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PINELIS I.*

ROSENTHAL-TYPE INEQUALITIES FOR MARTINGALES IN 2-SMOOTH BANACH SPACES¹⁾

Мы улучшаем ряд известных ранее верхних оценок для моментов норм мартингалов в 2-гладких банаховых пространствах. Некоторые из этих улучшений имеют место даже для сумм независимых действительных случайных величин. Приводятся применения к концентрации меры на произведениях пространств для функций, липшицевых по каждой переменной, в том числе применения, касающиеся центральных моментов нормы сумм независимых случайных векторов в произвольном сепарабельном банаховом пространстве.

Ключевые слова и фразы: вероятностные неравенства; оценки моментов; неравенство Розенталя; 2-гладкие банаховы пространства; гильбертовы пространства; мартингалы; суммы независимых случайных величин; концентрация меры; функции, являющиеся липшицевыми по каждой переменной; произведения пространств.

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1. Introduction, summary, and discussion. Let $(S_i)_{i=0}^\infty$ be a martingale in a separable Banach space $(\mathfrak{X}, \|\cdot\|)$ relative to a filter $(\mathcal{F}_i)_{i=0}^\infty$ of σ -algebras. Assume that $S_0 = 0$ and the differences $X_i := S_i - S_{i-1}$ satisfy the condition

$$\mathbf{E}_{i-1} \|X_i\|^2 \leq b_i^2 \quad (1)$$

almost surely for all $i \in 1, \dots, \infty$, where $\mathbf{E}_{i-1}$ denotes the conditional expectation given $\mathcal{F}_{i-1}$ and the b_i 's are some positive real numbers. Here and in what follows, for any m and n in $\mathbf{Z} \cup \{\infty\}$ we let $m, \dots, n := \{i \in \mathbf{Z}: m \leq i \leq n\}$. Also introduce

$$a_i^{(t)} := \mathbf{E} \|X_i\|^t, \quad A_n^{(t)} := \sum_{i \in 1, \dots, n} a_i(t), \quad \text{and} \quad B_n := \sqrt{\sum_{i \in 1, \dots, n} b_i^2}$$

for all real $t \geq 0$, $i \in 1, \dots, \infty$, and $n \in 0, \dots, \infty$; as usual, assume that the sum and product of an empty family are 0 and 1, respectively; let also $0^0 := 1$.

Assume further that the Banach space $(\mathfrak{X}, \|\cdot\|)$ is 2-smooth (or, more exactly, $(2, D)$ -smooth, for some $D = D(\mathfrak{X}) > 0$), in the following sense:

$$\|x + y\|^2 + \|x - y\|^2 \leq 2\|x\|^2 + 2D^2\|y\|^2 \quad (2)$$

for all x and y in $\mathfrak{X}$. The importance of the 2-smooth spaces was elucidated in [27]: they play the same role with respect to the vector martingales as the spaces of type 2 do with respect to the sums of independent random vectors. The definition of 2-smooth spaces assumed in this paper is the same as that in [17], which is slightly different from the one given by Pisier in [27] — which latter required only that (2) hold for an equivalent norm; the reason for the modified definition is that we would like to follow the dependence of certain constants on D , the constant of 2-smoothness. Substituting λx for y in (2), where $\lambda \in \mathbf{R}$, one observes that necessarily $D \geq 1$, except for $\mathfrak{X} = \{0\}$. As shown in [17], [5], the space $L^p(T, \mathcal{A}, \nu)$ is $(2, \sqrt{p-1})$ -smooth for any $p \geq 2$ and any measure space $(T, \mathcal{A}, \nu)$. In particular, what is obvious and well known, if $\mathfrak{X}$ is a Hilbert space, then it is $(2, 1)$ -smooth.

