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**PARTIALLY COMPLETE SUFFICIENT STATISTICS
ARE JOINTLY COMPLETE¹⁾**

Теорема, сформулированная в названии статьи (удивительно, что результат не был замечен на протяжении многих лет), существенно дополняет теорию достаточных статистик в части, относящейся к полноте по Леману–Шеффе.

Приводятся два доказательства, одно из которых, аналитическое, следует работе Ландерса и Рогге [23], а другое, чисто статистическое, основано на теории оптимального несмещенного оценивания в сравнительно мало известной ее версии, усовершенствованной Шметтером и Штрассером [42].

Обсуждаются связи с предшествующими результатами и приводятся иллюстрирующие примеры.

Ключевые слова и фразы: полнота по Леману–Шеффе, оптимальное несмещенное оценивание, профильная достаточность, модели усечения, несмещенные оценки с равномерно минимальной дисперсией (UMVUEs).

1. Introduction, main results, and discussion. The main purpose of this paper is to provide with Theorem 1.1 below a result yielding (Lehmann–Scheffé-) completeness in possibly complicated statistical models as a consequence of completeness in suitable submodels, and to illustrate the use of this theorem with short proofs of some classical results. The latter include complete sufficiency in models involving truncation, see Examples 4.4 and 4.5, Subexample 4.6, and Subsubexample 4.7, and we present short and natural proofs of the requisite auxiliary results — Lemma 4.1, Corollary 4.1, and Lemma 4.3. Example 4.5 might be new in its present natural generality.

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The conclusion of joint sufficiency, absent from Theorem 1.1, can be added under a homogeneity assumption, see Theorem 1.4 and Counterexample 5.5.

A secondary purpose of this paper is to correct or refute related completeness claims from the literature, see below Theorem 1.2 and the two paragraphs following it.

We present two proofs of Theorem 1.1, namely in Section 2 a short and direct one, generalizing the original proof of the prototypical Example 4.1 given by Landers and Rogge [23], and, at the end of Section 3, a very short and purely statistical one, utilizing the theory of optimal unbiased estimation as completed by Schmetterer and Strasser [42]. The perhaps surprising possibility of the second proof rests on the apparently not too well-known fact that optimality of an unbiased estimator, in the sense of Definition 3.1, is always equivalent to its measurability with respect to a certain complete but not necessarily sufficient sub- σ -algebra, namely the σ -algebra $\mathcal{O}$ associated to the model $\mathcal{P}$ in the known Theorem 3.1. Theorem 3.1 and the trivial but useful and known Lemma 3.1 easily yield with Theorem 3.2 a lower bound for $\mathcal{O}$ in terms of the σ -algebras $\mathcal{O}_\eta$ corresponding to submodels $\mathcal{P}_\eta$ forming an exhaustion of $\mathcal{P}$; and Theorem 3.2 in turn allows a very short and computation free second proof of Theorem 1.1.

To be more precise, let us introduce some notation and recall basic definitions. With $\text{Prob}(\mathcal{X}, \mathcal{A})$ denoting the set of all laws on the measurable space $(\mathcal{X}, \mathcal{A})$, every set $\mathcal{P} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ is a (*statistical*) *model* on $(\mathcal{X}, \mathcal{A})$, and then every $\mathcal{Q} \subseteq \mathcal{P}$ is a *submodel*, and every family $(\mathcal{P}_\eta: \eta \in H)$ of submodels $\mathcal{P}_\eta$ of $\mathcal{P}$ with $\bigcup_{\eta \in H} \mathcal{P}_\eta = \mathcal{P}$ will here be called a (*parametrized*) *exhaustion* of $\mathcal{P}$. The most common example of the latter is given for a model $\mathcal{P} = \{P_\vartheta: \vartheta \in \Theta\}$ with $\Theta = \Theta_1 \times \Theta_2$ by $\mathcal{P}_\eta := \{P_{\vartheta_1, \eta}: \vartheta_1 \in \Theta_1\}$ for $\eta \in \Theta_2$, as occurring in Corollary 1.1 and there in particular in assumption (i), but other cases as in Example 4.4 are not uncommon.

Let $\mathcal{P} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be a model. A sub- σ -algebra $\mathcal{C}$ of $\mathcal{A}$ is *complete* for $\mathcal{P}$ if every $\mathcal{C}$ -measurable function $h: \mathcal{X} \rightarrow \mathbf{R}$ with vanishing expectations under $\mathcal{P}$, that is,

$$Ph = 0 \quad \text{for } P \in \mathcal{P}, \quad (1)$$

already satisfies

$$h = 0 \quad \mathcal{P}\text{-a.s.}, \quad (2)$$

that is, $h = 0$ P -a.s. for every $P \in \mathcal{P}$. The model $\mathcal{P}$ itself is *complete* if $\mathcal{A}$ is complete for $\mathcal{P}$. A statistic S from $(\mathcal{X}, \mathcal{A})$ to some measurable space $(\mathcal{Y}, \mathcal{B})$ is *complete* for $\mathcal{P}$ if the σ -algebra $\sigma(S)$ it generates on $\mathcal{X}$ is complete for $\mathcal{P}$. While logically unnecessary, statistics are common and

often very convenient for describing sub- σ -algebras in concrete examples, such as Subsubexample 4.7 below, and hence they, rather than the sub- σ -algebras, occur in the title of the present paper.

Completeness as a tool for statistical theory was introduced systematically by Lehmann and Scheffé [26], [27], after special cases had been considered before by Wald [47], [48], Scheffé [40], and Halmos [15]. Its classical use in estimation or testing theories is well known, see [24], [25], [34] for textbook treatments, and [31, Lemma 4.2] for a simple recent example in a rather applied setting. Again recently, completeness has also been used in the econometric literature for studying identifiability problems in instrumental regression models, see, for example, [11].

However, for any given model and sub- σ -algebra, completeness can be difficult to verify even if strongly suspected. Hence sufficient criteria like the following main result of this paper can be useful.

Theorem 1.1. *Let $\mathcal{P} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be a model, and let I be a set. For each $i \in I$, let $\mathcal{C}_i$ be a sub- σ -algebra of $\mathcal{A}$ and $(\mathcal{P}_{i,\eta}: \eta \in \mathbf{H}_i)$ an exhaustion of $\mathcal{P}$ with $\mathcal{C}_i$ complete sufficient for each $\mathcal{P}_{i,\eta}$. Then $\bigvee_{i \in I} \mathcal{C}_i$ is complete for $\mathcal{P}$.*

Here, of course, $\bigvee_{i \in I} \mathcal{C}_i$ denotes the supremum of $\{\mathcal{C}_i: i \in I\}$ in the set of all sub- σ -algebras of $\mathcal{A}$ partially ordered by inclusion.

The hypothesis (1) in the definition of completeness entails that h belongs to

$$\mathcal{L}^1(\mathcal{P}) := \bigcap_{P \in \mathcal{P}} \mathcal{L}^1(P),$$

the set of all functions integrable with respect to every $P \in \mathcal{P}$. There are obvious analogues of Theorem 1.1 and the other results in this paper involving p -completeness with some $p \in]1, \infty[$, where the implication (1) $\Rightarrow$ (2) is only required for $h \in \mathcal{L}^p(\mathcal{P}) := \bigcap_{P \in \mathcal{P}} \mathcal{L}^p(P)$, or bounded completeness, which are not spelled out here except for one remark after Theorem 1.2.

We present two proofs for Theorem 1.1 in Sections 2 and 3 below. Examples and counterexamples are collected in Sections 4 and 5. Let us proceed here by stating explicitly the most transparent nontrivial special case of Theorem 1.1, where $I = \{1, 2\}$ and $\mathcal{P}$ is parametrized by a cartesian product of two sets.

Corollary 1.1. *Let $\mathcal{P} = \{P_{\vartheta}: \vartheta \in \Theta\} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be a model with $\Theta = \Theta_1 \times \Theta_2$ and let $\mathcal{C}_1, \mathcal{C}_2$ be sub- σ -algebras with the properties:*

- (i) *For each $\vartheta_2 \in \Theta_2$, $\mathcal{C}_1$ is complete sufficient for $\{P_{\vartheta_1, \vartheta_2}: \vartheta_1 \in \Theta_1\}$;*
- (ii) *for each $\vartheta_1 \in \Theta_1$, $\mathcal{C}_2$ is complete sufficient for $\{P_{\vartheta_1, \vartheta_2}: \vartheta_2 \in \Theta_2\}$.*

Then $\mathcal{C}_1 \vee \mathcal{C}_2$ is complete for $\mathcal{P}$.

One might rephrase for example assumption (i) above as « $\mathcal{C}_1$ is partially complete sufficient for ϑ_1 », hence the title of the present paper, and instead of «partially», some would prefer «profile».

Theorem 1.1 contains the classical Example 4.1 of Landers and Rogge [23], except for the latter's rather trivial «only if» claim, which cannot be added to Theorem 1.1 by Counterexample 5.1. Similarly, the «if» claim of the special case of Example 4.1, where $I = \{1, 2\}$, is contained in Corollary 1.1, in a result of Rüschendorf [37, Lemma 1] concerning products involving Markov kernels, and in the following Theorem 1.2. We recall that a statistical model is called *homogeneous*, if its members are mutually absolutely continuous.

