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COLORED MAXIMAL BRANCHING PROCESS

Рассматривается версия многотипного максимального ветвящегося процесса, введенного недавно А. В. Лебедевым. Основным результатом работы является предельная теорема для эмпирических частот типов. Показано, как изменяется со временем первоначальное распределение типов под воздействием механизма селекции среди соревнующихся индивидуумов максимального ветвящегося процесса.

Ключевые слова и фразы: многотипные максимальный ветвящийся процесс, максимальный ветвящийся процесс, асимптотическое поведение цепей Маркова, аддитивные функционалы на цепях Маркова.

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1. Introduction. The main goal of this paper is to study the mechanism of the selection of the ancestor in a certain version of the multi-type maximal branching process (following [18] we will use the abbreviation MTMBP for these processes). More precisely, we consider a version of the MTMBP, where particles are colored at random and their offspring distribution depends on the color. Our main result (stated in Theorems 2.2 and 2.3 below) is a limit theorem for the distribution of the color of the direct ancestor of the n -th generation, as n goes to infinity.

Maximal branching processes (MBP) were introduced by Lamperti in [7]. An MBP is a Markov chain on the set of nonnegative integers with a unique absorbing state at zero and transition kernel determined by the following recursive equation:

$$Z_{n+1} = \max_{1 \leq k \leq Z_n} X_{n,k}, \quad Z_n > 0,$$

where the random variables $X_{n,k}$ are independent identically distributed, nonnegative, and integer-valued. The process can be thus described as an «extremal analogue» of the Galton–Watson branching processes, where the next generation is formed by the offspring of a most productive individual. The MBP is an elegant mathematical construction, and their theory turns out to be closely related to a general problem of the study of asymptotic behavior of (Markov) random processes in a half-line with asymptotically vanishing drift. The latter is sometimes referred to as Lamperti’s problem (cf. [19], [20]) to acknowledge the contribution of Lamperti’s pioneering works [9]–[11].

A generalization of the MBP from integer-valued population processes to their real-valued analogue is considered in a series of papers by Lebedev [12]–[14], see also a review of his results in [15]. An application of these processes to the queueing theory (for gated infinite-server queues, cf. [2]) is discussed in [13]. More recently, in [16]–[18] Lebedev studied an extension of the MBP to a multi-type setting. This paper intends to contribute to the understanding of the MTMBP by considering certain aspect of their asymptotic behavior in a setting where explicit computation is possible through a link to the one-dimensional theory of the MBP which has been developed by Lamperti in [7], [8].

The *colored maximal branching process* (CMBP for short) that we consider in this work is a vector-valued Markov chain $Z_n = (Z_n^{(1)}, Z_n^{(2)}, \dots, Z_n^{(d)})$ which describes evolution of a population of individuals of d different types (colors) in discrete time $n = 0, 1, \dots$. We use the term *color* as an alternative to the *type*, to distinguish our model from a more general one of the MTMBP, introduced by Lebedev in [16]–[18]. The integer $d \geq 1$ is fixed and $Z_n^{(i)}$ represents the number of individuals of type i present in n -th generation. Transitions of the Markov chain Z_n consist of two stages: at the first stage the total size of the population $Y_n := \sum_{i=1}^d Z_n^{(i)}$ in the generation n is determined, and at the second stage colors are randomly assigned to the Y_n individuals who form the n -th generation. That is where our model differs from the one considered in [16]–[18], where, similarly to the ordinary multi-type Galton–Watson branching process, individuals are born being already of a certain type.

Let $\mathcal{D} := \{1, \dots, d\}$ and let $\mathbf{Z}_+$ denote the set of nonnegative integers. The dynamics of Y_n is determined by the equation

$$Y_{n+1} = \max_{1 \leq i \leq d} \max_{1 \leq k \leq Z_n^{(i)}} X_{n,k}^{(i)},$$

where $X_{n,k}^{(i)}$ represents the number of children of the k -th individual of type i . We assume that the random variables $X_{n,k}^{(i)}$ take values in $\mathbf{Z}_+$, are independent and, moreover, $\{X_{n,k}^{(i)} : n \in \mathbf{Z}_+, k \in \mathbf{N}\}$ are identically distributed for each fixed $i \in \mathcal{D}$. We denote the common distribution function of $X_{n,k}^{(i)}$ by F_i .

Once a new generation is formed, a color from the set $\mathcal{D}$ is assigned to the individuals, independently of each other and of the previous history. Let $\chi_{n,k}$ denote the color of the k -th individual in the n -th generation. We assume that $\chi_{n,k}, n \in \mathbf{Z}_+, k \in \mathbf{N}$ are independent

and identically distributed $\mathcal{D}$ -valued random variables, and denote

$$\mu_i := \mathbf{P}\{\chi_{n,k} = i\}, \quad i \in \mathcal{D}, \quad (1.1)$$

and assume throughout that $\mu_i > 0$ for all $i \in \mathcal{D}$. The number of individuals of type i in the n -th generation thus can be written as

$$Z_n^{(i)} = \sum_{k=1}^{Y_n} \mathbf{1}_{\{\chi_{n,k}=i\}},$$

where $\mathbf{1}_A$ stands for the indicator of the event A in the underlying probability space.

