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MAXIMUM INEQUALITIES FOR REARRANGEMENTS OF SUMMANDS AND ASSIGNMENTS OF SIGNS¹⁾

В статье изучается взаимосвязь между расстановками знаков и перестановками слагаемых в форме вероятностных максимальных неравенств. Взаимосвязь основана на лемме, сводящей задачу о перестановках к задаче о выборе знаков. Наряду с упрощением известных доказательств, она позволяет находить новые факты, а также решения в более общих постановках. Одним из главных результатов статьи является следующее неравенство. Пусть $x_1, \dots, x_n$, $\sum_{k=1}^n x_k = 0$, — набор элементов нормированного пространства X . Тогда для любого набора знаков $\theta = (\theta_1, \dots, \theta_n)$ и любого $t > 0$

$$\mathbf{P}\left\{\pi: \max_{1 \leq k \leq n} \left\| \sum_1^k x_{\pi(i)} \right\| > t\right\} \leq C \mathbf{P}\left\{\pi: \max_{1 \leq k \leq n} \left\| \sum_1^k x_{\pi(i)} \theta_i \right\| > \frac{t}{C}\right\},$$

где $\pi \in \Pi_n$, Π_n — группа всех перестановок отрезка $\{1, \dots, n\}$, $\mathbf{P}$ — равномерное распределение на Π_n и C — абсолютная константа. Приведенное неравенство неумлучшаемо (обратное неравенство также справедливо с некоторой другой абсолютной константой). Оно обобщает известные результаты Гарсия, Морэ и Пизье, Кашина и Саакяна,

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Чобаняна и Салеги, и Левенталя. Все результаты настоящей статьи могут быть сформулированы на языке перестановочных случайных величин.

Ключевые слова и фразы: перестановки слагаемых, расстановки знаков, вероятностные максимальные неравенства, перестановочные случайные величины.

1. Introduction. The main aim of this paper is to study the distribution of the random variable

$$|x_\pi|_n = \max_{1 \leq k \leq n} \left\| \sum_1^k x_{\pi(i)} \right\|,$$

where $x_1, \dots, x_n \in X$, X is a normed space, that is defined on Π_n , the set of all permutations π of $\{1, \dots, n\}$, with the uniform probability $\mathbf{P}$ on it. We suggest a method that allows to reduce a rearrangement problem to a problem of choosing signs (Lemma 1). It allows to simplify proofs and find new facts and statements. Although the results are stated for vectors, all of them are purposeful for the case of reals.

There are a series of problems and results in analysis where rearrangement maximum inequalities involving $|x_\pi|_n$ are used, see, e.g., the following sources and the literature therein: the Lévy–Steinitz theorem on the sum range of a conditionally convergent series (see [1]), Nikishin type theorems on a.s. convergence of a rearranged functional series (see [2]), orthogonal series (see [3]), Kolmogorov conjecture on systems of convergence (see [4]), the Ulyanov problem on the uniform convergence of a rearrangement of the trigonometric Fourier series of a periodic continuous function (see [5], [6], and [7]) and the applications of compact vector summation in scheduling theory (see [8]).

The first result in this direction was found by M. Kadets [9] who was solving the Steinitz problem for the L_p -spaces. Garsia in [10] and [11] proved that in the one-dimensional case ($X = \mathbf{R}^1$) for any $x_1, \dots, x_n \in X$ with $\sum_1^n x_i = 0$

$$\mathbf{E}_\pi |x_\pi|_n^p \leq C_p \left(\sum_1^n |x_i|^2 \right)^{p/2},$$

where $\mathbf{E}_\pi$ is the average (expectation), $p \geq 1$, and C_p is a constant dependent only on p . This inequality led to the well-known Garsia theorem on a.s. convergent rearrangement of an orthogonal series.