Theorem 1.2 (essentially Cramer, Kamps, Schenk [10]). *Let Θ_1, Θ_2 be sets and let*

$$\begin{aligned}\mathcal{Q} &:= \{Q_{\vartheta_1} : \vartheta_1 \in \Theta_1\} \subseteq \text{Prob}(\mathcal{X}_1, \mathcal{A}_1), \\ \mathcal{R} &:= \{R_{\vartheta_1, \vartheta_2} : \vartheta_1 \in \Theta_1, \vartheta_2 \in \Theta_2\} \subseteq \text{Prob}(\mathcal{X}_2, \mathcal{A}_2), \\ \mathcal{P} &:= \{Q_{\vartheta_1} \otimes R_{\vartheta_1, \vartheta_2} : \vartheta_1 \in \Theta_1, \vartheta_2 \in \Theta_2\}\end{aligned}$$

be models with these properties:

- (i) $\mathcal{Q}$ is complete;
- (ii) for each $\vartheta_1 \in \Theta_1$, $\{R_{\vartheta_1, \vartheta_2} : \vartheta_2 \in \Theta_2\}$ is complete;
- (iii) for each $\vartheta_2 \in \Theta_2$, $\{R_{\vartheta_1, \vartheta_2} : \vartheta_1 \in \Theta_1\}$ is homogeneous;
- (iv) $\mathcal{L}^1(\mathcal{P}) = \mathcal{L}^1(\{Q_{\vartheta_1} \otimes R_{\vartheta'_1, \vartheta_2} : \vartheta_1, \vartheta'_1 \in \Theta_1, \vartheta_2 \in \Theta_2\})$.

Then $\mathcal{P}$ is complete.

Theorem 1.2 is proved in Subexample 4.2, just after explaining how it implies Example 4.1 with $I = \{1, 2\}$.

Except for a slightly different notation, Theorem 1.2 in its present formulation differs from a claim of [10, Theorem 2 and Remark 2] exactly by the addition of the integrability assumption (iv). The relevance of this assumption in the two proofs of Theorem 1.2 known to us is explained by giving the new one of them in Subexample 4.2 and commenting on the other one afterwards. Whether Theorem 1.2 would remain true if (iv) were omitted seems to be unknown. Counterexample 5.3 shows that the homogeneity assumption (iii) cannot be omitted. A version of Theorem 1.2 proposed by San Martin and Mouchart [39, Theorem 2.1] remains wrong even if assumption (iv) is added, see Counterexample 5.4.

The problem with assumption (iv) in Theorem 1.2 is a good illustration of the fact that bounded completeness, where the implication (1) $\Rightarrow$ (2) is required only for bounded $\mathcal{C}$ -measurable functions h , is often much simpler to treat: If we replace in Theorem 1.2 every «complete» by «boundedly complete», then assumption (iv) can be omitted without substitute, as becomes clear by considering either proof of Theorem 1.2.

To ease now the comparison of Corollary 1.1 with Theorem 1.2, let us rewrite the latter in the style of the former, while reformulating part of the hypothesis using Basu theorems due to Basu [6] and Kagan [17].

Theorem 1.3 (a rewrite of Theorem 1.2). *Let $\mathcal{P} = \{P_\vartheta: \vartheta \in \Theta\} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be a model with $\Theta = \Theta_1 \times \Theta_2$, and let $\mathcal{C}_1, \mathcal{C}_2$ be sub- σ -algebras with the following properties:*

- (i) *For each $\vartheta_2 \in \Theta_2$, $\mathcal{C}_1$ is complete for $\{P_{\vartheta_1, \vartheta_2}: \vartheta_1 \in \Theta_1\}$;*
- (ii) *for each $\vartheta_1 \in \Theta_1$, $\mathcal{C}_1$ is ancillary and $\mathcal{C}_2$ is complete sufficient for $\{P_{\vartheta_1, \vartheta_2}: \vartheta_2 \in \Theta_2\}$;*
- (iii) *for each $\vartheta_2 \in \Theta_2$, $\{P_{\vartheta_1, \vartheta_2}|_{\mathcal{C}_2}: \vartheta_1 \in \Theta_1\}$ is homogeneous;*
- (iv) $\mathcal{L}^1(\{P_\vartheta|_{\mathcal{C}_1} \otimes P_\vartheta|_{\mathcal{C}_2}: \vartheta \in \Theta\}) = \mathcal{L}^1(\{P_\vartheta|_{\mathcal{C}_1} \otimes P_{\vartheta'}|_{\mathcal{C}_2}: \vartheta, \vartheta' \in \Theta\})$.

Then $\mathcal{C}_1 \vee \mathcal{C}_2$ is complete for $\mathcal{P}$.

Thus Theorem 1.3 has, in comparison to Corollary 1.1, the advantage of no sufficiency condition on $\mathcal{C}_1$, but the disadvantage of the ancillarity condition on $\mathcal{C}_1$ in assumption (ii) and the additional assumptions (iii), (iv). Of course, as for Theorem 1.2, it appears unknown whether assumption (iv) may be omitted in Theorem 1.3.

While it seems to us that Corollary 1.1 is more frequently applicable than Theorem 1.2 in either formulation, one might try to look for a natural common generalization. Counterexample 5.2 shows that it is not possible to just omit the sufficiency assumption concerning $\mathcal{C}_1$ in Corollary 1.1, even if the conditions (iii) and (iv) of Theorem 1.3 were added. In other words: Theorem 1.3 would become false if the ancillarity condition in its assumption (ii) were omitted.

As remarked in the previous paragraph, even in the special situation of Corollary 1.1, sufficiency may not be omitted in the hypothesis. Without any additional assumption, it may neither be added in the conclusion by Counterexample 5.5. This suggests that it should be impossible to state Theorem 1.1 just for the case of $I = \{1, 2\}$ and refer to a simple induction argument for the case of a general finite I . It further shows that the homogeneity assumption in Kagan's [18, Theorem 2.1] sharpening of the factorization theorem cannot be omitted. Assuming then homogeneity and a certain connectedness property of our exhaustions, we get the following result, of which, to our surprise, we could not even find its part (a) in the literature.

Theorem 1.4. *Let $\mathcal{P} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be an homogeneous model, and let I be a set. For each $i \in I$, let $\mathcal{C}_i$ be a sub- σ -algebra of $\mathcal{A}$ and $(\mathcal{P}_{i, \eta}: \eta \in H_i)$ an exhaustion of $\mathcal{P}$. Assume that the following property holds: If $P', P'' \in \mathcal{P}$, then there exist $n \in \mathbb{N}$ and $P_1, \dots, P_n \in \mathcal{P}$ with $P_1 = P', P_n = P''$, and such that for each $k \in \{1, \dots, n-1\}$ there exist $i \in I$ and $\eta \in H_i$ with $P_k, P_{k+1} \in \mathcal{P}_{i, \eta}$.*

(a) *For each i , let $\mathcal{C}_i$ be sufficient for each $\mathcal{P}_{i, \eta}$. Then $\bigvee_{i \in I} \mathcal{C}_i$ is sufficient for $\mathcal{P}$.*

(b) *For each i , let $\mathcal{C}_i$ be minimal sufficient for each $\mathcal{P}_{i, \eta}$. Then $\bigvee_{i \in I} \mathcal{C}_i$ is minimal sufficient for $\mathcal{P}$.*

(c) For each i , let $\mathcal{C}_i$ be complete sufficient for each $\mathcal{P}_{i,\eta}$. Then $\bigvee_{i \in I} \mathcal{C}_i$ is complete sufficient for $\mathcal{P}$.

Note that the connectedness assumption concerning the exhaustions in Theorem 1.4 in particular holds if I is finite, $\mathcal{P} = \{P_\vartheta: \vartheta \in \times_{i \in I} \Theta_i\}$, $H_i = \times_{j \in I \setminus \{i\}} \Theta_j$, and each $\mathcal{P}_{i,\eta}$ is obtained by fixing all but the i th of the coordinates of ϑ to coincide with those of η , as in Corollary 1.1, where $I = \{1, 2\}$. In the latter case, by an obvious modification of the proof of Theorem 1.4 given below, the hypothesis in 1.4(a) can in fact be weakened to assuming sufficiency of $\mathcal{C}_1$ for $\{P_{\vartheta_1, \vartheta_2}: \vartheta_1 \in \Theta_1\}$ for each $\vartheta_2 \in \Theta_2$ as before, but sufficiency of $\mathcal{C}_2$ for $\{P_{\vartheta_1, \vartheta_2}: \vartheta_2 \in \Theta_2\}$ for just one $\vartheta_1 \in \Theta_1$.

Let us finally mention that the paper of Oosterhoff and Schriever [33] contains many interesting examples loosely related to the topic of the present paper.

2. Proofs: Theorem 1.1 from scratch, equivalence of Theorems 1.2 and 1.3, Theorem 1.4. Here we present our first proof for Theorem 1.1, which generalizes the original proof of Example 4.1 without being any longer.