Theorem 1. *Take any real $t \geq 0$ and $n \in 0, \dots, \infty$. Then*

$$\begin{aligned} \mathbf{E} \|S_n\|^t &\leq \sum_{j \in 0, \dots, m-1} c_j^{(t)} \sum_{J \in \mathcal{J}_{n,j}} A_{\mu_n(J)-1}^{(t-2j)} \prod_{i \in J} b_i^2 \\ &\quad + \widetilde{c}_m^{(t)} \sum_{J \in \mathcal{J}_{n,m}} \left(A_{\mu_n(J)-1}^{(2)} \right)^{t/2-m} \prod_{i \in J} b_i^2, \end{aligned} \quad (3)$$

where $m := \lfloor t/2 \rfloor$, $\mathcal{J}_{n,j}$ denotes the set of all subsets of the set $1, \dots, n$ of cardinality j , $\mu_n(J) := \min J$ if $J \neq \emptyset$, $\mu_n(\emptyset) := n+1$,

$$\begin{aligned} c_j^{(t)} &:= \frac{t-2j-2+D^2}{t-2j-1} q(t-2j) \prod_{k \in 0, \dots, j-1} \frac{(t-2k)(t-2k-2+D^2)p(t-2k)}{2}, \\ \widetilde{c}_m^{(t)} &:= \prod_{j \in 0, \dots, m-1} \frac{(t-2j)(t-2j-2+D^2)p(t-2j)}{2}, \end{aligned}$$

and $p(\cdot)$ and $q(\cdot)$ are any functions such that

$$\begin{aligned} p(2) + q(2) &\geq 1, \quad p(2) > 0, \quad q(2) > 0, \\ p(s) &\geq 1 \quad \text{and} \quad q(s) \geq 1 \quad \text{for all } s > 2, \\ p(s)^{1/(3-s)} + q(s)^{1/(3-s)} &\leq 1 \quad \text{for all } s > 3. \end{aligned} \quad (4)$$

Almost the same result was obtained in [25] in the special case when $\mathfrak{X}$ is a Hilbert space (in which case one can take $D = 1$) — except that, in place of the term $A_{\mu_n(J)-1}^{(2)}$ in (3) above, the bound in [25] contained the larger term $A_n^{(2)} - \sum_{i \in J} a_i^{(2)}$.

The proofs, whenever necessary, are deferred to Section 3.

Corollary 1. *In the conditions of Theorem 1, let $t > 2$ and take any positive real numbers λ_j for all $j \in 0, \dots, m-1$. Then*

$$\mathbf{E} \|S_n\|^t \leq C_A^{(t)} A_n^{(t)} + C_B^{(t)} B_n^t, \quad (5)$$

where

$$\begin{aligned} C_A^{(t)} &:= \sum_{j=0, \dots, m-1} c_j^{(t)} \frac{t-2j-2}{t-2} \frac{1}{\lambda_j^{2j} j!}, \\ C_B^{(t)} &:= \tilde{c}_m^{(t)} \prod_{j=1, \dots, m} \frac{1}{t/2 - m + j} + \sum_{j=0, \dots, m-1} c_j^{(t)} \frac{2j}{t-2} \frac{\lambda_j^{t-2j-2}}{j!}. \end{aligned} \quad (6)$$

In the special case when $\mathfrak{X}$ is a Hilbert space, a similar but somewhat less precise bound was obtained in [25, Corollary]. Namely, in place of the terms $(t-2j-2)/(t-2)$, $1/(t/2-m+j)$, and $2j/(t-2)$ in (6), the bound in [25, Corollary] contains the larger terms 1, $1/j$, and 1, respectively.

Results somewhat similar to Corollary 1 were also obtained in [17], by rather different methods. Particularly, [17, Theorem 4.1] implies the «spectrum» of inequalities

$$\mathbf{E} \|S_n\|^t \leq K^t \left(c^t A_n^{(t)} + c^{t/2} e^{t^2/c} D^t B_n^t \right) \quad (7)$$

for some positive absolute constant K and all real $t \geq 2$, depending on the freely chosen value of the «balancing» parameter $c \in [1, t]$. One can use this freedom to minimize or quasi-minimize the bound in (7) in $c \in [1, t]$, depending on the value of the ratio $A_n^{(t)}/B_n^t$ — as is done in [17]; see Theorem 6.1 and the definition of B_p^* on page 1693 there. In fact, [17, Theorems 6.1 and 6.2] show that