For $n \in \mathbf{Z}_+$, let $\mathcal{X}_n := \{X_{n,k}^{(i)}: k \in \mathbf{N}, i \in \mathcal{D}\}$, $\mathcal{C}_n = \{\chi_{n,k}: k \in \mathbf{N}\}$, and let $\mathcal{F}_n = \sigma(\mathcal{X}_k, \mathcal{C}_k: k \leq n) = \sigma(\mathcal{X}_k, \mathcal{C}_k, Z_k: k \leq n)$ be the σ -algebra of the «events up to time n ». To describe vectors Z_n when Y_n is given, we introduce the sets

$$\mathcal{M}_y = \left\{ (k_1, \dots, k_d) \in \mathbf{Z}_+^d: \sum_{i \in \mathcal{D}} k_i = y \right\}, \quad y \in \mathbf{Z}_+.$$

Transition kernel of the Markov chain Z_n on $\mathbf{Z}_+^d$ is formally defined by the following equations:

$$\mathbf{P}\left\{Y_{n+1} \leq y \mid \mathcal{F}_n, Z_n^{(i)} = k_i, i \in \mathcal{D}\right\} = \prod_{i=1}^d \left(F_i(y)\right)^{k_i}, \quad y, k_i \in \mathbf{Z}_+,$$

and, for $(k_1, \dots, k_d) \in \mathcal{M}_y$,

$$\mathbf{P}\left\{Z_{n+1}^{(i)} = k_i, i \in \mathcal{D} \mid \mathcal{F}_n, Y_{n+1} = y\right\} = \frac{y!}{k_1! \cdots k_d!} \mu_1^{k_1} \cdots \mu_d^{k_d}.$$

Throughout the paper we will assume that $Y_0 = 1$ and the color of the first particle is chosen at random according to the distribution defined in (1.1). To support all the random variables defined above, the underlying probability space can be chosen as for the multi-type Galton–Watson branching processes. It can be formally constructed using the recipe given in [6, Chapter VI].

2. Statement of results. Following [7], we will say that the chain $(Z_n)_{n \in \mathbf{Z}_+}$ belongs to the class R if $\mathbf{P}\{Y \rightarrow \infty\} = 0$, and to the class T otherwise. Here and henceforth $\{Y \rightarrow \infty\}$ serves as a shortcut for $\{\lim_{n \rightarrow \infty} Y_n = +\infty\}$. First, we will obtain criteria for the classification of the Markov chain Z_n . It turns out (cf. [7], [8]) that the asymptotic behavior of Z_n is best understood in terms of the dynamics of the random sequence

$$L_n := \ln Y_n, \quad n \in \mathbf{Z}_+.$$

Observe that both $(Y_n)_{n \geq 0}$ and $(L_n)_{n \geq 0}$ are Markov chains. In particular, transition kernel of the latter is determined by the following family of distribution functions:

$$\begin{aligned} H_L(\xi, \eta) &:= \mathbf{P}\left\{L_{n+1} \leq \eta \mid L_n = \xi\right\} = \mathbf{P}\left\{Y_{n+1} \leq e^\eta \mid Y_n = e^\xi\right\} \\ &= \mathbf{E}\left[\prod_{i=1}^d \left(F_i(e^\eta)\right)^{Z_n^{(i)}} \mid Y_n = e^\xi, (Z_n^{(i)})_{i=1}^d\right] \\ &= \sum_{k_1, \dots, k_d} \frac{(e^\xi)!}{k_1! \cdots k_d!} \prod_{i=1}^d \left(\mu_i F_i(e^\eta)\right)^{k_i} \\ &= \left(\sum_{i=1}^d \mu_i F_i(e^\eta)\right)^{e^\xi} = \left(1 - \mathbf{E}\left[G_\chi(e^\eta)\right]\right)^{e^\xi}, \end{aligned} \quad (2.1)$$

where χ is a generic random color in $\mathcal{D}$ with the same distribution as $\chi_{n,k}$, and $G_\chi(x) := 1 - F_\chi(x)$.

Taking in account that $\sum_{i=1}^d \mu_i F_i(\cdot) = \mathbf{E}[F_\chi(\cdot)]$ is a distribution function and that $Z_n^{(i)}$ is conditionally independent of $\mathcal{F}_n$ given Y_n , the corresponding Lamperti's results in [7], [8] yield the following recurrence-transience criteria for Z_n . Let γ denote Euler's constant, that is $\gamma = \lim_{n \rightarrow \infty} (\sum_{k=1}^n k^{-1} - \ln n) \approx 0.57721$.