FIRST PROOF OF THEOREM 1.1. Let $h: \mathcal{X} \rightarrow \mathbf{R}$ be measurable with respect to $\mathcal{C} := \bigvee_{i \in I} \mathcal{C}_i$ and satisfy (1).

Let $i \in I$. For $\eta \in H_i$, the sufficiency of $\mathcal{C}_i$ for $\mathcal{P}_{i,\eta}$ yields a $g_\eta \in \bigcap_{P \in \mathcal{P}_{i,\eta}} P(h | \mathcal{C}_i)$, for which (1) yields

$$Pg_\eta = 0 \quad \text{for } P \in \mathcal{P}_{i,\eta}$$

and hence, using the completeness of $\mathcal{C}_i$ for $\mathcal{P}_{i,\eta}$,

$$g_\eta = 0 \quad \mathcal{P}_{i,\eta}\text{-a.s.},$$

and thus $P\mathbf{1}_Ch = P\mathbf{1}_Cg_\eta = 0$ for $C \in \mathcal{C}_i$ and $P \in \mathcal{P}_{i,\eta}$. Hence, since $(\mathcal{P}_{i,\eta}: \eta \in H_i)$ exhausts $\mathcal{P}$, for every $C \in \mathcal{C}_i$ the assumption (1) also holds with h replaced by $\mathbf{1}_Ch$.

Inductively repeating the above argument for different i yields

$$P\mathbf{1}_Eh = 0 \quad \text{for } E \in \left\{ \bigcap_{i \in I_0} C_i : I_0 \subseteq I \text{ finite, } C_i \in \mathcal{C}_i \right\} =: \mathcal{E} \text{ and } P \in \mathcal{P}$$

and hence, as $\mathcal{E}$ is a $\cap$ -stable generator of $\mathcal{C}$ with $\mathcal{X} \in \mathcal{E}$, $h = 0$ P -a.s. for $P \in \mathcal{P}$.

PROOF THAT THEOREMS 1.2 AND 1.3 ARE EQUIVALENT. There is clearly no loss of generality in assuming $\mathcal{C}_1 \vee \mathcal{C}_2 = \mathcal{A}$ in Theorem 1.3, which we will do in this proof. Using [8, Theorems 1 and 3], with the first cited theorem due to Basu (see [6, Theorem 2]) and the second due to Kagan [17] and also proved by Barra (see [5, Theorem 3 on pages 26, 27 of the 1981 English edition]), we note that condition 1.3(ii) is equivalent to

(ii') $\mathcal{C}_1, \mathcal{C}_2$ are $\mathcal{P}$ -independent and, for each $\vartheta_1 \in \Theta_1$, $\mathcal{C}_1$ is ancillary and $\mathcal{C}_2$ is complete for $\{P_{\vartheta_1, \vartheta_2}: \vartheta_2 \in \Theta_2\}$.

Although not needed here, let us mention that, if we assume 1.4(iii), then (ii') is further equivalent to the following condition.

(ii'') $\mathcal{C}_1, \mathcal{C}_2$ are $\mathcal{P}$ -independent and, for each $\vartheta_1 \in \Theta_1$, $\mathcal{C}_2$ is complete sufficient for $\{P_{\vartheta_1, \vartheta_2}: \vartheta_2 \in \Theta_2\}$.

To check this, one can apply [8, Theorem 2], say in the version of Koehn and Thomas [21, Corollary], noting that there the nonsplitting assumption may equivalently be imposed on the model restricted to the sufficient σ -algebra, as in Basu's [7] original version.

Theorem 1.2 follows from Theorem 1.3 with (ii') in place of (ii) by letting $\mathcal{C}_i$ denote the σ -algebra generated by the i th coordinate projection in $(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$.

Conversely Theorem 1.3 with (ii') in place of (ii) follows from Theorem 1.2 with $Q_{\vartheta_1} := P_{\vartheta_1, \vartheta_2}|_{\mathcal{C}_1}$ and $R_{\vartheta_1, \vartheta_2} := P_{\vartheta_1, \vartheta_2}|_{\mathcal{C}_2}$ for $\vartheta \in \Theta$, and by observing that every $\mathcal{C}_1 \vee \mathcal{C}_2$ -measurable $h: \mathcal{X} \rightarrow \mathbf{R}$ is of the form $h(x) = g(f_1(x), f_2(x))$ for $x \in \mathcal{X}$ with a $\mathcal{C}_1 \otimes \mathcal{C}_2$ -measurable function $g: \mathcal{X} \times \mathcal{X} \rightarrow \mathbf{R}$ and f_i denoting the identity from $(\mathcal{X}, \mathcal{A})$ to $(\mathcal{X}, \mathcal{C}_i)$.

P r o o f o f T h e o r e m 1.4. By homogeneity, there is a σ -finite measure μ on $(\mathcal{X}, \mathcal{A})$ such that each $P \in \mathcal{P}$ has some $]0, \infty[$ -valued μ -density f_P . Then, by Bahadur's [1, Section 6] version of a result of Lehmann and Scheffé [27, Part I, section 6) as presented in [46, p. 69, Theorem 1.5.9], the σ -algebra

$$\mathcal{C} := \sigma(\mathcal{F}) \quad \text{with } \mathcal{F} := \left\{ \frac{f_{P'}}{f_{P''}}: P', P'' \in \mathcal{P} \right\}$$

is minimal sufficient for $\mathcal{P}$, and, for each $i \in I$ and $\eta \in \mathbf{H}_i$,

$$\mathcal{C}_{i, \eta} := \sigma(\mathcal{F}_{i, \eta}) \quad \text{with } \mathcal{F}_{i, \eta} := \left\{ \frac{f_{P'}}{f_{P''}}: P', P'' \in \mathcal{P}_{i, \eta} \right\}$$

is minimal sufficient for $\mathcal{P}_{i, \eta}$.

(a) For each $i \in I$, the sufficiency assumption on $\mathcal{C}_i$ yields for each $\eta \in \mathbf{H}_i$ first $\mathcal{C}_{i, \eta} \subseteq \mathcal{C}_i [\mathcal{P}_{i, \eta}]$, and then by homogeneity even $\mathcal{C}_{i, \eta} \subseteq \mathcal{C}_i [\mathcal{P}]$, and hence we get

$$\mathcal{C}' := \bigvee_{i \in I} \bigvee_{\eta \in \mathbf{H}_i} \mathcal{C}_{i, \eta} \subseteq \bigvee_{i \in I} \mathcal{C}_i [\mathcal{P}].$$

Let now $g = f_{P'}/f_{P''} \in \mathcal{F}$. Choose $n \in \mathbf{N}$ and $P_1 = P', \dots, P_n = P''$ as assumed to exist. Then $g = \prod_{k=1}^{n-1} f_{P_k}/f_{P_{k+1}}$ is a product of functions each belonging to some $\mathcal{F}_{i, \eta}$, and hence g is $\mathcal{C}'$ -measurable. Thus $\mathcal{C} \subseteq \mathcal{C}'$. Hence $\bigvee_{i \in I} \mathcal{C}_i$ inherits sufficiency for $\mathcal{P}$ from its almost sure sub- σ -algebra $\mathcal{C}'$.

(b) Keeping the notation of part (a), we also have $\mathcal{C}' \subseteq \mathcal{C}$ trivially and hence $\mathcal{C} = \mathcal{C}'$. The stronger minimal sufficiency assumption on $\mathcal{C}_i$ now even

yields $\mathcal{C}_{i,\eta} = \mathcal{C}_i[\mathcal{P}]$ for each i and η , and hence $\mathcal{C}' = \bigvee_{i \in I} \mathcal{C}_i[\mathcal{P}]$. Hence $\bigvee_{i \in I} \mathcal{C}_i$ inherits minimal sufficiency for $\mathcal{P}$ from $\mathcal{C}$.

(c) Clear by combining part (a) with Theorem 1.1.

Theorem 1.4 is proved.

3. A shorter proof of Theorem 1.1 via optimal unbiased estimation. In this section, we give our second and very short proof of Theorem 1.1 by using what we regard as the main version of the basic theory of optimal mean unbiased estimation, for univariate estimands, as completed by Schmetterer and Strasser [42]. In spite of its conciseness and elegance, this theory in its entirety appears to be not widely known, and it is indeed not presented even in the union of the books on mathematical statistics we are aware of and which, like [41], [44], [49], [34], [50], [24], [4], treat unbiased estimation more thoroughly than others. Hence we proceed to give a brief summary in Theorem 3.1 below.