- (i) for each $t \geq 2$ and each value of the ratio $A_n^{(t)}/B_n^t$, the minimum in $c \in [1, t]$ of the upper bound in (7) is an optimal (up to a factor of the form K^t) upper bound on $\mathbf{E} \|S_n\|^t$;
- (ii) the spectrum of the bounds in (7) is also «minimal» in the sense that for each $c \in [1, t]$ there exists a value of the ratio $A_n^{(t)}/B_n^t$ such that the corresponding «individual» bound in (7) is the best possible (again up to a factor of the form K^t);
- (iii) the above statements (i) and (ii) hold if the term(s) $A_n^{(t)}$ and/or B_n^t are/is replaced, respectively, by $\mathbf{E} \max_{i \in 1, \dots, n} \|X_i\|^t$ and/or $\mathbf{E} (\sum_{i=1}^n \mathbf{E}_{i-1} \|X_i\|^2)^{t/2}$.

One can similarly use the balancing parameters λ_j in (6) to minimize or quasi-minimize the bound in (5). Moreover, by Remark 6.8 in [17] (with details given by [20, Proposition 9.2]), the minimum of the bound on $\mathbf{E} \|S_n\|^t$ in [25, Theorem 1] with respect to the corresponding balancing parameters is equivalent (once again up to a factor of the form K^t) to the minimum in $c \in [1, t]$ of the bound in (7) — at least when the condition (1) holds; hence, the same is true for the bounds in (5) and (3), which therefore possess the same up-to-the- K^t -factor optimality property.

Thus, the main advantage of inequality (7) over (5) and (3) is that the former does not require the condition (1). On the other hand, the bounds in (5) and (3) are quite explicit and do not contain unspecified constants such as K in (7). If one follows the lines of the proof of (7) in [17] without serious efforts at modification, the resulting value of K turns out to be large, namely equal to 120, quite in contrast with the constant factors in (5) and (3) — cf., e.g., the bound in (12) below, which is a particular case of (5). Moreover, (5) and (3) appear to provide a better dependence of the bounds on the 2-smoothness constant D .

As for the condition (1), it turns out quite naturally satisfied in applications to concentration of measure on product spaces for separately Lipschitz functions and, in particular, for the norm of the sums of independent random vectors — cf., e.g., [23] and further bibliography there. Such an application is provided in Section 2 of the present note.

Special cases of Corollary 1 are the following inequalities:

$$\mathbf{E} \|S_n\|^t \leq \frac{t-2+D^2}{t-1} \left(A_n^{(t)} + (t-1)B_n^t \right) \quad \text{for all } t \in (2, 3], \quad (8)$$

$$\mathbf{E} \|S_n\|^t \leq \frac{t-2+D^2}{t-1} \left(\frac{A_n^{(t)}}{\alpha^{t-3}} + (t-1) \frac{B_n^t}{(1-\alpha)^{t-3}} \right) \quad \text{for all } t \in [3, 4] \quad (9)$$

and $\alpha \in (0, 1)$. Minimizing the upper bound in (9) in $\alpha \in (0, 1)$, one can combine (8) and (9) into the inequality

$$\mathbf{E} \|S_n\|^t \leq \frac{t-2+D^2}{t-1} \left[\left(A_n^{(t)} \right)^{1/s_t} + (t-1)^{1/s_t} B_n^{t/s_t} \right]^{s_t} \quad \text{for all } t \in (2, 4], \quad (10)$$

where $s_t := \max(1, t-2)$.

Note that the coefficient of $A_n^{(t)}$ in (8) cannot be less than 1, whatever the coefficient of B_n^t in (8) may be; for instance, take $n = 1$ and let X_1 be such that $\mathbf{P}(X_1 = 0) = 1 - 2p$ and $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = p$, with $p \downarrow 0$. So, the coefficient of $A_n^{(t)}$ in (8) takes the optimal value 1 when $D = 1$.