Lemma 2.1. *Let $(Z_n)_{n \geq 0}$ be a CMBP that satisfies Assumption 2.1. Then*

- (i) *if $\limsup_{x \rightarrow \infty} x \cdot \mathbf{E}[G_\chi(x)] < e^{-\gamma}$, the chain Z_n is in the class R;*
- (ii) *if $\liminf_{x \rightarrow \infty} x \cdot \mathbf{E}[G_\chi(x)] > e^{-\gamma}$, the chain Z_n is in the class T;*
- (iii) *if for some constant $\theta \in \mathbf{R}$,*

$$\mathbf{E}[G_\chi(x)] = \frac{e^{-\gamma}}{x} + \frac{\theta + o(1)}{x \log x}, \quad x \rightarrow \infty, \quad (2.2)$$

and $\theta < \pi^2 e^{-\gamma}/12$, the chain Z_n is in the class R;

- (iv) *if (2.2) holds with $\theta > \pi^2 e^{-\gamma}/12$, the chain Z_n is in the class T.*

In the rest of the paper we focus on the growing to infinity processes in a «critical» regime. More precisely, we will impose the following basic set of assumptions.

Assumption 2.1. *Let Z_n be a CMBP as described in Section 1. Suppose that*

- (A1) $\mathbf{P}\{X_{n,1}^{(i)} = 0\} = 0$ for all $i \in \mathcal{D}$;
- (A2) for all $i \in \mathcal{D}$ there exists the limit $\alpha_i := \lim_{x \rightarrow \infty} x(1 - F_i(x))$;
- (A3) $\alpha_i \in (0, \infty)$ for all $i \in \mathcal{D}$. Furthermore, $\beta \in (e^{-\gamma}, \infty)$, where

$$\beta := \sum_{i=1}^d \mu_i \alpha_i = E[\alpha_\chi]. \quad (2.3)$$

In particular, these assumptions imply that $\mathbf{P}\{Y \rightarrow \infty\} = 1$. The following proposition is immediate from (2.1).

Proposition 2.1. *Let Assumption 2.1 hold. Then, for any $\lambda \in \mathbf{R}$,*

$$\lim_{\xi \rightarrow \infty} H_L(\xi, \xi + \lambda) = J_\beta(\lambda),$$

where $J_\beta(\lambda) := \exp(-\beta e^{-\lambda})$, with β being defined in (2.3).

Heuristically, the proposition suggests that the asymptotic behavior of the Markov chain L_n is that of a transient random walk with the increments distributed according to J_β . This intuition was made precise in the «comparison lemma» of [7], where a coupling of two processes is explicitly constructed.

We now turn to a law of large numbers for $\ln Z_n^{(i)}$, which can be obtained using the comparison with a random walk. First, observe that the law of large numbers for triangular arrays implies that for all $i \in \mathcal{D}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{X_{n,k}=i\}} = \mu_i \quad \mathbf{P}\text{-a.s.},$$

where μ_i is introduced in (1.1). By passing in this limit result to the random subsequence of integers Y_n we obtain the following statement.

Proposition 2.2. *Let Assumption 2.1 hold. Then for any $i \in \mathcal{D}$,*

$$\lim_{n \rightarrow \infty} \frac{Z_n^{(i)}}{Y_n} = \mu_i \quad \mathbf{P}\text{-a.s.}$$

For a single-type process, the analogue of the following result is stated in Section 3 of [8]. The multi-type version follows from its single-type prototype by applying the latter to the MBP associated with the distribution function $\mathbf{E}[F_\chi(\cdot)]$ and using Proposition 2.2.

Lemma 2.2. *Let Assumption 2.1 hold. Then for any $i \in \mathcal{D}$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln Z_n^{(i)} = \gamma + \ln \beta \quad \mathbf{P}\text{-a.s.}$$

We remark that the random sequence $\ln Z_n^{(i)}$, $n \in \mathbf{Z}_+$, satisfies a large deviation principle under $\mathbf{P}$ (see, for instance, Remark (ii) in [21, p. 594]).

Further we state a central limit theorem for $\ln Z_n^{(i)}$ in the transient case. The «border-line» case $\beta = e^{-\gamma}$ and (2.2) holds with $\theta > \pi^2 e^{-\gamma}/12$ was studied by Lamperti in [8] (see Theorem 2 in [8]) using the theory he developed in [11]. A related result corresponding to the case $F_i(x) \sim 1 - \alpha_i x^{-\varepsilon}$ with $\varepsilon \in (0, 1)$ was obtained in [20, Theorem 2.5]. In what follows we consider the case $F_i(x) \sim 1 - \alpha_i x^{-1}$ with $\beta > e^{-\gamma}$.

Let $D([0, 1]; \mathbf{R}^d)$ denote the set of $\mathbf{R}^d$ -vector valued càdlàg functions on $[0, 1]$, endowed with the Skorokhod J_1 -topology. Let

$$S_n(t) = \sqrt{\frac{6}{\pi^2 n}} \sum_{k=1}^{[nt]} (L_k - \gamma - \ln \beta), \quad t \in [0, 1]. \quad (2.4)$$

Theorem 2.1. *Let Assumption 2.1 hold. Suppose in addition that (2.2) holds for some $\theta \in \mathbf{R}$. Then S_n converges weakly in $D([0, 1]; \mathbf{R}^d)$, as $n \rightarrow \infty$, to a standard d -dimensional Brownian motion.*

The proof of Theorem 2.1 given in the Appendix uses a standard martingale technique and relies on certain moment estimates obtained in [8] (this is where the full extent of the extra condition (2.2) is exploited). We remark that Theorem 2.1 is «in spirit» of the results of [8], and even though it is not stated there it seems quite likely that the result was known to Lamperti.