Definition 3.1. Let $\mathcal{P} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be a model, and let

$$\mathcal{E} := \mathcal{L}^1(\mathcal{P}) = \bigcap_{P \in \mathcal{P}} \mathcal{L}^1(P)$$

be the vector space of all measurable functions $g: \mathcal{X} \rightarrow \mathbf{R}$ being integrable with respect to every $P \in \mathcal{P}$. Then, for any function $\kappa: \mathcal{P} \rightarrow \mathbf{R}$, the elements of $\mathcal{E}_\kappa := \{g \in \mathcal{E}: Pg = \kappa(P) \text{ for } P \in \mathcal{P}\}$ are called *unbiased estimators* of the *estimand* κ , and a $\hat{\kappa} \in \mathcal{E}_\kappa$ is called *optimal unbiased* for κ , if

$$P\varphi \circ (\hat{\kappa} - \kappa(P)) \leq P\varphi \circ (g - \kappa(P)) \quad \text{for } g \in \mathcal{E}_\kappa \text{ and } P \in \mathcal{P}$$

holds for every convex function $\varphi: \mathbf{R} \rightarrow \mathbf{R}$. Finally, a $g \in \mathcal{E}$ is called *optimal unbiased*, without reference to any estimand, if g is optimal unbiased for its own expectation $P \mapsto Pg$.

Theorem 3.1 (Rao, Blackwell, Lehmann, Scheffé, Bahadur, Schmetterer, Strasser). *Let $\mathcal{P} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be a model, and let*

$$\mathcal{O} := \{A \in \mathcal{A}: P\mathbf{1}_A h = 0 \text{ for } h \in \mathcal{E}_0 \text{ and } P \in \mathcal{P}\},$$

where $\mathcal{E}_0$ is as in Definition 3.1 with $\kappa = 0$.

(a) $\mathcal{O}$ is a sub- σ -algebra of $\mathcal{A}$ and contains all $\mathcal{P}$ -null sets.

(b) An estimator $\hat{\kappa} \in \mathcal{E}$ is optimal unbiased iff it is $\mathcal{O}$ -measurable, and this is the case iff $\hat{\kappa} \in \bigcap_{P \in \mathcal{P}} P(\hat{\kappa} | \mathcal{O})$ holds for every $\tilde{\kappa} \in \mathcal{E}$ with $P\hat{\kappa} = P\tilde{\kappa}$ for $P \in \mathcal{P}$.

(c) $\mathcal{O}$ is (Lehmann–Scheffé-)complete. If $\mathcal{C} \subseteq \mathcal{A}$ is sufficient, then $\mathcal{O} \subseteq \mathcal{C}[\mathcal{P}]$.

(d) The following statements are equivalent:

- (i) Every unbiasedly estimable parameter has an optimal unbiased estimator.
- (ii) There exists a complete sufficient sub- σ -algebra.
- (iii) $\mathcal{O}$ is sufficient.

If these statements are true, then every complete sufficient sub- σ -algebra $\mathcal{C}$ satisfies $\mathcal{C} = \mathcal{O} [\mathcal{P}]$ and $\mathcal{C} \subseteq \mathcal{O}$.

P r o o f. (a) $\mathcal{O}$ is a Dynkin system and, since $A \in \mathcal{O}$ implies $\mathbf{1}_A h \in \mathcal{E}_0$ for $h \in \mathcal{E}_0$, also $\cap$ -stable. The null set claim is trivial.

(b) The first «only if» follows from [42, Satz 2, the special case of $p = 1$] applied to, say, $W(t) := |t| - \ln(1 + |t|)$ for $t \in \mathbf{R}$. The second «only if» is clear since the definition of $\mathcal{O}$ applied to $h := \tilde{\kappa} - \hat{\kappa}$ yields $0 \in \bigcap_{P \in \mathcal{P}} P(\tilde{\kappa} - \hat{\kappa} \mid \mathcal{O})$. Finally, the first property, namely optimality of $\hat{\kappa}$, follows from the last by the conditional Jensen inequality argument familiar from the proof of the Rao–Blackwell theorem.

(c) $\mathcal{O}$ is complete by the uniqueness theorem for integrals. If $\mathcal{C} \subseteq \mathcal{A}$ is sufficient and $A \in \mathcal{O}$, then $\mathbf{1}_A$ is optimal and its Rao–Blackwellization with respect to $\mathcal{C}$ is better, hence also optimal and hence equal to $\mathbf{1}_A$ almost surely, yielding $A \in \mathcal{C} [\mathcal{P}]$.

(d) [42, Sätze 4 and 5, the special case of $p = 1$].

Theorem 3.1 is proved.

Thus $\mathcal{O}$ above is the σ -algebra generated by all optimal unbiased estimators in the model $\mathcal{P}$, by Theorem 3.1 (b) and by considering the estimators $\mathbf{1}_A$ with $A \in \mathcal{O}$, so let us here briefly call $\mathcal{O}$ the *optimal* σ -algebra of $\mathcal{P}$.

Key sources of Theorem 3.1 include the ones leading to the Rao–Blackwell–Lehmann–Scheffé theorem in 1950, for which partial credit is also due to the noneponymous Halmos, Hodges, and Barankin, in view of the references given by Pfanzagl [34, p. 105–107]. Afterwards, a fundamental idea of Rao [36, p. 30, 31], first made rigorous by Bahadur [3] and later more generally by Torgersen [45] in the mathematically inconvenient and practically less important setting of the UMVU theory, finally led to the present result in the hands of Schmetterer and Strasser [42], after earlier work of themselves and of Padmanabhan, Linnik, and Rukhin cited by them. Further developments include [3], [22], [19], [20].

Turning now to exhaustions of models, there is a trivial but useful remark of Bondesson [9], stated here in the version of Pfanzagl [34, p. 108, Remark 3.2.8].

Lemma 3.1. *Let $(\mathcal{P}_\eta; \eta \in \mathbf{H})$ be an exhaustion of the model $\mathcal{P} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$. If $\hat{\kappa}$ is an optimal unbiased estimator in each of the submodels $\mathcal{P}_\eta$, then so it is in $\mathcal{P}$.*

P r o o f. Absolutely trivial by Definition 3.1 and the definition of «exhaustion».

Combining Theorem 3.1 with Lemma 3.1 yields the following result.

Theorem 3.2. Let $(\mathcal{P}_\eta; \eta \in H)$ be an exhaustion of the model $\mathcal{P}$, and let $\mathcal{O}$ and $\mathcal{O}_\eta$, respectively, denote the optimal σ -algebras of $\mathcal{P}$ and $\mathcal{P}_\eta$ for $\eta \in H$. Then $\bigcap_{\eta \in H} \mathcal{O}_\eta \subseteq \mathcal{O}$.

P r o o f. If $A \in \bigcap_{\eta \in H} \mathcal{O}_\eta$, then Theorem 3.1 (b) yields that $\mathbf{1}_A$ is an optimal unbiased estimator in each of the models $\mathcal{P}_\eta$, and hence in $\mathcal{P}$ by Lemma 3.1, and hence is $\mathcal{O}$ -measurable by Theorem 3.1 (b) again. Thus $\bigcap_{\eta \in H} \mathcal{O}_\eta \subseteq \mathcal{O}$. Theorem 3.2 is proved.

Finally, Theorems 3.1 and 3.2 yield:

S e c o n d p r o o f o f T h e o r e m 1.1. Let $\mathcal{O}$ and $\mathcal{O}_{i,\eta}$ denote the optimal σ -algebras of $\mathcal{P}$ and $\mathcal{P}_{i,\eta}$ for $i \in I$ and $\eta \in H_i$. Then $\mathcal{C}_i \subseteq \mathcal{O}_{i,\eta}$ whenever $\eta \in H_i$, by Theorem 3.1 (d) applied to $\mathcal{P}_{i,\eta}$ and its complete sufficient sub- σ -algebra $\mathcal{C}_i$. Hence $\mathcal{C}_i \subseteq \bigcap_{\eta \in H_i} \mathcal{O}_{i,\eta} \subseteq \mathcal{O}$ for every $i \in I$, by Theorem 3.2, and hence $\bigvee_{i \in I} \mathcal{C}_i \subseteq \mathcal{O}$. As $\mathcal{O}$ is complete for $\mathcal{P}$ by Theorem 3.1 (c), so is its sub- σ -algebra $\bigvee_{i \in I} \mathcal{C}_i$. Theorem 1.1 is proved.

4. Examples, including a proof of Theorem 1.2.

E x a m p l e 4.1 (product models, Landers and Rogge [23]). Let I be a set. For each $i \in I$, let $\mathcal{P}_i \subseteq \text{Prob}(\mathcal{X}_i, \mathcal{A}_i)$ be a model. Then $\mathcal{P} := \{\bigotimes_{i \in I} P_i; P_i \in \mathcal{P}_i \text{ for } i \in I\}$ is complete if each $\mathcal{P}_i$ is complete, and the converse is also true if the $\mathcal{P}_i$ are assumed to be nonempty.

P r o o f. Let $(\mathcal{X}, \mathcal{A})$ be the product of the $(\mathcal{X}_i, \mathcal{A}_i)$. For $i \in I$, let $\mathcal{C}_i$ be the sub- σ -algebra of $\mathcal{A}$ generated by the i th coordinate projection $\pi_i: \mathcal{X} \rightarrow \mathcal{X}_i$, $H_i := \bigtimes_{j \in I \setminus \{i\}} \mathcal{P}_j$, and

$$\mathcal{P}_{i,\eta} := \left\{ \bigotimes_{k \in I} P_k; P_i \in \mathcal{P}_i, P_j = Q_j \text{ for } j \in I \setminus \{i\} \right\}$$

for $\eta = (Q_j; j \in I \setminus \{i\}) \in H_i$.