Maurey and Pisier [12] were the first who showed the relationship between the permutations and signs. They proved that in the general case of a normed space X for any $x_1, \dots, x_n \in X$ with $\sum_1^n x_i = 0$ and for any $p \geq 1$

$$\mathbf{E}_\pi |x_\pi|_n^p \sim \mathbf{E}_r \left\| \sum_1^n x_i r_i \right\|^p, \quad (1.1)$$

where $r_1, \dots, r_n$ are Rademacher random variables. The relation of equivalence in (1.1) means that ratios are bounded by positive constants dependent only on p . It is amazing that for a long time (until early 90s), the result of Maurey and Pisier remained unknown. Meanwhile Kashin and Saakyan [3] have proved the following result for $X = \mathbf{R}^1$ in terms of distributions: There is a universal constant C such that for any $t > 0$ and any reals $x_1, \dots, x_n$ with $\sum_1^n x_i = 0$ the right-hand side fragment of the following inequality holds:

$$\mathbf{P}_r \left\{ \omega: \left\| \sum_1^n x_i r_i(\omega) \right\| > 2t \right\} \leq \mathbf{P}_\pi \{ \pi: |x_\pi|_n > t \} \leq C \mathbf{P}_r \left\{ \omega: \left\| \sum_1^n x_i r_i(\omega) \right\| > \frac{t}{C} \right\}. \quad (1.2)$$

However, the method used in [3] does not work in the case of vectors. Then Chobanyan [13] and Chobanyan and Salehi [14] used a different method to assert the left-hand side

part and prove the two-sided inequality (1.2) for a general normed space X . In [15] for the case $x_i = \pm 1$ and in [16] for the case of $\mathbf{R}^1$ the inequality (1.2) was given the following form: There are universal constants C_1 and C_2 such that for any reals $x_1, \dots, x_n$ with $\sum_1^n x_i = 0$, any collection of signs $\theta = (\theta_1, \dots, \theta_n)$, and any $t > 0$ the following two-sided inequality holds:

$$C_1 \mathbf{P} \left\{ \pi: |x_\pi \theta|_n > \frac{t}{C_1} \right\} \leq \mathbf{P} \{ \pi: |x_\pi|_n > t \} \leq C_2 \mathbf{P} \left\{ \pi: |x_\pi \theta|_n > \frac{t}{C_2} \right\}, \quad (1.3)$$

where

$$|x_\pi \theta|_n = \max_{1 \leq k \leq n} |x_{\pi(1)} \theta_1 + \dots + x_{\pi(k)} \theta_k|.$$

Inequality (1.3) can be regarded as a refinement of (1.2) because its most important right-hand fragment holds for each θ , no average is taken with respect to θ . In this paper we extend this result to the case of a normed space X . As a corollary we single out the following curious result: There exist universal constants C_1 and C_2 such that for any collection $x = (x_1, \dots, x_n)$ with $\sum_1^n x_i = 0$ in an arbitrary normed space, any collection of signs $\theta = (\theta_1, \dots, \theta_n)$ and any $t > 0$ the following two-sided inequalities hold for the distribution of a Rademacher sum:

$$C_1 \mathbf{P} \left\{ \pi: |x_\pi \theta|_n > \frac{t}{C_1} \right\} \leq \mathbf{P} \left\{ \omega: \left\| \sum_1^n x_i r_i(\omega) \right\| \geq t \right\} \leq C_2 \mathbf{P} \left\{ \pi: |x_\pi \theta|_n > \frac{t}{C_2} \right\}.$$

Let us note that inequality (1.3) (as well as the majority of inequalities of this paper) can be expressed in terms of exchangeable random variables: There are universal constants C_1 and C_2 such that for any finite exchangeable system $\xi = (\xi_1, \dots, \xi_n)$ with $\sum_1^n \xi_i = 0$ of X -valued random variables, any collection of signs $\theta = (\theta_1, \dots, \theta_n)$ and any $t > 0$ the following two-sided inequality holds:

$$\begin{aligned} C_1 \mathbf{P} \left\{ \omega: \max_{1 \leq k \leq n} \left\| \sum_1^k \xi_l(\omega) \theta_l \right\| > \frac{t}{C_1} \right\} &\leq \mathbf{P} \left\{ \omega: \max_{1 \leq k \leq n} \left\| \sum_1^k \xi_l(\omega) \right\| > t \right\} \\ &\leq C_2 \mathbf{P} \left\{ \omega: \max_{1 \leq k \leq n} \left\| \sum_1^k \xi_l(\omega) \theta_l \right\| > \frac{t}{C_2} \right\}. \end{aligned} \quad (1.3')$$

In Section 2 we give a proposition (Lemma 1) first proved in [13] and [14], which in particular simplifies the proofs of moment maximum inequalities (among them the Maurey and Pisier core result [12] and Garsia's inequalities [11], see Section 3), and allows to state them in terms of general convex increasing functions. Theorem 1 of Section 3 represents a new idea in maximum inequalities. It suggests also a maximum inequality for finite exchangeable systems of random variables. The idea is developed in Section 4, where we give the proof of one of the main theorem on two-sided inequalities for the tail probabilities that lead via the integration by parts formula to moment inequalities for arbitrary increasing continuous functions.

