda)$ is an orthogonally scattered random measure and P_j a periodic sequence with period T . Let $a_j = |B_j|$, $a_j^t = t - s_{j-1}$ for $t \in B_j$. As by (3.4), $X_j^p(t) \equiv N_j(a_j^t)$ and $N_j(y)$ is a discrete PC process with period T for fixed y , so by Proposition 2.1 $X_j^p(t) = \int_0^{2\pi} e^{i\lambda j/T} \zeta_j(d\lambda, t)$ for $t \in B_j$, where $\zeta_j(d\lambda, t) = Q(d\lambda) \hat{P}_{j, a_j^t}$ and $\hat{P}_{j, y}$ is a periodic function in j with period T for fixed y . Using the relation (3.4) we have $X_j^p \equiv X_j^p(s_j)$, so $\hat{P}_{j, a_j} \equiv P_j$. By a similar method as in (3.11) $X_j^p(t) = \int_0^{2\pi} e^{i\lambda j} \Upsilon(d\lambda, a_j^t)$ for $t \in B_j$, where $\Upsilon(d\lambda, a_j^t)$ has the same structure as $\vartheta(d\lambda)$ in (3.12), by replacing $\tilde{P}_k$ with $\tilde{P}_{k, a_k^t}$, where $\hat{P}_{j, a_j^t} = \sum_{k=0}^{T-1} \tilde{P}_{k, a_k^t} e^{i2\pi j k/T}$. So (3.3) implies that

$$\Theta_{j,k}(d\lambda, d\omega, a_j^t, a_k^u) = \langle \Upsilon(d\lambda, a_j^t), \Upsilon(d\omega, a_k^u) \rangle = \begin{cases} \frac{\lambda a_j^t a_j^u + (1-\lambda) a_j a_j^t}{a_j^2} \tilde{\theta}(d\lambda, d\omega), & j = k, \\ \frac{a_j^t a_k^u}{a_j a_k} \tilde{\theta}(d\lambda, d\omega), & j \neq k, \end{cases}$$

which is bi-periodic with period T in j and k , and period S in t and u ($t \leq u$), where, as in (3.11), $\tilde{\theta}(d\lambda, d\omega) = \langle \vartheta(d\lambda), \vartheta(d\omega) \rangle$ has Ψ as its support.

We describe the spectral representation for $X_j^{ls}(\cdot)$ by the following remark.

R e m a r k 3.3. Spectral representation of the process $X_j^{ls}(\cdot)$, $j \in \mathbf{N}$, defined by (3.6) is

$$X_j^{ls}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \Phi_j(d\lambda, t), \quad t \in B_j, \quad (3.12)$$

where $\Phi_j(d\lambda, t) = U^{j-1}(t)\eta_{j-1}(d\lambda) + U^j(t)\eta_j(d\lambda)$, and η_j is the orthogonally scattered random measure in the spectral representation of the stationary process $Y_j^s(\cdot)$. Also by the independence of $\{U^j(\cdot)\}$ and $\{Y_j^s(\cdot)\}$, the cross spectral covariance $F_{j,k}(d\lambda, d\lambda, t, u) = \langle \Phi_j(d\lambda, t), \Phi_k(d\lambda, u) \rangle$ for $\lambda \neq \omega$ can be written as

$$F_{j,k}(d\lambda, d\lambda, t, u) = \psi_{j-1}(t+u)G_{j-1}(d\lambda)I_{\{k,k+1\}}(j) + \psi_j(t+u)G_j(d\lambda)I_{\{k,k-1\}}(j),$$

where $\psi_j(t+u) = \langle U^j(t), U^j(u) \rangle$, $G_j(d\lambda) = \mathbf{E}|\eta_j(d\lambda)|^2$, and $F_{j,k}(d\lambda, d\omega, t, u) = 0$.