Assume that each $\mathcal{P}_i$ is complete. Then, for $i \in I$, $(\mathcal{P}_{i,\eta}; \eta \in H_i)$ exhausts $\mathcal{P}$ and, for $\eta \in H_i$, $\mathcal{C}_i$ is complete for $\mathcal{P}_{i,\eta}$, since (1) for h $\mathcal{C}_i$ -measurable, and thus $h = g \circ \pi_i$ for some $\mathcal{A}_i$ -measurable g , here yields $0 = (\bigotimes_{k \in I} P_k)h = P_i g$ for $P_i \in \mathcal{P}_i$, hence $g = 0$ $\mathcal{P}_i$ -a.s., hence $h = 0$ $\mathcal{P}$ -a.s., and $\mathcal{C}_i$ is sufficient for $\mathcal{P}_{i,\eta}$ by [8, Theorem 3], since $\mathcal{D}_i := \bigvee_{j \in I \setminus \{i\}} \mathcal{C}_j$ is ancillary (under $\mathcal{P}_{i,\eta}$) and $\mathcal{C}_i, \mathcal{D}_i$ are independent with $\mathcal{C}_i \vee \mathcal{D}_i$ sufficient. Hence $\bigvee_{i \in I} \mathcal{C}_i = \mathcal{A}$ is complete for $\mathcal{P}$ by Theorem 1.1.

Assume that $\mathcal{P} \neq \emptyset$ is complete, $i \in I$, and $P_i h = 0$ for $P_i \in \mathcal{P}_i$. Then $P h \circ \pi_i = 0$ for $P \in \mathcal{P}$, hence $h \circ \pi_i = 0$ $\mathcal{P}$ -a.s., and hence, using $\mathcal{P} \neq \emptyset$, $h = 0$ $\mathcal{P}_i$ -a.s. Example 4.1 is proved.

We recall that Example 4.1 is the basic tool for proving complete sufficiency of «the vector of order statistics» in certain nonparametric models, see, for example, [28], [34, p. 21], and [29, p. 1267], where also [32] should have been cited as explained in [30, p. 405].

Let us also mention that it took some twenty years from the desire to have Example 4.1 at least for finite I , shining through analogous results involving a more restrictive assumption of «strong completeness» in [27, Part II, section 7] or [14, p. 26], to the the proof in [23], and that even the analogue involving bounded completeness was provided only about one year earlier in [35] with a somewhat complicated proof.

The special case of $I = \{1, 2\}$ of the «if»-statement of Example 4.1 is contained in Theorem 1.2, as essentially already remarked in [10]: If, in the notation of Theorem 1.2, the laws do not actually depend on ϑ_1 , then conditions (iii) and (iv) are trivially fulfilled, and assumption (ii) is just the completeness of $\mathcal{R}$. Conversely, but less obviously, one can go the other way round.

S u b e x a m p l e 4.2. Theorem 1.2 can be deduced from Example 4.1.

P r o o f. Under the assumptions of Theorem 1.2, let $h: \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathbf{R}$ satisfy (1). Then

$$\int \left(\int h(x_1, x_2) dQ_{\vartheta_1}(x_1) \right) dR_{\vartheta_1, \vartheta_2}(x_2) = 0 \quad \text{for } \vartheta_1 \in \Theta_1 \text{ and } \vartheta_2 \in \Theta_2$$

by Fubini. For each $\vartheta_1 \in \Theta_1$, the completeness assumption (ii) yields

$$\int h(x_1, x_2) dQ_{\vartheta_1}(x_1) = 0 \tag{3}$$

first for $\{R_{\vartheta_1, \vartheta_2}: \vartheta_2 \in \Theta_2\}$ -a.e. x_2 , and then, using the homogeneity assumption (iii), even for $\{R_{\vartheta'_1, \vartheta_2}: \vartheta'_1 \in \Theta_1, \vartheta_2 \in \Theta_2\}$ -a.e. x_2 . Hence

$$\int \left(\int h(x_1, x_2) dQ_{\vartheta_1}(x_1) \right) dR_{\vartheta'_1, \vartheta_2}(x_2) = 0 \quad \text{for } \vartheta_1, \vartheta'_1 \in \Theta_1, \vartheta_2 \in \Theta_2.$$

Now, thanks to assumption (iv), the $\mathcal{P}$ -integrable function h is also integrable with respect to each $Q_{\vartheta_1} \otimes R_{\vartheta'_1, \vartheta_2}$ and hence Fubini yields

$$\int h dQ_{\vartheta_1} \otimes R_{\vartheta'_1, \vartheta_2} = 0 \quad \text{for } \vartheta_1, \vartheta'_1 \in \Theta_1, \vartheta_2 \in \Theta_2 \tag{4}$$

and hence, applying Example 4.1 for $I = \{1, 2\}$ to the models $\mathcal{Q}$ and $\mathcal{R}$, which are complete by (i) and (ii), we get $h = 0$ $Q_{\vartheta_1} \otimes R_{\vartheta'_1, \vartheta_2}$ -a.s. for $\vartheta_1, \vartheta'_1 \in \Theta_1$ and $\vartheta_2 \in \Theta_2$, and hence in particular (2). Subexample 4.2 is proved.

The proof in Cramér et al. [10, pages 273, 274]), valid under the present additional assumption (iv), uses the original proof of Example 4.1, rather than the result, and is hence a bit longer. On the other hand, starting from (3), instead of having to conclude (4), they would only need to justify the second equality in

$$\begin{aligned} 0 &= \int_B \left(\int h(x_1, x_2) dQ_{\vartheta_1}(x_1) \right) dR_{\vartheta'_1, \vartheta_2}(x_2) \\ &= \int \left(\int_B h(x_1, x_2) dR_{\vartheta'_1, \vartheta_2}(x_2) \right) dQ_{\vartheta_1}(x_1) \quad \text{for } B \in \mathcal{A}_2, \end{aligned}$$

which, in view of an example in [13], might also hold for some functions h not being $Q_{\vartheta_1} \otimes R_{\vartheta'_1, \vartheta_2}$ -integrable.

Example 4.3. Completeness for multiparameter exponential models, with natural parameter spaces with nonempty interiors, follows from the one-parameter case.

Proof. Let $\mathcal{P} = \{P_\alpha: \alpha \in A\}$ be a k -parameter exponential model with natural parameter space $A \subseteq \mathbf{R}^k$, that is, each P_α has a μ -density $x \mapsto c(\alpha)h(x) \exp(\sum_{i=1}^k \alpha_i T_i(x))$, and let $A_0 \subseteq A$ be nonempty and open. Then Theorem 1.1, applied to $\mathcal{P}_0 := \{P_\alpha: \alpha \in A_0\}$, $I := \{1, \dots, k\}$, $\mathcal{C}_i := \sigma(T_i)$, and the $\mathcal{P}_{i,\eta}$ being the one-parameter exponential models obtained from $\mathcal{P}_0$ by varying α_i while keeping all other parameter coordinates fixed, yields the completeness of $\bigvee_{i \in I} \mathcal{C}_i$ for $\mathcal{P}_0$ and hence, by homogeneity of $\mathcal{P}$, also for $\mathcal{P}$. Example 4.3 is proved.

The above reduction from the multiparameter to the one-parameter case appears to be shorter than the one in [34, p. 26, 27].

The following lemma and its corollary, provided here in preparation for Example 4.5 and Subexample 4.6, contain in particular the determination of complete sufficient statistics in discrete as well as continuous and even mixed «taxi problem models» with unknown lower and upper bounds, compare [12, Example (e) on p. 226 and the Exercises 8, 9 on pages 237, 238] for the discrete case, without having to calculate the joint laws of sample minima and maxima as apparently intended in [24, Problem 6.30 on p. 72].

If $(\mathcal{X}, \mathcal{A}, \mu)$ is a measure space and $E \in \mathcal{A}$ with $0 < \mu(E) < \infty$, then we consider the conditional law $\mu(\cdot | E) := \mu(\cdot \cap E)/\mu(E)$.

Lemma 4.1. Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space, $\mathcal{E} \subseteq \mathcal{A}$, $n \in \mathbf{N}$, $\mathcal{P} := \{\mu(\cdot | E)^{\otimes n}: E \in \mathcal{E}, 0 < \mu(E) < \infty\}$, and $\mathcal{C} := \sigma(E^n: E \in \mathcal{E})$.

(a) Let μ be σ -finite. Then $\mathcal{C}$ is sufficient for $\mathcal{P}$.

(b) Let $\mathcal{E}$ be $\cap$ -stable. Then $\mathcal{C}$ is complete for $\mathcal{P}$.

Proof. Let $\mathcal{E}_0 := \{E \in \mathcal{E}: 0 < \mu(E) < \infty\}$ and $P_E := \mu(\cdot | E)^{\otimes n}$ for $E \in \mathcal{E}_0$.

(a) For $E \in \mathcal{E}_0$, the function $(1/\mu(E)^n) \mathbf{1}_{E^n}$ is a $\mathcal{C}$ -measurable density of P_E , with respect to the σ -finite measure $\mu^{\otimes n}$. Hence $\mathcal{C}$ is sufficient for $\mathcal{P}$ by the factorization criterion.