The main results of this paper are stated in the following two theorems. Let τ_n denote the set of colors present among individuals at generation n with the maximum number of offspring. That is,

$$i \in \tau_n \iff Y_{n+1} = X_{n,k}^{(i)} \quad \text{for some } k = 1, \dots, Z_n^{(i)}.$$

First, we study the asymptotic distribution of the colors in τ_n . The proof of the following theorem is given in Section 3.

Theorem 2.2. *Let Assumption 2.1 hold. Then for all $i \in \mathcal{D}$,*

$$\lim_{n \rightarrow \infty} \mathbf{P}\{i \in \tau_n\} = \frac{\mu_i \alpha_i}{\beta}.$$

Our following result confirms the intuition that the limiting distributions found in Theorem 2.2 coincide with the asymptotic frequencies of the colors of the most productive individuals who serve as ancestors of the next generation.

Theorem 2.3. *Let Assumption 2.1 hold. Then for all $i \in \mathcal{D}$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{i \in \tau_k\}} = \frac{\mu_i \alpha_i}{\beta} \quad \mathbf{P}\text{-a.s.}$$

The proof of Theorem 2.2 is given in Section 4.

Using the same decoupling techniques as in the proof of Theorem 2.3 one can establish the asymptotic of the «time of the rule of the longest-reigning dynasty» in the CMBP. More precisely, for $n \in \mathbf{Z}_+$ and $i \in \mathcal{D}$ let

$$r_n^{(i)} = \max \left\{ k \leq n+1 : \prod_{s=n-k+1}^n \mathbf{1}_{\{i \in \tau_s\}} = 1 \right\}$$

and

$$R_n^{(i)} = \max_{0 \leq k \leq n} r_k^{(i)}.$$

Theorem 2.4. *Let Assumption 2.1 hold. Then for all $i \in \mathcal{D}$,*

$$\lim_{n \rightarrow \infty} \frac{R_n^{(i)}}{\ln n} = -\frac{1}{\ln(1 - \mu_i \alpha_i \beta^{-1})} \quad \mathbf{P}\text{-a.s.}$$

Furthermore, for all $i \in \mathcal{D}$,

$$\limsup_{n \rightarrow \infty} \frac{r_n^{(i)}}{\ln n} = -\frac{1}{\ln(1 - \mu_i \alpha_i \beta^{-1})} \quad \text{while} \quad \liminf_{n \rightarrow \infty} \frac{r_n^{(i)}}{\ln n} = 0 \quad \mathbf{P}\text{-a.s.}$$

The above theorem states in fact that $R_n^{(i)}$ and $r_n^{(i)}$ exhibit the same almost sure asymptotic behavior as, respectively, the longest run and the current run of heads in a series of independent coin tossing trials, where the probability of heads equals $\mu_i \alpha_i / \beta$. See, for instance, [5] and [3, p. 54, 55] for the corresponding results for the coin tossing. Although the proof of Theorem 2.4 is very similar to that of Theorem 2.3, for the sake of completeness and reader's convenience it is included in Section 5.

3. Proof of Theorem 2.2. To prove the theorem, it suffices to show that

$$\lim_{y \rightarrow \infty} \mathbf{P}\{i \in \tau_n \mid Y_n = y\} = \frac{\mu_i \alpha_i}{\beta}. \quad (3.1)$$

Indeed, from $\mathbf{P}(Y \rightarrow \infty) = 1$ and (3.1) it follows that $\lim_{n \rightarrow \infty} \mathbf{P}\{i \in \tau_n \mid Y_n\} = c\mu_i \alpha_i / \beta$, $\mathbf{P}$ -a.s., and hence

$$\lim_{n \rightarrow \infty} \mathbf{P}\{i \in \tau_n\} = \lim_{n \rightarrow \infty} \mathbf{E} \left[\mathbf{P}(i \in \tau_n \mid Y_n) \right] = \frac{\mu_i \alpha_i}{\beta},$$

by the bounded convergence theorem.

For any integer $y \geq 2$, similarly to (2.1), we have

$$\begin{aligned} & \mathbf{P}\{i \in \tau_n \mid Y_n = y\} \\ &= \sum_{x=1}^{\infty} \sum_{k_1, \dots, k_d} \frac{y!}{k_1! \cdots k_d!} \prod_{j \neq i} \left(\mu_j F_j(x) \right)^{k_j} \left\{ \left(\mu_i F_i(x) \right)^{k_i} - \left(\mu_i F_i(x-1) \right)^{k_i} \right\} \\ &= \sum_{x=1}^{\infty} \left\{ \left(\sum_j \mu_j F_j(x) \right)^y - \left(\sum_{j \neq i} \mu_j F_j(x) + \mu_i F_i(x-1) \right)^y \right\}. \end{aligned} \quad (3.2)$$