2. Preliminaires. Everywhere in this paper X will stand for a normed linear space, real or complex, Π_n for all permutations π of $\{1, \dots, n\}$, $\mathbf{P}$ for the uniform distribution on Π_n , and Θ_n for all collections of signs $\theta = (\theta_1, \dots, \theta_n)$, $\theta_i = \pm 1$, $i = 1, \dots, n$, $n \in \mathbf{N}$.

Given $x = (x_1, \dots, x_n) \in X^n$, $\pi \in \Pi_n$, and $\theta = (\theta_1, \dots, \theta_n) \in \Theta_n$ we denote

$$|x_\pi \theta|_n = \max_{1 \leq k \leq n} \|x_{\pi(1)} \theta_1 + \dots + x_{\pi(k)} \theta_k\|.$$

In particular,

$$|x|_n = \max_{1 \leq k \leq n} \|x_1 + \dots + x_k\|.$$

Let $\pi \in \Pi_n$ be a permutation with $\pi = (k_1, \dots, k_n)$ and let $\theta \in \Theta_n$ with

$$\theta_{u_1} = +1, \dots, \quad \theta_{u_s} = +1; \quad \theta_{v_1} = -1, \dots, \quad \theta_{v_t} = -1,$$

where $u_1 < u_2 < \dots < u_s$ and $v_1 < v_2 < \dots < v_t$; $s + t = n$ are disjoint subsets of $\{1, \dots, n\}$. Below we use the permutation $\pi^*(\pi, \theta)$ defined as follows:

$$\pi^* = (k_{u_1}, k_{u_2}, \dots, k_{u_s}, k_{v_t}, k_{v_{t-1}}, \dots, k_{v_1}).$$

$\pi_o \in \Pi_n$ will denote an *optimal* permutation, i.e., a permutation such that $|x_{\pi_o}|_n \leq |x_\pi|_n$ for any $\pi \in \Pi_n$.

In proofs below we will use repeatedly the following lemma (see [13], [14]).

Lemma 1. (i) If $(x_1, \dots, x_n) \in X^n$ with $\sum_1^n x_i = 0$ and $\theta \in \Theta_n$, then

$$|x_\pi|_n + |x_\pi \theta|_n \geq 2|x_{\pi^*}|_n \geq |x_\pi \theta|_n.$$

(ii) For any $\theta \in \Theta_n$

$$|x_{\pi_o}| \leq |x_{\pi_o} \theta|.$$

To make the presentation self-contained we repeat a simple proof here.

P r o o f. Using the fact that $|\cdot|_n$ is a norm in X^n we get

$$|x_\pi|_n + |x_\pi \theta|_n \geq 2|x_\pi^+|_s; \quad |x_\pi|_n + |x_\pi \theta|_n \geq 2|x_\pi^-|_t, \quad (2.1)$$

where $x_\pi^+ = (x_{\pi(u_1)}, \dots, x_{\pi(u_s)})$ and $x_\pi^- = (x_{\pi(v_1)}, \dots, x_{\pi(v_t)})$, where $u_1 < \dots < u_s$ and $v_1 < \dots < v_t$ are the indices, where θ is positive and negative, respectively.

Now (2.1) implies

$$|x_\pi|_n + |x_\pi \theta|_n \geq 2 \max(|x_\pi^+|_s, |x_\pi^-|_t). \quad (2.2)$$

Obviously, we also have $|x_\pi \theta|_n \leq 2 \max(|x_\pi^+|_s, |x_\pi^-|_t)$. It is easy to make sure that $\max(|x_\pi^+|_s, |x_\pi^-|_t) = |x_{\pi^*}|_n$. This equality along with (2.2) gives (i). Statement (ii) is an immediate consequence of (i).