(b) Let $h: \mathcal{X} \rightarrow \mathbf{R}$ be $\mathcal{C}$ -measurable with (1). Let $E_0 \in \mathcal{E}_0$. For $E \in \mathcal{E}$, we then have

$$\int_{(E \cap E_0)^n} h \, dP_{E_0} = \int h \, d\left(\mathbf{1}_{E \cap E_0} \mu(\cdot | E_0)\right)^{\otimes n} = 0,$$

namely trivially if $\mu(E \cap E_0) = 0$, and otherwise by (1), since then $(\mathbf{1}_{E \cap E_0} \mu(\cdot | E_0))^{\otimes n} = (\mu(E \cap E_0)/\mu(E_0))^n P_{E \cap E_0}$ and $P_{E \cap E_0} \in \mathcal{P}$. Now on E_0^n , $\{(E \cap E_0)^n: E \in \mathcal{E}\}$ is a $\cap$ -stable generator of the trace of $\mathcal{C}$, and contains E_0^n , and hence we get $h = 0$ P_{E_0} -a.s. Lemma 4.1 is proved.

By Counterexample 5.6, the above σ -finiteness assumption cannot be omitted.

In Corollary 4.1, Example 4.4, and Subexample 4.6 below, $\mathcal{X}$ is assumed to be a subset of the extended real line $\overline{\mathbf{R}}$. We then call a set $J \subseteq \mathcal{X}$ an *upray in $\mathcal{X}$* , if $x \in J, y \in \mathcal{X}, x \leq y$ jointly imply $y \in J$. We analogously define *downray in $\mathcal{X}$* . And we call I an *interval in $\mathcal{X}$* , if $x, z \in I, y \in \mathcal{X}, x \leq y \leq z$ jointly imply $y \in I$. Clearly, in $\mathcal{X}$, every ray is an interval, and I is an interval iff $I = J \cap K$ for an upray J and a downray K .

If $(X_i: i \in I)$ is a family of functions X_i , all with the same domain of definition Ω , then we write $(\downarrow X_i: i \in I)$ for the function $\Omega \ni \omega \mapsto (X_i(\omega): i \in I)$.

Corollary 4.1 (Completeness of min and max in truncation models). *Let μ be a measure on a measurable subspace $(\mathcal{X}, \mathcal{A})$ of $\overline{\mathbf{R}}$, $n \in \mathbf{N}$, and $(\downarrow X_1, \dots, \downarrow X_n) := \text{id}_{\mathcal{X}^n}$.*

(a) $\min_{i=1}^n X_i$ is complete sufficient for $\{\mu(\cdot | J)^{\otimes n}: J \text{ upray in } \mathcal{X}, 0 < \mu(J) < \infty\}$.

(b) $\max_{i=1}^n X_i$ is complete sufficient for $\{\mu(\cdot | K)^{\otimes n}: K \text{ downray in } \mathcal{X}, 0 < \mu(K) < \infty\}$.

(c) $(\downarrow \min_{i=1}^n X_i, \max_{i=1}^n X_i)$ is complete for

$$\{\mu(\cdot | I)^{\otimes n}: I \text{ interval in } \mathcal{X}, 0 < \mu(I) < \infty\}$$

and, if μ is assumed to be σ -finite, also sufficient.

P r o o f. We apply Lemma 4.1 with, respectively, (a) $\mathcal{E} :=$ set of all uprays in $\mathcal{X}$ and $\sigma(E^n: E \in \mathcal{E}) = \sigma((\min_{i=1}^n X_i)^{-1}(E): E \in \mathcal{E}) = \sigma(\min_{i=1}^n X_i)$, where the last identity holds since $\mathcal{E}$ generates $\mathcal{A}$,

(b) $\mathcal{E} :=$ set of all downrays in $\mathcal{X}$ and $\sigma(E^n: E \in \mathcal{E}) = \sigma(\max_{i=1}^n X_i)$,

(c) $\mathcal{E} :=$ set of all intervals in $\mathcal{X}$ and

$$\begin{aligned} \sigma(E^n: E \in \mathcal{E}) &= \sigma((J \cap K)^n: J \text{ upray}, K \text{ downray}) \\ &= \sigma\left(\left(\min_{i=1}^n X_i, \max_{i=1}^n X_i\right)^{-1}(J \times K): J \text{ upray}, K \text{ downray}\right) \\ &= \sigma\left(\min_{i=1}^n X_i, \max_{i=1}^n X_i\right), \end{aligned}$$

and, in parts (a) and (b), with μ replaced by the σ -finite measure $\mu(\cdot \cap \mathcal{X}_0)$ with $\mathcal{X}_0 := \bigcup\{E \in \mathcal{E}: \mu(E) < \infty\}$.

E x a m p l e 4.4. The completeness assertion in Corollary 4.1(c) also follows from Corollary 4.1, (a), (b) via Theorem 1.1, without using Lemma 4.1(b).

P r o o f. Let $\mathcal{I} := \{I \subseteq \mathcal{X}: I \text{ interval in } \mathcal{X}, 0 < \mu(I) < \infty\}$ and

$$\Theta := \{(J, K): J \text{ upray in } \mathcal{X}, K \text{ downray in } \mathcal{X}, J \cap K \in \mathcal{I}\}.$$

For $(J, K) \in \Theta$, let $\pi_1(J, K) := J$ and $\pi_2(J, K) := K$. Let H_2, H_1 , in this order, denote the images of the coordinate projections π_1, π_2 . If

$K \in H_1$ is fixed, then $\min_{\nu=1}^n X_\nu$ is complete sufficient for $\mathcal{P}_{1,K} := \{\mu(\cdot | J \cap K)^{\otimes n} : (J, K) \in \Theta\}$, by Corollary 4.1(a) with K and $\mu(\cdot | K)$ in the roles of $\mathcal{X}$ and μ . Analogously, if $J \in H_2$ is fixed, then $\max_{\nu=1}^n X_\nu$ is complete sufficient for $\mathcal{P}_{2,J} := \{\mu(\cdot | J \cap K)^{\otimes n} : (J, K) \in \Theta\}$. As each of $(\mathcal{P}_{1,K} : K \in H_1)$ and $(\mathcal{P}_{2,J} : J \in H_2)$ is an exhaustion of the model $\mathcal{P}$ of Corollary 4.1(c), completeness of $(\min_{\nu=1}^n X_\nu, \max_{\nu=1}^n X_\nu)$ follows from Theorem 1.1. Example 4.4 is proved.

We next recall as Lemma 4.3 below part of a result by Smith [43] about weighted models, for which [38] may serve as an introduction. We provide a short proof for convenience, and also since our part (b), being slightly more general than the original, might appear to contradict the correct remark in [43, p. 248, second line after Theorem]. To this end, we need the following perhaps not too well-known probabilistic property of conditional expectations implicitly proved in [43, p. 249].

Lemma 4.2 (strictness in the isotonicity of conditional expectations; Smith, 1957). *Let X, Y be $\bar{\mathbf{R}}$ -valued random variables on $(\Omega, \mathcal{A}, \mathbf{P})$ with $X \leq Y$ a.s. and with existing expectations, possibly infinite. Let $\mathcal{C}$ be a sub- σ -algebra of $\mathcal{A}$ and let $X_0 \in \mathbf{E}(X | \mathcal{C})$ and $Y_0 \in \mathbf{E}(Y | \mathcal{C})$. Then $X_0 \leq Y_0$ a.s. If in addition neither $\mathbf{E}X = \mathbf{E}Y = \infty$ nor $\mathbf{E}X = \mathbf{E}Y = -\infty$, then $X_0 < Y_0$ a.s. on $\{X < Y\}$.*

P r o o f. The first claim is of course standard, see, e.g., [16, v. 1, p. 452]. Under the additional assumption, $Y - X$ and $Y_0 - X_0$ are defined a.s., and using $\{X_0 = Y_0\} \in \mathcal{C}$ yields

$$\begin{aligned} \mathbf{E}(Y - X)\mathbf{1}_{\{X_0=Y_0\}} &= \mathbf{E}Y\mathbf{1}_{\{X_0=Y_0\}} - \mathbf{E}X\mathbf{1}_{\{X_0=Y_0\}} \\ &= \mathbf{E}Y_0\mathbf{1}_{\{X_0=Y_0\}} - \mathbf{E}X_0\mathbf{1}_{\{X_0=Y_0\}} = 0 \end{aligned}$$

and hence $X = Y$ a.s. on $\{X_0 = Y_0\}$, which, by contraposition and since $X \leq Y$ and $X_0 \leq Y_0$ a.s., yields the second claim. Lemma 4.2 is proved.