Lemma 3.3. *Under the assumptions of Remarks 3.2 and 3.3, the cross covariance of the processes $X_j^p(t)$ and $Y_r^s(u)$ is bi-periodic, with period S in t and u and period T in j and r , $\langle X_j^p(t), Y_r^s(u) \rangle = \langle X_{j+T}^p(t), Y_{r+T}^s(u+S) \rangle$, where $t \in B_j$ and $u \in B_r$, if and only if*

$$\langle X_j^p(t), Y_r^s(u) \rangle = \frac{a_j^t}{a_j} \int_0^{2\pi} \int_{-\infty}^{\infty} e^{i(j\lambda - u\omega)} \Delta_r(d\lambda, d\omega), \quad (3.13)$$

where support of $\Delta_r(d\lambda, d\omega) = \langle \vartheta(d\lambda), \eta_r(d\omega) \rangle$ is $\Gamma = \{(\lambda, \omega): \lambda T = S\omega - 2\pi l, l \in \mathbf{Z}\}$.

P r o o f. By (3.9) $\langle X_j^p(t), Y_r^s(u) \rangle = (a_j^t/a_j) \langle X_j^p, Y_r^s(u) \rangle$, by (3.11) $X_j^p = \int_0^{2\pi} e^{i\lambda j} \vartheta(d\lambda)$, and by (3.6) $Y_r^s(u) = \int_{-\infty}^{\infty} e^{i\omega u} \eta_r(d\omega)$, where $\eta_{r+T}(\cdot) = \eta_r(\cdot)$. So we have (3.13), and one can easily verify that if Γ is the support of the cross spectral covariance, then the covariance function has bi-periodic property. On the other hand, by assuming that the cross covariance is bi-periodic and using (3.9), equality of $\theta_r(j, u) = \theta_{r+T}(j+T, u+S)$ implies that for any natural number N ,

$$\begin{aligned} \theta_r(j, u) &= \frac{1}{2N+1} \sum_{k=-N}^N \theta_{r+kT}(j+kT, u+kS) \\ &= \int_0^{2\pi} \int_{-\infty}^{\infty} e^{i(j\lambda - u\omega)} D_N(T\lambda - S\omega) \Delta_r(d\lambda, d\omega), \end{aligned}$$

where $D_N(2\pi l) = 1$, $l \in \mathbf{Z}$, and $D_N(\sigma) = (\sin(N+1/2)\sigma)/(2(N+1/2)\sin(\sigma/2))$ converges to zero for $\sigma \neq 2\pi\mathbf{Z}$. This implies that the support of cross spectral density must be contained in Γ .

Theorem 3.2. *The spectral representation of the multi-component PC-LS process $X(t) = X^{ls}(t) + X^p(t)$, where $X^p(t)$ and $X^{ls}(t)$ are defined by (3.11), (3.5), and (3.6), can be written as*

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ_{m,t}(\lambda), \quad t \in B_m, \quad (3.14)$$

where

$$dZ_{m,t}(\lambda) = \Phi_m(d\lambda, t) + I_{[0,2\pi)}(\lambda) e^{i\lambda(m-t)} [\Upsilon(d\lambda, a_m^t) + A_{t,m} e^{-i\lambda} d\vartheta],$$

with ϑ , Φ_m , and Υ defined by (3.12) and Remark 3.2. The cross spectral covariance for $t \in B_m$, $u \in B_n$ is given by

$$\begin{aligned} \langle dZ_{m,t}(\lambda), dZ_{n,u}(\omega) \rangle &= F_{m,n}(d\lambda, d\omega, t, u) \\ &+ K(m, n, \lambda, \omega) \left[e^{-i(\lambda-\omega)} A_{t,m} A_{u,n} \tilde{\theta}(d\lambda, d\omega) + \Theta_{m,n}(d\lambda, d\omega, a_m^t, a_n^u) \right] \\ &+ \frac{aa_n^u}{a_n} \Lambda_{m,n}(d\omega, d\lambda, t) + \frac{aa_m^t}{a_m} \Lambda_{n,m}(d\lambda, d\omega, u) \\ &+ A_{u,n} a \Lambda_{m,n-1}(d\omega, d\lambda, t) e^{-i\omega} + A_{t,m} a \Lambda_{n,m-1}(d\lambda, d\omega, u) e^{-i\lambda}, \end{aligned}$$

where $A_{t,m} = 1 - a_m^t/a_m$,

$$\Lambda_{l,k}(d\theta, d\eta, v) = e^{i(\theta k - v\eta)} (\Delta_l(d\theta, d\eta) + \Delta_{l-1}(d\theta, d\eta)) I_{[0,2\pi)}(d\theta), \quad l, k = m, n,$$

and Θ , F , and Δ are defined in Remark 3.2, Remark 3.3, and Lemma 3.3, respectively, and $K(m, n, \lambda, \omega) = e^{i\lambda(m-t) - i\omega(n-u)} I_{[0,2\pi)}(\lambda) I_{[0,2\pi)}(\omega)$.