3. The moment maximum inequalities for exchangeable random variables.

A finite collection $\xi = (\xi_1, \dots, \xi_n)$ of X -valued random variables is called *exchangeable*, if for each $\pi \in \Pi_n$ the rearranged collection $(\xi_{\pi(1)}, \dots, \xi_{\pi(n)})$ has the same distribution in X^n as $(\xi_1, \dots, \xi_n)$.

Theorem 1. Let $\xi = (\xi_1, \dots, \xi_n)$ be an exchangeable system of X -valued random variables with $\sum_1^n \xi_i = 0$, and let $\Phi: [0, \infty) \rightarrow [0, \infty)$ be a convex increasing function. Then

(i) For any collection $\theta = (\theta_1, \dots, \theta_n) \in \Theta_n$ the following two-sided inequality holds:

$$\mathbf{E} \Phi \left(\frac{1}{2} |\xi \theta|_n \right) \leq \mathbf{E} \Phi(|\xi|_n) \leq \mathbf{E} \Phi(|\xi \theta|_n),$$

where as before $|\xi \theta|_n = \max_{1 \leq k \leq n} \left\| \sum_1^k \xi_i \theta_i \right\|$.

(ii) The following two-sided inequality holds:

$$\mathbf{E} \mathbf{E}_r \Phi \left(\frac{1}{2} \left\| \sum_1^n \xi_i r_i \right\| \right) \leq \mathbf{E} \Phi(|\xi|_n) \leq 2 \mathbf{E} \mathbf{E}_r \Phi \left(\left\| \sum_1^n \xi_i r_i \right\| \right),$$

where $r_1, \dots, r_n$ is a system of Rademacher random variables (independent random variables taking values $+1$ and -1 with probabilities $1/2$ and $1/2$, respectively) independent of ξ .

P r o o f. Since Φ is monotone and convex, we get using Lemma 1

$$\Phi(|\xi \theta|_n) \geq \Phi(2|\xi_{\pi^*}|_n - |\xi|_n) \geq 2\Phi(|\xi_{\pi^*}|_n) - \Phi(|\xi|_n).$$

Taking expectation of both sides and using the fact that ξ is an exchangeable system, we come to the right-hand fragment of (i). The left-hand fragment of (i) also follows from Lemma 1:

$$\mathbf{E} \Phi \left(\frac{1}{2} |\xi \theta|_n \right) \leq \mathbf{E} \Phi \left(\frac{1}{2} (|\xi^+|_s + |\xi^-|_t) \right) \leq \mathbf{E} \Phi(|\xi_{\pi^*}|_n).$$

Part (ii) follows from (i) by integrating with respect to the Rademacher random variables and using the Lévy inequality.

R e m a r k. The fact that for exchangeable X -valued random variables $\xi_1, \dots, \xi_n$ with $\sum_1^n \xi_i = 0$

$$c_p \leq \frac{\mathbf{E} \max_{1 \leq k \leq n} \|\sum_1^k \xi_i\|^p}{\mathbf{E} \mathbf{E}_r \|\sum_1^n \xi_k r_k\|^p} \leq C_p,$$

where $1 \leq p < \infty$, $c_p > 0$, $C_p > 0$ are constants dependent only on p , was found by Maurey and Pisier [12] by use of combinatorial arguments.

Let $x = (x_1, \dots, x_n) \in X^n$ and consider the following random variables on the probability space (Π_n, μ) , where μ is the uniform distribution on Π_n :

$$\xi_k(\pi) = x_{\pi(k)}, \quad \pi \in \Pi_n, \quad k = 1, \dots, n.$$

Obviously, $\xi = (\xi_1, \dots, \xi_n)$ is a system of exchangeable random variables and Theorem 1 for it can be rewritten as follows.