Fix any $\varepsilon > 0$. Then (compare with Proposition 2.1),

$$\begin{aligned} & \sum_{x=1}^{\lfloor \varepsilon y \rfloor} \left\{ \left(\sum_j \mu_j F_j(x) \right)^y - \left(\sum_{j \neq i} \mu_j F_j(x) + \mu_i F_i(x-1) \right)^y \right\} \\ & \leq \sum_{x=1}^{\lfloor \varepsilon y \rfloor} \left\{ \left(\sum_j \mu_j F_j(x) \right)^y - \left(\sum_j \mu_j F_j(x-1) \right)^y \right\} \\ & = \left(\sum_j \mu_j F_j(\lfloor \varepsilon y \rfloor) \right)^y \rightarrow e^{-\beta/\varepsilon} \quad \text{as } y \rightarrow \infty. \end{aligned} \quad (3.3)$$

By Taylor's expansion formula, for any $A > 0$ and $b \in (0, A)$, we have

$$(A-b)^y = A^y - byA^{y-1} + \frac{b^2}{2} y(y-1)(A-c)^{y-2}$$

for some $c \in (0, b)$. In particular,

$$byA^{y-1} - \frac{b^2 y^2}{2} A^{y-2} \leq A^y - (A-b)^y \leq byA^{y-1}. \quad (3.4)$$

Therefore, letting

$$A(x) := \sum_j \mu_j F_j(x) \quad \text{and} \quad b_i(x) := F_i(x) - F_i(x-1), \quad (3.5)$$

we obtain from (3.2) and (3.4) that

$$\begin{aligned}
 & \mathbf{P}\{i \in \tau_n \mid Y_n = y\} \\
 & \geq \sum_{x=1+\lfloor \varepsilon y \rfloor}^{\infty} \left\{ \left(\sum_j \mu_j F_j(x) \right)^y - \left(\sum_{j \neq i} \mu_j F_j(x) + \mu_i F_i(x-1) \right)^y \right\} \\
 & = \sum_{x=1+\lfloor \varepsilon y \rfloor}^{\infty} y \mu_i (A(x))^{y-1} b_i(x) - \sum_{x=1+\lfloor \varepsilon y \rfloor}^{\infty} \mu_i (A(x))^{y-2} \frac{(b_i(x)y)^2}{2} \\
 & := I_1(y, \varepsilon) - I_2(y, \varepsilon).
 \end{aligned} \tag{3.6}$$

To evaluate $I_1(y, \varepsilon)$ and $I_2(y, \varepsilon)$ we will exploit the following implication of a general property of regularly varying sequences (see, for instance, [1], [4]):

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{x^2 b_i(x)}{\alpha_i} &= \lim_{x \rightarrow \infty} \frac{x \{ (1 - F_i(x-1)) - (1 - F_i(x)) \}}{1 - F_i(x)} \\
 &= \lim_{x \rightarrow \infty} \frac{\lfloor x \rfloor \{ (1 - F_i(\lfloor x \rfloor - 1)) - (1 - F_i(\lfloor x \rfloor)) \}}{1 - F_i(\lfloor x \rfloor)} \\
 &= \lim_{n \rightarrow \infty} \frac{n \{ (1 - F_i(n-1)) - (1 - F_i(n)) \}}{1 - F_i(n)} = 1.
 \end{aligned} \tag{3.7}$$

Since $(A(x))^{y-1} b_i(x)$ is a step function, for the first term in (3.6) we have

$$\begin{aligned}
 I_1(y, \varepsilon) &= \sum_{x=1+\lfloor \varepsilon y \rfloor}^{\infty} y \mu_i (A(x))^{y-2} \frac{(b_i(x)y)^2}{2} \\
 &\geq \int_{\varepsilon+1/y}^{\infty} y^2 \mu_i (A(yt))^{y-1} b_i(yt) dt.
 \end{aligned} \tag{3.8}$$

By (3.7), $y^2 \mu_i (A(yt))^{y-1} b_i(yt)$ converges, as $y \rightarrow \infty$, to the continuous distribution function $\alpha_i \mu_i e^{-\beta/t} t^{-2}$. Furthermore, the convergence is uniform on compact intervals. Therefore, first truncating the integral in the last display and then taking the limit as $y \rightarrow \infty$,

$$\liminf_{y \rightarrow \infty} I_1(y, \varepsilon) \geq \lim_{M \rightarrow \infty} \int_{\varepsilon}^M \alpha_i \mu_i e^{-\beta/t} t^{-2} dt = \int_{\varepsilon}^{\infty} \alpha_i \mu_i e^{-\beta/t} t^{-2} dt. \tag{3.9}$$

Similarly,

$$\begin{aligned}
 I_2(y, \varepsilon) &= \sum_{x=1+\lfloor \varepsilon y \rfloor}^{\infty} y \mu_i (A(x))^{y-2} \frac{(b_i(x)y)^2}{2} \\
 &\leq \int_{\varepsilon}^{\infty} \mu_i (A(yt))^{y-2} \frac{(b_i(yt)y)^2}{2} dt,
 \end{aligned}$$

and hence, by Fatou's lemma,

$$\limsup_{y \rightarrow \infty} \{I_2(y, \varepsilon)y\} \leq \int_{\varepsilon}^{\infty} \mu_i \alpha_i^2 e^{-\beta/t} t^{-4} dt = \int_0^{\varepsilon^{-1}} \mu_i \alpha_i^2 e^{-\beta s} s^2 ds. \tag{3.10}$$