Proof. By the result of Remarks 3.2, 3.3 and relations (3.5), (3.6) we have that $\{X(t), t \in \mathbf{R}^+\}$ has the time varying spectral representation (3.14). Also by (3.9), Lemma 3.3, and Remark 3.2, 3.3, the last assertion of the theorem can easily be obtained as the cross spectral covariance.

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REFERENCES

1. Berg C., Christensen J. P. R., Ressel P. Harmonic Analysis on Semigroups. New York: Springer-Verlag, 1984, 289 p.
2. Bibi A., Lescheb I. On general periodic time-varying bilinear processes. — *Econom. Lett.*, 2012, v. 114, p. 353–357.
3. Cox D. R., Miller H. D. The Theory of Stochastic Processes. New York: Wiley, 1965, 398 p.
4. Dahlhaus R., Polonik W. Empirical spectral processes for locally stationary time series. — *Bernoulli*, 2009, v. 15, № 1, p. 1–39.
5. Dehay D., Hurd H. L. Representation and estimation for periodically and almost periodically correlated random processes. — *Cyclostationarity in Communications and Signal Processing*. Ed. by W. A. Gardner. IEEE Press, 2004, p. 295–326.
6. Ehm W., Genton M. G., Gneiting T. Stationary covariances associated with exponentially convex functions. — *Bernoulli*, 2003, v. 9, № 4, p. 607–615.
7. Girardin V., Senoussi R. Semigroup stationary processes and spectral representation. — *Bernoulli*, 2003, v. 9, № 5, p. 857–876.
8. Гладышев Е.Г. Периодические и почти-периодически коррелированные случайные процессы с непрерывным временем. — *Теория вероятн. и ее примен.*, 1963, т. 8, в. 2, с. 184–189.
9. Hurd H. L., Miamer A. G. Periodically Correlated Random Sequences. Spectral Theory and Practice. Hoboken: Wiley, 2007, 353 p.
10. Javorskyj I., Leskow J., Kravets I., Isayev I., Gajecka E. Linear filtration methods for statistical analysis of periodically correlated random processes. Part II: Harmonic series representation. — *Signal Processing*, 2011, v. 91, № 11, p. 2506–2519.
11. Javorskyj I., Leskow J., Kravets I., Isayev I., Gajecka E. Linear filtration methods for statistical analysis of periodically correlated random processes. Part I: Coherent and component methods and their generalization. — *Signal Processing*, 2012, v. 92, № 7, p. 1559–1566.
12. Loève M. Probability Theory. II. New York–Heidelberg: Springer-Verlag, 1978, 413 p.
13. Mallat S. G., Papanicolaou G., Zhang Z. Adaptive covariance estimation of locally stationary processes. — *Ann. Statist.*, 1998, v. 26, № 1, p. 1–47.
14. Oxley M. E., Reid T. F., Suter B. W. Locally stationary processes. — *Proceedings of the Tenth IEEE Workshop on Statistical Signal and Array Processes*, 2000, p. 257–261.
15. Palma W., Olea R., Ferreira G. Estimation and forecasting of locally stationary processes. — *J. Forecast.*, 2013, v. 32, № 1, p. 86–96.
16. Soltani A. R., Hashemi M. Periodically correlated autoregressive Hilbertian processes. — *Statist. Inference Stoch. Process.*, 2011, v. 14, № 2, p. 177–188.
17. Soltani A. R., Parvardeh A. Simple random measures and simple processes. — *Теория вероятн. и ее примен.*, 2005, т. 50, в. 3, с. 533–548.
18. Wahlberga P., Schreier P. J. On Wiener filtering of certain locally stationary stochastic processes. — *Signal Processing*, 2010, v. 90, № 3, p. 885–890.

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