Corollary 1. Let $\Phi: [0, \infty) \rightarrow [0, \infty)$ be a convex increasing function, let $x = (x_1, \dots, x_n)$ be a collection of elements in X with $\sum_1^n x_i = 0$. Then

(i) For any collection $\theta = (\theta_1, \dots, \theta_n) \in \Theta_n$ the following two-sided inequality holds:

$$\mathbf{E}_\pi \Phi\left(\frac{1}{2}|x_\pi \theta|_n\right) \leq \mathbf{E}_\pi \Phi(|x_\pi|_n) \leq \mathbf{E}_\pi \Phi(|x \theta|_n);$$

(ii) The following two-sided inequality holds:

$$\mathbf{E}_r \Phi\left(\frac{1}{2}\left\|\sum_1^n x_i r_i\right\|\right) \leq \mathbf{E}_\pi \Phi(|x_\pi|_n) \leq 2\mathbf{E}_r \Phi\left(\left\|\sum_1^n x_i r_i\right\|\right).$$

The estimating quantity $\mathbf{E} \Phi(\|\sum_1^n x_k r_k\|)$ can be expressed effectively through the coefficients $x_k, k = 1, \dots, n$, for Banach lattices do not containing l_∞^n uniformly (see [2] and the references therein). In the following corollary we give only the simple and popular case of a Hilbert space involving well-known Garsia's maximum inequalities.

Corollary 2. Let $\mathbf{H}$ be a Hilbert space and $x = (x_1, \dots, x_n) \in \mathbf{H}^n$. Then for any p , $1 \leq p < \infty$, the following maximum inequality holds:

$$c_p \left(\sum_1^n \|x_k\|^2 \right)^{p/2} \leq \frac{1}{n!} \sum_\pi |x_\pi|_n^p \leq C_p \left(\sum_1^n \|x_k\|^2 \right)^{p/2}, \quad (3.1)$$

where c_p and C_p are constants.

The right-hand side fragment of inequality (3.1) was proved by Garsia [10] for the case of real x_k .

In the following section we give a maximum inequality for the tail probabilities that allow to get two-sided maximum inequalities (analogs of Theorem 1) for any continuous increasing Φ (not only convex).

4. The main theorem. In this section we prove the main theorem of this note.

Theorem 2. Let $\xi = (\xi_1, \dots, \xi_n)$ be an exchangeable system of X -valued random variables with $\sum_1^n \xi_i = 0$. Then for any collection $\theta = (\theta_1, \dots, \theta_n) \in \Theta_n$ and any $t > 0$ the following two-sided inequality holds:

$$\mathbf{P}_1(|\xi \theta|_n > 2t) \leq \mathbf{P}_1(|\xi|_n > t) \leq 2\mathbf{P}_1\left(|\xi \theta|_n > \frac{t}{22}\right).$$

We will prove the following equivalent form of Theorem 2.

Theorem 2'. Let $x = (x_1, \dots, x_n) \in X^n$ be such that $\sum_1^n x_i = 0$. Then for any collection $\theta = (\theta_1, \dots, \theta_n) \in \Theta_n$ and any $t > 0$ the following two-sided inequality holds:

$$\mathbf{P}\{\pi: |x_\pi \theta|_n > 2t\} \leq \mathbf{P}\{\pi: |x_\pi|_n > t\} \leq 2\mathbf{P}\left\{\pi: |x_\pi \theta|_n > \frac{t}{22}\right\}.$$

In Theorem 2' $\mathbf{P}$, as before, stands for the uniform distribution on Π_n ; in Theorem 2 $\mathbf{P}_1$ is the probability of the underlying probability space.

The proof of the left-hand side inequality is an immediate consequence of Lemma 1 (i). The proof of the right-hand side inequality of Theorem 2' is based on the following Lemmas 2-6 in what follows. Let $y = (y_1, \dots, y_n) \in X^n$ and let $t > 0$ be a fixed number. Denote by $\mathcal{R}$ the set of all rearrangements of $(y_1, \dots, y_n)$ and introduce the cyclic transformation T by

$$T((z_1, \dots, z_n)) = (z_n, z_1, \dots, z_{n-1}), \quad z \in \mathcal{R}.$$

Lemma 2. Assume $\|y_1 + \dots + y_n\| < t$ and let $z \in \mathcal{R}$ be such that $|z - y| < Ct$, where C is a positive number. Then there is a number k , $k \in \{0, 1, \dots, n-1\}$, such that

- (i) $|y - T^k y|_n \geq |y|_n - t$;
- (ii) $|z - T^k z|_n \geq |z|_n - (1 + 4C)t$.