Since $\varepsilon > 0$ is an arbitrary positive real, it follows from (3.9), (3.10), and (3.6) that

$$\begin{aligned}
 \liminf_{y \rightarrow \infty} \mathbf{P}\{i \in \tau_n \mid Y_n = y\} &\geq \int_0^{\infty} \alpha_i \mu_i e^{-\beta/t} t^{-2} dt \\
 &= \int_0^{\infty} \alpha_i \mu_i e^{-\beta s} ds = \frac{\mu_i \alpha_i}{\beta}.
 \end{aligned} \tag{3.11}$$

On the other hand, using (3.2), (3.3), and the upper bound in (3.4), we obtain that

$$\limsup_{y \rightarrow \infty} \mathbf{P}\{i \in \tau_n \mid Y_n = y\} \leq \limsup_{\varepsilon \rightarrow 0} \limsup_{y \rightarrow \infty} I_1(y, \varepsilon).$$

To conclude the proof of the theorem, observe that, similarly to (3.8) and (3.9),

$$I_1(y, \varepsilon) \leq \int_{\varepsilon}^{\infty} y^2 \mu_i (A(yt))^{y-1} b_i(yt) dt \rightarrow \int_{\varepsilon}^{\infty} \alpha_i \mu_i e^{-\beta/t} t^{-2} dt \quad \text{as } y \rightarrow \infty.$$

Since

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \alpha_i \mu_i e^{-\beta/t} t^{-2} dt = \int_0^{\infty} \alpha_i \mu_i e^{-\beta/t} t^{-2} dt = \frac{\mu_i \alpha_i}{\beta},$$

this implies

$$\limsup_{y \rightarrow \infty} \mathbf{P}\{i \in \tau_n \mid Y_n = y\} \leq \frac{\mu_i \alpha_i}{\beta}.$$

In view of (3.11) this completes the proof of (3.1). Theorem 2.2 is proved.

4. Proof of Theorem 2.3. Let $\xi_n^{(i)} := \mathbf{1}_{\{i \in \tau_n\}}$, $n \in \mathbf{Z}_+$, $i \in \mathcal{D}$, and $\xi_n := (\xi_n^{(1)}, \dots, \xi_n^{(d)})$. Observe that $K_n^{(i)} := \sum_{k=1}^n \xi_k^{(i)}$ is an additive functional of the Markov chain Y_n . More precisely,

$$\begin{aligned} \mathbf{P}\{\xi_n^{(i)} = s, Y_{n+1} = z \mid (Y_k, \xi_k)_{k < n}, Y_n = y\} \\ = \mathbf{P}\{Y_{n+1} = z \mid Y_n = y\} \cdot \mathbf{P}\{\xi_n^{(i)} = s \mid Y_n = y, Y_{n+1} = z\}, \end{aligned} \quad (4.1)$$

for any $y, z \in \mathbf{N}$ and $s \in \{0, 1\}$. This implies that the random sequence formed by the triples (Y_n, Y_{n+1}, ξ_n) , $n \in \mathbf{Z}_+$ is a Markov chain (in fact, a hidden Markov model with ξ_n playing role of the «observable variables») and that one can generate a realization of this sequence step by step, at each step first generating the value of Y_{n+1} given Y_n and then the value of ξ_n given Y_n and Y_{n+1} . Heuristically, in view of Proposition 2.1 and Theorem 2.2, this suggests that the asymptotic behavior of the sequence $\xi_n^{(i)}$ along a typical trajectory of the chain Y_n is similar to that of the sequence of outcomes of independent identically distributed coin tossing trials with the probability of heads equal to $\mu_i \alpha_i / \beta$. In what follows we will derive a formal version of this heuristic argument and deduce from it the result stated in Theorem 2.3.

Note that, similarly to (3.2),

$$\begin{aligned} \mathbf{P}\{\xi_n^{(i)} = 1 \mid Y_n = y, Y_{n+1} = z\} \\ = \frac{(\sum_j \mu_j F_j(z))^y - (\sum_{j \neq i} \mu_j F_j(z) + \mu_i F_i(z-1))^y}{(\sum_j \mu_j F_j(z))^y - (\sum_j \mu_j F_j(z-1))^y}. \end{aligned}$$