Lemma 4.3 (permanence of sufficiency and complete sufficiency under a fixed weighing; Smith, 1957). *Let $\mathcal{P} \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be a statistical model, $q: \mathcal{X} \rightarrow [0, \infty[$ $\mathcal{P}$ -integrable, and $\mathcal{P}_q := \{P_q: P \in \mathcal{P}, Pq > 0\}$ with P_q denoting the q -weighted version of P , given by $P_q(A) := P\mathbf{1}_A q / Pq$ for $A \in \mathcal{A}$.*

(a) *Let $\mathcal{C}$ be sufficient for $\mathcal{P}$. Then $\mathcal{C}$ is sufficient for $\mathcal{P}_q$.*

(b) *Let $\mathcal{C}$ be complete sufficient for $\mathcal{P}$. Then $\mathcal{C}$ is complete sufficient for $\mathcal{P}_q$.*

P r o o f. (a) There is an $f \in \bigcap_{P \in \mathcal{P}} P(q | \mathcal{C})$. For $P_q \in \mathcal{P}_q$, we then have $P_q(C) = P\mathbf{1}_C q / Pq = P\mathbf{1}_C f / Pq$ for $C \in \mathcal{C}$, and so f/Pq is a $P|_{\mathcal{C}}$ -density of $P_q|_{\mathcal{C}}$. Now let $A \in \mathcal{A}$ be given. With $g \in \bigcap_{P \in \mathcal{P}} P(\mathbf{1}_A q | \mathcal{C})$, we let $h := g/f$ with $0/0 := 0$, and, since $\mathbf{1}_A q \leq q$, we $\mathcal{P}$ -a.s. have $g \leq f$ and hence the implication $f = 0 \Rightarrow g = 0$. Hence, for $P \in \mathcal{P}_q$, we have $P_q\mathbf{1}_C h = P\mathbf{1}_C h f / Pq = P\mathbf{1}_C g / Pq = P\mathbf{1}_C \mathbf{1}_A q / Pq = P_q\mathbf{1}_C \mathbf{1}_A$ for $C \in \mathcal{C}$, and hence $h \in \bigcap_{P_q \in \mathcal{P}_q} P_q(A | \mathcal{C})$.

(b) Let $h: \mathcal{X} \rightarrow \mathbf{R}$ be $\mathcal{C}$ -measurable with $P_q h = 0$ for $P_q \in \mathcal{P}_q$. Then, trivially for $P \in \mathcal{P}$ with $Pq = 0$ and hence for every $P \in \mathcal{P}$, we have $0 = Pqh = Pgh$ with $g \in \bigcap_{P \in \mathcal{P}} P(q|\mathcal{C})$ chosen by sufficiency. By completeness of $\mathcal{C}$ for $\mathcal{P}$ and by $\mathcal{C}$ -measurability of gh , we have, for every $P \in \mathcal{P}$, first $gh = 0$ P -a.s. and then, since $g > 0$ P -a.s. on $\{q > 0\}$ by Lemma 4.2 applied to $X := 0$ and $Y := q$, we have $h = 0$ P -a.s. on $\{q > 0\}$, and finally $h = 0$ $\mathcal{P}_q$ -a.s. Lemma 4.3 is proved.

We remark that Smith [43] considers also minimal sufficiency.

Example 4.5 (modification of complete sufficiency under an unknown truncation). Let $\mathcal{P}_0 \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be a model, $\mathcal{E} \subseteq \mathcal{A}$ $\cap$ -stable, and $P_E := P(\cdot | E)$ for $P \in \mathcal{P}_0$ and $E \in \mathcal{E}$ with $P(E) > 0$. Let $n \in \mathbf{N}$ and $\mathcal{C}$ complete sufficient for $\{P^{\otimes n}: P \in \mathcal{P}_0\}$. Then $\mathcal{C} \vee \sigma(E^n: E \in \mathcal{E})$ is complete sufficient for $\{P_E^{\otimes n}: P \in \mathcal{P}_0, E \in \mathcal{E}, P(E) > 0\}$.

Proof. We apply Theorem 1.1 with $I := \{1, 2\}$: Let $H_1 := \mathcal{E}$ and $\mathcal{P}_{1,E} := \{P_E^{\otimes n}: P \in \mathcal{P}_0, P(E) > 0\}$ for $E \in H_1$. Then $\mathcal{C}_1 := \mathcal{C}$ is complete sufficient for each $\mathcal{P}_{1,E}$, by Lemma 4.3 applied to $\{P^{\otimes n}: P \in \mathcal{P}_0\}$ and $q := \mathbf{1}_{E^n}$. Let $H_2 := \mathcal{P}_0$ and $\mathcal{P}_{2,P} := \{P_E^{\otimes n}: E \in \mathcal{E}, P(E) > 0\}$ for $P \in \mathcal{P}_0$. Then $\mathcal{C}_2 := \sigma(E^n: E \in \mathcal{E})$ is complete sufficient for each $\mathcal{P}_{2,P}$, by Lemma 4.1 with $\mu = P$. Finally, each $(\mathcal{P}_{i,\eta}: \eta \in H_i)$ exhausts $\mathcal{P}$. Example 4.5 is proved.

Subexample 4.6 (truncated exponential families on the line). Let $(\mathcal{X}, \mathcal{A})$ be a measurable subspace of $\overline{\mathbf{R}}$, and let $\mathcal{P}_0 \subseteq \text{Prob}(\mathcal{X}, \mathcal{A})$ be a k -parameter exponential family in the statistics $T_1, \dots, T_k: \overline{\mathbf{R}} \rightarrow \mathbf{R}$ and the natural parameters $a_1, \dots, a_k: \mathcal{P}_0 \rightarrow \mathbf{R}$ with $\{(a_1(P), \dots, a_k(P)): P \in \mathcal{P}_0\}$ having nonempty interior. Let $n \in \mathbf{N}$, $(X_1, \dots, X_n) := \text{id}_{\mathcal{X}^n}$, and $S := (\sum_{i=1}^n T_1(X_i), \dots, \sum_{i=1}^n T_k(X_i))$.

(a) $(S, \min_{i=1}^n X_i)$ is complete sufficient for

$$\{P(\cdot | J)^{\otimes n}: P \in \mathcal{P}_0, J \text{ upray in } \mathcal{X}, P(J) > 0\}.$$

(b) $(S, \max_{i=1}^n X_i)$ is complete sufficient for

$$\{P(\cdot | K)^{\otimes n}: P \in \mathcal{P}_0, K \text{ downray in } \mathcal{X}, P(K) > 0\}.$$

(c) $(S, \min_{i=1}^n X_i, \max_{i=1}^n X_i)$ is complete sufficient for

$$\{P(\cdot | I)^{\otimes n}: P \in \mathcal{P}_0, I \text{ interval in } \mathcal{X}, P(I) > 0\}.$$

Proof. Example 4.5 with $\mathcal{C} := \sigma(S)$ and Corollary 4.1.

Subsample 4.7 (see, e.g., Lehmann and Casella [24, p. 43]). Let $X_1, \dots, X_n$ be i.i.d. according to the shifted exponential law, with λ -density given by $f_{a,b}(x) = (1/b) \exp(-(x-a)/b) \mathbf{1}_{[a,\infty[}(x)$ for $x \in \mathbf{R}$, where $a \in \mathbf{R}$ and $b \in]0, \infty[$. Then $\min_{i=1}^n X_i$ and $\sum_{i=1}^n X_i$ are jointly complete sufficient.

Proof. Let us first replace $\ll a \in \mathbf{R} \gg$ by $\ll a > a_0 \gg$ with $a_0 \in \mathbf{R}$ fixed. Then the claim is a special case of Subexample 4.6 (a), with $\mathcal{X} = \mathbf{R}$,

$\mathcal{P}_0 := \{f_{a_0, b} \lambda : b \in]0, \infty[, k = 1, T_1(x) = x$. The claim with « $a \in \mathbf{R}$ » then follows by writing the model in question as a union of the increasing sequence of the models with « $a > a_0$ » with $a_0 \in -\mathbf{N}$. Subsubexample 4.7 is proved.

The proof of Subsubexample 4.7 in [24, p. 43] uses the independence of $X_{(1)} := \min_{i=1}^n X_i$ and $\sum_{i=1}^n (X_i - X_{(1)})$. Note that, nevertheless, the Landers–Rogge theorem Example 4.1 would not apply, since the law of $X_{(1)}$ depends on both parameters a and b . On the other hand, Theorem 1.2 does apply, as intended in [10, p. 271], but to check condition 1.3(iv), one apparently has to do some computation like the following: The densities of $\sum_{i=1}^n (X_i - X_{(1)})$ and $X_{(1)}$ are, up to constants depending only on the parameters a and b , given by $g_b(x) := x^{n-2} \exp(-x/b) \mathbf{1}_{]a, \infty[}(x)$ and $h_{a,b}(y) := \exp(-ny/b) \mathbf{1}_{]0, \infty[}(y)$, and we have $g_b(x) h_{a,b'}(y) \leq g_{b \wedge b'}(x) h_{a, b \wedge b'}(y)$.

5. Counterexamples. We cannot add an «only if» statement in Theorem 1.1, as present in its special case Example 4.1, not even in Corollary 1.1 and with «sufficiency» strengthened to «minimal sufficiency» in the hypothesis.