P r o o f o f L e m m a 2. (i) We have

$$\frac{1}{n} \sum_{k=0}^{n-1} |y - T^k y|_n \geq \left| y - \frac{1}{n} \sum_{k=0}^{n-1} T^k y \right|_n \geq |y|_n - \left\| \sum_{k=1}^n y_k \right\| > |y|_n - t.$$

The second inequality follows from

$$\left| \sum_{k=0}^{n-1} T^k y \right|_n = \left| \left(\sum_1^n y_k, \dots, \sum_1^n y_k \right) \right|_n = n \left\| \sum_1^n y_k \right\|.$$

This implies (i) for some k , $k \in \{0, 1, \dots, n-1\}$.

(ii) We have according to (i):

$$\begin{aligned} |z - T^k z|_n &= |z - y + y - T^k y + T^k y - T^k z|_n \geq |y - T^k y|_n - |z - y|_n - |T^k y - T^k z|_n \\ &\geq |y|_n - t - Ct - |T^k y - T^k z|_n. \end{aligned} \quad (4.1)$$

Putting in the last term $u = y - z$, we get using the fact that $\sum_1^n u_i = 0$:

$$\begin{aligned} |T^k(u)|_n &= |(u_{n-k+1}, u_{n-k+2}, \dots, u_n, u_1, u_2, \dots, u_{n-k})|_n \\ &\leq \max_{1 \leq k \leq l \leq n} \left\| \sum_{i=k}^l u_i \right\| \leq 2|u|_n = 2|y - z|_n. \end{aligned}$$

Now (4.1) gives

$$|z - T^k z|_n \geq |y|_n - t - Ct - 2|y - z|_n \geq |y|_n - (3C + 1)t.$$

To get (ii) we use the fact that $|y - z|_n < Ct$.

Let us introduce necessary notation: Given $\theta = (\theta_1, \dots, \theta_n) \in \Theta_n$ we put

$$\{+\} = \{i: \theta_i = +1\}, \quad \{-\} = \{i: \theta_i = -1\}.$$

For each $y = (y_1, \dots, y_n) \in X^n$ put

$$y^+ = y_i I_{\{+\}}(i), \quad y^- = y_i I_{\{-\}}(i), \quad i = 1, \dots, n.$$

Given $x = (x_1, \dots, x_n)$ with $\sum_1^n x_i = 0$ define

$$\mathcal{A} = \{x_\pi: \pi \in \Pi_n\}, \quad \mathcal{A}_x = \{x_\pi: \pi \in \Pi_n, x_\pi^+ = x^+\}, \quad \Lambda_x = \{z \in \mathcal{A}_x: |z\theta|_n < t\}.$$

Lemma 3. *There exists an injection $S: \Lambda_x \rightarrow \mathcal{A}_x$ such that*

$$|(Sz)^- - z^-|_n \geq |z^-|_n - 8.5t, \quad z \in \Lambda_x. \quad (4.2)$$

P r o o f. Let $z \in \Lambda_x$. We claim that

$$\left\| \sum_{i \in \{-\}} z_i^- \right\| < \frac{t}{2}. \quad (4.3)$$

Indeed, since $z \in \Lambda_x$, we have

$$\left\| \sum_{\{-\}} z_i^- - \sum_{\{+\}} z_i^+ \right\| \leq t; \quad \left\| \sum_{\{-\}} z_i^- + \sum_{\{+\}} z_i^+ \right\| = 0.$$

These relations imply (4.3). We also have for $y \in \Lambda_x$

$$|y^- - z^-|_n = |z\theta - y\theta|_n \leq |y\theta|_n + |z\theta|_n \leq 2t.$$

According to Lemma 2, for the cyclic permutation T of $\{-\}$, some integer $k \leq \text{card}\{-\}$ and any $z \in \Lambda_x$ restricted to $\{-\}$ the following inequality holds:

$$|z^- - T^k z^-|_n \geq |z^-|_n - 8.5t.$$

(the role of t in Lemma 2 here is played by $t/2$, while that of C by 4). Finally let us put for any $z \in \Lambda_x$: $(Sz)_i = x_i$, if $i \in \{+\}$ and $(Sz)_i = (T^k(z^-))_i$, if $i \in \{-\}$.