Consider now the case when the latter condition holds, that is, when $\mathfrak{X}$ is a Hilbert space. Then the behavior of the upper bound in (10) depends mainly on the ratio $r := r_n^{(t)} := A_n^{(t)}/B_n^t$. If $r \rightarrow \infty$ (which happens, e.g., in the situation described in the previous paragraph), then the upper bound in (10) is asymptotic to $A_n^{(t)}$ and thus is asymptotically optimal. If $r \downarrow 0$ (which happens, e.g., in the important case when $n \rightarrow \infty$, the X_i 's are independent identically distributed (i.i.d.), t is fixed, and $\mathbf{E}|X_1|^t < \infty$), then the upper bound in (10) is asymptotic to $(t-1)B_n^t$. On the other hand, by the central limit theorem, this upper bound cannot be less than $\mathbf{E}|Z|^t B_n^t$, where Z is a standard normal random variable. Since $\mathbf{E}|Z|^t = t-1$ for $t \in \{2, 4\}$, it follows that the upper bound in (10) is asymptotically optimal when $r \downarrow 0$ and at that either $t \downarrow 2$ or $t \rightarrow 4$. The graph of the ratio of $t-1$ to $\mathbf{E}|Z|^t$ in Figure 1 shows that the asymptotic coefficient $t-1$ at B_n^t in (10) for $r \downarrow 0$ is rather close to optimality for all $t \in (2, 4]$.

Fig. 1. Graph of the ratio of $t-1$ to $\mathbf{E}|Z|^t$ for $t \in (2, 4]$.

Again when $\mathfrak{X}$ is a Hilbert space, (8) and (9) with $\alpha = 1/2$ yield

$$\mathbf{E} \|S_n\|^t \leq 2^{(t-3)+} \left(A_n^{(t)} + (t-1)B_n^t \right) \quad \text{for all } t \in (2, 4], \quad (11)$$

where $u_+ := \max(0, u)$. This latter result was obtained, by a more Stein-like method, in [4, Lemma 6.3] in the case when $\mathfrak{X} = \mathbf{R}$, $t \in (2, 3]$, and the X_i 's are independent. In the case when $\mathfrak{X} = \mathbf{R}$, $t = 3$, and the X_i 's are i.i.d., inequality (11) was stated without proof in [16]; in the more general case when $\mathfrak{X} = \mathbf{R}$, $t \in (2, 4]$, and the X_i 's are independent but

not necessarily identically distributed, the bound in (11) was obtained in [15, Lemma 13], but with the larger factor $t(t-1) \cdot 2^{-t/2}$ in place of $t-1$.

In the case $t = 3$, which is particularly important in applications to Berry–Esseen-type bounds (see, e.g., [4], [24]), (8) or (9) yields

$$\mathbf{E} \|S_n\|^3 \leq \frac{1+D^2}{2} (A_n^{(3)} + 2B_n^3). \quad (12)$$

Rosenthal-type inequalities and related results can be found, among other papers, in [2], [3], [6]–[9], [14], [18], [21], [22], [26], [28], [29].

2. Application to concentration of measure on product spaces for separately Lipschitz functions. In this section, let us redefine $X_1, \dots, X_n$ to be independent random variables with values in measurable spaces $\mathfrak{X}_1, \dots, \mathfrak{X}_n$, respectively. Let $g: \mathfrak{P} \rightarrow \mathbf{R}$ be a measurable function on the product space $\mathfrak{P} := \mathfrak{X}_1 \times \dots \times \mathfrak{X}_n$. Let us say (cf. [1], [19]) that g is *separately Lipschitz* if it satisfies a Lipschitz-type condition in each of its arguments:

$$|g(x_1, \dots, x_{i-1}, \tilde{x}_i, x_{i+1}, \dots, x_n) - g(x_1, \dots, x_n)| \leq \rho_i(\tilde{x}_i, x_i) \quad (13)$$

for some measurable functions $\rho_i: \mathfrak{X}_i \times \mathfrak{X}_i \rightarrow \mathbf{R}$ and all $i \in 1, \dots, n$, $(x_1, \dots, x_n) \in \mathfrak{P}$, and $\tilde{x}_i \in \mathfrak{X}_i$. Take now any separately Lipschitz function g and let

$$Y := g(X_1, \dots, X_n).$$

Suppose that the random variable Y has a finite mean. Take any real $t > 2$.