For $M > 0$ let $\Omega_M := \{(y, z) \in \mathbf{N}^2: z > y^{2/3} \text{ and } y, z > M\}$. Recall (3.5). By virtue of (3.4), we have for all $(y, z) \in \Omega_M$ with M sufficiently enough (namely, large enough to ensure that the denominator in (4.2) and the numerator in (4.3) below are positive),

$$\begin{aligned} \mathbf{P}\{\xi_n^{(i)} = 1 \mid Y_n = y, Y_{n+1} = z\} \\ \leq \frac{y \mu_i b_i(z) (A(z))^{y-1}}{y (\sum_j \mu_j b_j(z)) (A(z))^{y-1} - (y^2/2) (\sum_j \mu_j b_j(z))^2 (A(z))^{y-2}} \\ = \frac{\mu_i b_i(z) A(z)}{(\sum_j \mu_j b_j(z)) A(z) - (y/2) (\sum_j \mu_j b_j(z))^2}, \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} \mathbf{P}\{\xi_n^{(i)} = 1 \mid Y_n = y, Y_{n+1} = z\} \\ \geq \frac{y \mu_i b_i(z) (A(z))^{y-1} - (y^2/2) (\mu_i b_i(z))^2 (A(z))^{y-2}}{y (\sum_j \mu_j b_j(z)) (A(z))^{y-1}} \\ = \frac{\mu_i b_i(z) A(z) - (y/2) (\mu_i b_i(z))^2}{(\sum_j \mu_j b_j(z)) A(z)}. \end{aligned} \quad (4.3)$$

It follows from (4.2) and (4.3) that

$$\begin{aligned} & \lim_{M \rightarrow \infty} \sup_{(y,z) \in \Omega_M} \mathbf{P}\{\xi_n^{(i)} = 1 \mid Y_n = y, Y_{n+1} = z\} \\ &= \lim_{M \rightarrow \infty} \inf_{(y,z) \in \Omega_M} \mathbf{P}\{\xi_n^{(i)} = 1 \mid Y_n = y, Y_{n+1} = z\} = \frac{\mu_i \alpha_i}{\beta}. \end{aligned} \quad (4.4)$$

By Lemma 2.1, $\mathbf{P}\{Y_n > M \text{ i.o.}\} = 0$ for any $M > 0$. Furthermore, (2.1) yields

$$\mathbf{P}\{Y_{n+1} \leq y^{2/3} \mid Y_n = y\} = \left(1 - \mathbf{E}\left[G_\chi(y^{2/3})\right]\right)^y \leq \exp\left(-y \mathbf{E}\left[G_\chi(y^{2/3})\right]\right),$$

which, using Assumption 2.1 and Lemma 2.2, implies by a «conditional version» of the Borel–Cantelli lemma (see, for instance, [3, p. 240]) that

$$\lim_{M \rightarrow \infty} \mathbf{1}_{\{(Y_n, Y_{n+1}) \in \Omega_M\}} = 1 \quad \mathbf{P}\text{-a.s.} \quad (4.5)$$

This completes the proof of Theorem 2.3 by using (4.1) and the comparison with a sequence of independent coin tossing trials with the probability of heads equal to $\alpha_i \mu_i / \beta_i \pm \varepsilon$ with an arbitrary small $\varepsilon > 0$, whenever $(Y_n, Y_{n+1}) \in \Omega_M$. Theorem 2.3 is proved.

5. Proof of Theorem 2.4. We will continue to use notation introduced in the course of the proof of Theorem 2.3. Similarly to the proof of the latter, the proof of Theorem 2.4 is based on an application of (4.4) and (4.5).

Fix any $i \in \mathcal{D}$ and $\varepsilon > 0$ such that $\varepsilon < \max\{1 - \mu_i \alpha_i / \beta, \mu_i \alpha_i / \beta\}$. By virtue of (4.4) one can choose $M = M(\varepsilon) > 0$ be so large that

$$\sup_{(y,z) \in \Omega_{M(\varepsilon)}} \mathbf{P}\{\xi_n^{(i)} = 1 \mid Y_n = y, Y_{n+1} = z\} \leq \frac{\mu_i \alpha_i}{\beta} + \varepsilon$$

and

$$\inf_{(y,z) \in \Omega_{M(\varepsilon)}} \mathbf{P}\{\xi_n^{(i)} = 1 \mid Y_n = y, Y_{n+1} = z\} \geq \frac{\mu_i \alpha_i}{\beta} - \varepsilon.$$

In view of (4.5), there exists $N \in \mathbf{N}$ such that $(Y_n, Y_{n+1}) \in \Omega_{M(\varepsilon)}$ for all $n > N$. Therefore, using a standard coupling technique one can infer from (4.1) that the $\limsup$ and the $\liminf$, as $n \rightarrow \infty$, of both $R_n^{(i)} / \ln n$ and $r_n^{(i)} / \ln n$ are dominated with probability one by the corresponding quantities in a series of independent identically distributed coin tossing trials, with the probability of heads equal to $\mu_i \alpha_i / \beta \pm \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies the results in Theorem 2.4 which merely states that the $\limsup$'s and $\liminf$'s coincide with their counterparts for the biased coin tossing when the probability of heads is equal to $\mu_i \alpha_i / \beta$ (cf. [5]). Theorem 2.4 is proved.