Lemma 4. *Let $\Lambda = \{z \in \mathcal{A}: |\theta z|_n < t\}$. There is a mapping $U: \Lambda \rightarrow \mathcal{A}$ such that U is an injection and satisfies the following conditions:*

$$|z^- - (Uz)^-|_n \geq |z^-|_n - 8.5t \quad \text{and} \quad (Uz)^+ = z^+, \quad z \in \Lambda.$$

P r o o f. Obviously we can find a collection $x^1, \dots, x^k \in \mathcal{A}$ such that $\Lambda = \cup_{i=1}^k \Lambda_{x^i}$, where $\Lambda_{x^i} \cap \Lambda_{x^j} = \emptyset$, $i \neq j$. By Lemma 3 we have for any i , $1 \leq i \leq k$, a mapping $T_i: \Lambda_{x^i} \rightarrow \mathcal{A}_{x^i}$ satisfying (4.2). Finally we define

$$U(z) = T_i(z), \quad z \in \Lambda_{x^i}.$$

Lemma 5. *For every collection of signs $\theta = (\theta_1, \dots, \theta_n)$ and any $t \geq 0$ the following inequality holds:*

$$\mathbf{P}(|x_\pi|_n \geq 22t, |\theta x_\pi|_n < t) \leq \mathbf{P}(|\theta x_\pi| \geq t).$$

P r o o f. We need to show that if $z \in \Lambda$ satisfies $|z|_n \geq 22t$, then $|\theta \cdot U(z)|_n \geq t$. We calculate using Lemma 4:

$$|\theta \cdot U(z)|_n = |z^+ - (Uz)^-|_n \geq |z^- - (Uz)^-|_n - t \geq |z^-|_n - 9.5t \geq 10.5t - 9.5t = t,$$

where the last inequality is a consequence of $|z^+ + z^-|_n \geq 22t$ and $|z^+ - z^-|_n < t$, because they imply $|z^-|_n \geq 10.5t$.

The following lemma follows from Lemma 5 and actually is the left-hand side inequality of Theorem 2'.

Lemma 6. *For every collection of signs $\theta = (\theta_1, \dots, \theta_n)$ and any $t \geq 0$ the following inequality holds:*

$$\mathbf{P}(|x_\pi|_n \geq 22t) \leq 2\mathbf{P}(|\theta \cdot x_\pi| \geq t).$$

P r o o f. Lemma 5 gives

$$\begin{aligned} \mathbf{P}(|x_\pi|_n \geq 22t) &= \mathbf{P}(|x_\pi|_n \geq 22t, |\theta \cdot x_\pi| < t) + \mathbf{P}(|x_\pi|_n \geq 22t, |\theta \cdot x_\pi| \geq t) \\ &\leq \mathbf{P}(|\theta \cdot x_\pi| \geq t) + \mathbf{P}(|x_\pi|_n \geq 22t, |\theta \cdot x_\pi| \geq t) \leq 2\mathbf{P}(|\theta \cdot x_\pi| \geq t). \end{aligned}$$

Theorem 2 is proved.

In conclusion let us give a curious consequence of Theorem 2.

Corollary. Let $x = (x_1, \dots, x_n)$ be a collection of elements of X with $\sum_1^n x_i = 0$. Then

(i) There are universal constants $C_1 > 0$ and $C_2 > 0$ such that for any two collections $\theta = (\theta_1, \dots, \theta_n) \in \Theta_n$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \Theta_n$ the following inequality holds:

$$\mathbf{P}\{\pi: |x_\pi \theta| > t\} \leq C_1 \mathbf{P}\left\{\pi: |x_\pi \varepsilon| > \frac{t}{C_2}\right\}.$$

(ii) There are universal constants C_1 and C_2 such that for any $t > 0$ and any $\theta = (\theta_1, \dots, \theta_n) \in \Theta_n$

$$\frac{1}{2} \mathbf{P}\{\pi: |x_\pi| > 2t\} \leq \mathbf{P}\left\{\omega: \left\|\sum x_k r_k\right\| > t\right\} \leq C_1 \mathbf{P}\left\{\pi: |x_\pi \theta| > \frac{t}{C_2}\right\},$$

where $r_1, \dots, r_n$ is a system of Rademacher random variables.

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