C o u n t e r e x a m p l e 5.1. There exists a model $\mathcal{P} = \{P_\vartheta : \vartheta \in \Theta_1 \times \Theta_2\}$ with sub- σ -algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C}_1$ is minimal sufficient but incomplete for each $\{P_{\vartheta_1, \vartheta_2} : \vartheta_1 \in \Theta_1\}$ with $\vartheta_2 \in \Theta_2$, $\mathcal{C}_2$ is minimal sufficient but incomplete for each $\{P_{\vartheta_1, \vartheta_2} : \vartheta_2 \in \Theta_2\}$ with $\vartheta_1 \in \Theta_1$, and $\mathcal{C}_1 \vee \mathcal{C}_2$ is complete sufficient for $\mathcal{P}$.

P r o o f. We may take $\Theta_1 = \Theta_2 =]0, \infty[, n \in \mathbf{N}$, and P_ϑ the law of n i.i.d. normal random variables X_i , each with the density $\mathbf{R} \ni x \mapsto c(\vartheta) \exp(\vartheta_1 \vartheta_2 x - \vartheta_1^2 \vartheta_2^3 x^2)$, and $\mathcal{C}_1 = \mathcal{C}_2 = \sigma(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$. Here the first «minimal sufficient but incomplete» claim follows from the linear independence but algebraic dependence of $1, \vartheta_1 \vartheta_2, -\vartheta_1^2 \vartheta_2^3$ as functions of ϑ_1 , with ϑ_2 fixed, compare [34, Theorem 1.6.9 and Wijsman's Theorem 1.6.23]. The analogous second claim follows similarly. Completeness of $\mathcal{C}_1 \vee \mathcal{C}_2$ follows from the openness of $\{(\vartheta_1 \vartheta_2, -\vartheta_1^2 \vartheta_2^3) : \vartheta \in \Theta_1 \times \Theta_2\} =]0, \infty[\times]-\infty, 0[$. Counterexample 5.1 is proved.

Corollary 1.1 would become false if any of the two sufficiency assumptions were omitted, even if the assumptions (iii) and (iv) of Theorem 1.3 were added.

C o u n t e r e x a m p l e 5.2. Theorem 1.3 would become false if the ancillarity condition in its assumption (ii) were omitted.

P r o o f. We may take $\mathcal{X} := \{0, 1\}^2$, $\mathcal{A} := 2^{\mathcal{X}}$, $\Theta_1 := \{0, 1\}$, $\Theta_2 :=]0, 1/2[$, and, for $\vartheta \in \Theta$,

$$P_\vartheta := \begin{cases} ((1 - \vartheta_2)\delta_0 + \vartheta_2\delta_1)^{\otimes 2} & \text{if } \vartheta_1 = 0, \\ (\vartheta_2\delta_0 + (1 - \vartheta_2)\delta_1)^{\otimes 2} & \text{if } \vartheta_1 = 1, \end{cases}$$

$(X_1, X_2) := \text{id}_{\mathcal{X}}$, $\mathcal{C}_1 := \sigma(X_1)$, and $\mathcal{C}_2 := \sigma(X_1 + X_2)$. The completeness of

$\mathcal{C}_1$ required by 1.3(i) is easily checked, the complete sufficiency of $\mathcal{C}_2$ in 1.3 (ii) is a standard result for Bernoulli chains, and the assumptions 1.3 (iii), (iv) are obviously fulfilled. But $\mathcal{C}_1 \vee \mathcal{C}_2 = \sigma(X_1, X_2)$ is incomplete, since we have $P_\vartheta(X_1 - X_2) = 0$ for each $\vartheta \in \Theta$. Counterexample 5.2 is proved.

C o u n t e r e x a m p l e 5.3. Theorem 1.2 would become false if its homogeneity assumption (iii) were omitted.

P r o o f. Let $\mathcal{X}_1 := \mathcal{X}_2 := \Theta_1 := \Theta_2 := \{0, 1\}$, $\mathcal{A}_i$ the power set of $\mathcal{X}_i$, $Q_0 := \delta_0/3 + 2\delta_1/3$, $Q_1 := 2\delta_0/3 + \delta_1/3$, and $R_{\vartheta_1, \vartheta_2} := \delta_{\vartheta_1}$ for $\vartheta \in \Theta_1 \times \Theta_2$. Then all assumptions of Theorem 1.2 but (iii) are fulfilled, and with $h(x_1, x_2) := |x_1 - x_2| - 2/3$, we have (1) but not (2). Counterexample 5.3 is proved.

With an aim analogous to the above but concerning their unproven stronger version of Theorem 1.2, Cramer, Kamps, and Schenk [10, pages 275, 276] present an erroneous counterexample: Their assumption (*) holds if g is the signum function, but their conclusion « $g = 0$ » does not.

Again in their version of Theorem 1.2, Cramer, Kamps, and Schenk [10, p. 273] formulated the completeness assumption 1.2 (ii) slightly less explicitly than we did here, and this apparently led to the «clarification» refuted as follows:

C o u n t e r e x a m p l e 5.4. The claim of San Martin and Mouchart [39, Theorem 2.1], which is the present Theorem 1.2 with (ii) replaced by

(ii') $\{R_{\vartheta_1, \vartheta_2} : \vartheta_1 \in \Theta_1, \vartheta_2 \in \Theta_2\}$ is complete

and without assumption (iv), remains false even if assumption (iv) is added.

P r o o f. Let $\mathcal{X}_1 := \mathcal{X}_2 := \Theta_1 := \Theta_2 := \{0, 1\}$, $\mathcal{A}_i$ the power set of $\mathcal{X}_i$, $Q_0 := R_{0,0} := R_{0,1} := \delta_0/3 + 2\delta_1/3$, and $Q_1 := R_{1,0} := R_{1,1} := 2\delta_0/3 + \delta_1/3$. Then all the above assumptions are fulfilled, and with $h(x_1, x_2) := |x_1 - x_2| - 4/9$, we have (1) but not (2). Counterexample 5.4 is proved.

Without any additional assumption, sufficiency of $\bigvee_{i \in I} \mathcal{C}_i$ can not be added to the conclusion of Theorem 1.1, not even in the special case of Corollary 1.1:

C o u n t e r e x a m p l e 5.5. There exists a model $\mathcal{P}$ satisfying the assumptions of Corollary 1.1, but with $\mathcal{C}_1 \vee \mathcal{C}_2$ insufficient.

P r o o f. Let $\mathcal{X} := \{1, 2, 3\}$, $\mathcal{A} := 2^{\mathcal{X}}$, $\Theta_1 := \Theta_2 := \{1, 2\}$, and the $P_{\vartheta_1, \vartheta_2}$ defined by their densities $f_{\vartheta_1, \vartheta_2}$ with respect to counting measure given by

$$f_{1,1} := \frac{1}{3}, \quad f_{1,2} := f_{2,1} := \mathbf{1}_{\{3\}}, \quad f_{2,2}(x) := \frac{x}{6} \quad \text{for } x \in \mathcal{X}.$$

If $\vartheta_1 = 2$, then $\{P_{\vartheta_1, \vartheta_2} : \vartheta_2 \in \Theta_2\} = \{P_{2,1}, P_{2,2}\}$, and for this model, by the reference given in the proof of Theorem 1.4, the σ -algebra $\sigma(f_{2, \vartheta'_2}/f_{2, \vartheta''_2} : \vartheta'_2, \vartheta''_2 \in \Theta_2) = \{\emptyset, \{1, 2\}, \{3\}, \mathcal{X}\} =: \mathcal{C}_1$ is minimal sufficient and in fact easily checked to be complete. By similarly considering

the remaining three models occurring in the assumptions (i) and (ii) in Corollary 1.1, we see that the assumptions of Corollary 1.1 are fulfilled with $\mathcal{C}_2 := \mathcal{C}_1$. On the other hand, the σ -algebra $\sigma(f_{\vartheta'}/f_{\vartheta''}: \vartheta', \vartheta'' \in \Theta) = \mathcal{A}$ is minimal sufficient for $\mathcal{P}$, and hence $\mathcal{C}_1 \vee \mathcal{C}_2$, being not almost surely equal to $\mathcal{A}$, is insufficient. Counterexample 5.5 is proved.

C o u n t e r e x a m p l e 5.6. Lemma 4.1(a) would become false if its σ -finiteness assumption were omitted, and even so if the assumption of 4.1(b) were added.

P r o o f. Let $(\mathcal{X}, \mathcal{A}, \mu) = (\mathbf{R}, \mathcal{B}(\mathbf{R}), \#)$ be the real line with its Borel σ -algebra and counting measure, $\mathcal{E}$ consist of all singletons and the empty set, and $n = 1$. Then $\mathcal{E}$ is $\cap$ -stable. If $A \in \mathcal{A}$, then, for every $x \in \mathcal{X}$, $h \in \delta_x(A|\mathcal{E})$ implies $h(x) = \mathbf{1}_A(x)$, and hence $h \in \bigcap_{P \in \mathcal{P}} P(A|\mathcal{E})$ would imply $h = \mathbf{1}_A$, but the latter is not $\mathcal{E}$ -measurable if A is neither countable nor co-countable. So $\mathcal{E}$ is not sufficient. Counterexample 5.6 is proved.

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