6. Appendix A. Proof of Theorem 2.1. For $m \in \mathbf{N}$ and $x \in \mathbf{R}$ let

$$\nu_m(x) = \mathbf{E}\left[(L_{n+1} - L_n)^m \mid L_n = x\right],$$

and set

$$M_n = L_n - \sum_{k=0}^{n-1} \nu_1(L_k), \quad n \in \mathbf{N}.$$

Then $(M_n, \mathcal{F}_n)_{n \in \mathbf{N}}$ is a martingale, and the corresponding martingale difference sequence is

$$\Delta_n := M_n - M_{n-1} = L_n - L_{n-1} - \nu_1(L_{n-1}), \quad n \in \mathbf{N}.$$

To apply a standard functional central limit theorem for martingales to M_n , it suffices to verify the following two conditions (see, for instance, [3, p. 414]):

- (i) $A_n(t) := n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \mathbf{E}[\Delta_k^2 \mid \mathcal{F}_{k-1}] \rightarrow \pi^2 / (6t)$ in probability for all $t \in [0, 1]$ and

(ii) $B_n(\varepsilon) := n^{-1} \sum_{k=1}^n \mathbf{E} [\Delta_k^2 \mathbf{1}_{\{|\Delta_k| > \varepsilon \sqrt{n}\}} | \mathcal{F}_{k-1}] \rightarrow 0$ in probability for any $\varepsilon > 0$.

The following asymptotic formulas (as $x \rightarrow \infty$) for the first two conditional moments are obtained in [8] (see Lemmas 1 and 2 in [8]):

$$\nu_1(x) = \gamma + \ln \beta + \frac{\theta}{\beta x} + o(x^{-1}) \quad \text{and} \quad \nu_2(x) = (\gamma + \ln \beta)^2 + \frac{\pi^2}{6} + o(1),$$

where the constant parameter θ is introduced in (2.2).

Since $\mathbf{E} [\Delta_k^2 | \mathcal{F}_{k-1}] = \nu_2(L_{k-1}) - (\nu_1(L_{k-1}))^2$, these two formulas together with Lemma 2.1 imply that $\mathbf{E} [\Delta_k^2 | \mathcal{F}_{k-1}] = \pi^2/6 + o(1)$ as $k \rightarrow \infty$, and hence the first condition above holds.

To verify the second condition we will use the fact (see [8, Lemma 1]) that $\nu_4(x)$ is uniformly bounded in a neighborhood of infinity. More precisely, let $K > 0$ be a positive real such that $\nu_4(x) < K$ for all $x = \ln k$, $k \in \mathbf{N}$. By using first the Cauchy–Schwarz inequality and then Chebyshev’s bound $\mathbf{P}\{X > a\} \leq a^{-4} \mathbf{E}[X^4]$,

$$\begin{aligned} \mathbf{E} [\Delta_k^2 \mathbf{1}_{\{|\Delta_k| > \varepsilon \sqrt{n}\}} | \mathcal{F}_{k-1}] &\leq \left(\mathbf{E} [\Delta_k^4 | \mathcal{F}_{k-1}] \cdot \mathbf{P}\{|\Delta_k| > \varepsilon \sqrt{n} | \mathcal{F}_{k-1}\} \right)^{1/2} \\ &\leq \frac{1}{\varepsilon^2 n} \mathbf{E} [\Delta_k^4 | \mathcal{F}_{k-1}]. \end{aligned}$$

By Minkowski’s inequality, the right-most term in the last display is bounded with probability one by $\varepsilon^{-2} n^{-1} (\nu_4(L_{k-1}) + (\nu_1(L_{k-1}))^4)^{1/4}$, which implies that the second condition above for the martingale differences is satisfied for Δ_n .

To complete the proof, observe that the asymptotic formula for $\nu_1(x)$ yields

$$\frac{1}{\sqrt{n}} \left(L_n - (\gamma + \ln \beta)n \right) = \frac{1}{\sqrt{n}} \left(M_n + O(\ln n) \right),$$

where $O(x)$ is a function such that $O(x)/x$ is bounded away from both zero and infinity in a neighborhood of infinity. Theorem 2.1 is proved.

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MAXIMUM INEQUALITIES FOR REARRANGEMENTS OF SUMMANDS AND ASSIGNMENTS OF SIGNS¹⁾

В статье изучается взаимосвязь между расстановками знаков и перестановками слагаемых в форме вероятностных максимальных неравенств. Взаимосвязь основана на лемме, сводящей задачу о перестановках к задаче о выборе знаков. Наряду с упрощением известных доказательств, она позволяет находить новые факты, а также решения в более общих постановках. Одним из главных результатов статьи является следующее неравенство. Пусть $x_1, \dots, x_n$, $\sum_{k=1}^n x_k = 0$, — набор элементов нормированного пространства X . Тогда для любого набора знаков $\theta = (\theta_1, \dots, \theta_n)$ и любого $t > 0$

$$\mathbf{P}\left\{\pi: \max_{1 \leq k \leq n} \left\| \sum_1^k x_{\pi(i)} \right\| > t\right\} \leq C \mathbf{P}\left\{\pi: \max_{1 \leq k \leq n} \left\| \sum_1^k x_{\pi(i)} \theta_i \right\| > \frac{t}{C}\right\},$$

где $\pi \in \Pi_n$, Π_n — группа всех перестановок отрезка $\{1, \dots, n\}$, $\mathbf{P}$ — равномерное распределение на Π_n и C — абсолютная константа. Приведенное неравенство неумлучшаемо (обратное неравенство также справедливо с некоторой другой абсолютной константой). Оно обобщает известные результаты Гарсия, Морэ и Пизье, Кашина и Саакяна,

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