Corollary 2. *For each $i \in 1, \dots, n$, take any x_i and y_i in $\mathfrak{X}_i$. Then*

$$\mathbf{E} |Y - \mathbf{E} Y|^t \leq C_t C_A^{(t)} \sum_1^n \mathbf{E} \rho_i(X_i, x_i)^t + C_B^{(t)} \left(\sum_1^n \mathbf{E} \rho_i(X_i, y_i)^2 \right)^{t/2},$$

where $C_A^{(t)}$ and $C_B^{(t)}$ are as in (6) with $D = 1$,

$$C_t := R(t, b_t),$$

and b_t is a unique maximizer of

$$R(t, b) := (b^{t-1} + (1-b)^{t-1}) \left(b^{1/(t-1)} + (1-b)^{1/(t-1)} \right)^{t-1}$$

over all $b \in [0, 1/2]$.

This follows immediately from Corollary 1 and [23, Corollary 3.1].

An example of separately Lipschitz functions $g: \mathfrak{X}^n \rightarrow \mathbf{R}$ is given by the formula

$$g(x_1, \dots, x_n) = \|x_1 + \dots + x_n\|$$

for all $x_1, \dots, x_n$ in any (not necessarily 2-smooth) separable Banach space $(\mathfrak{X}, \|\cdot\|)$. In this case, one may take $\rho_i(\tilde{x}_i, x_i) \equiv \|\tilde{x}_i - x_i\|$. Thus, one immediately obtains the following corollary.

Corollary 3. *Let $X_1, \dots, X_n$ be independent random vectors in a separable Banach space $(\mathfrak{X}, \|\cdot\|)$. Let here $Y := \|X_1 + \dots + X_n\|$. For each $i \in 1, \dots, n$, take any x_i and y_i in $\mathfrak{X}$. Then, with $C_A^{(t)}$, $C_B^{(t)}$, and C_t as in Corollary 2,*

$$\mathbf{E} |Y - \mathbf{E} Y|^t \leq C_t C_A^{(t)} \sum_1^n \mathbf{E} \|X_i - x_i\|^t + C_B^{(t)} \left(\sum_1^n \mathbf{E} \|X_i - y_i\|^2 \right)^{t/2}.$$

In particular, $C_3 < 1.316$ and hence, by (12) with $D = 1$,

$$\mathbf{E} |Y - \mathbf{E} Y|^3 \leq 1.316 \sum_1^n \mathbf{E} \|X_i - x_i\|^3 + 2 \left(\sum_1^n \mathbf{E} \|X_i - y_i\|^2 \right)^{3/2};$$

this improves the corresponding bound in [23], which had the constant factor 3 in place of 2.

The separate-Lipschitz condition (13) is easier to check than a joint-Lipschitz one. Also, the former is more generally applicable. On the other hand, when a joint-Lipschitz condition is satisfied, one can generally obtain better bounds. Literature on the concentration of measure phenomenon, almost all of it for joint-Lipschitz settings, is vast; let us mention here only [10]–[13].

3. Proofs. The proof of Theorem 1 is similar to that of [25, Theorem 1]; instead of [25, Lemma 1] for Hilbert spaces one should now use the following more general lemma, valid for general 2-smooth Banach spaces.

Lemma 1. *Suppose that $t \geq 2$, and let $p(t)$, $q(t)$ satisfy conditions (4). Suppose also that the function $Q(\cdot) := \|\cdot\|^2$ is twice differentiable everywhere on $\mathfrak{X}$ — which may be assumed without loss of generality in view of [17, Remark 2.4]. Then for any x and y in $\mathfrak{X}$*

$$\begin{aligned} \|x + y\|^t - \|x\|^t &\leq \frac{t}{2} \|x\|^{t-2} Q'(x)(y) + \frac{t(t-2+D^2)}{2} p(t) \|x\|^{t-2} \|y\|^2 \\ &\quad + \frac{t-2+D^2}{t-1} q(t) \|y\|^t. \end{aligned}$$

Proof of Lemma 1. Take indeed any x and y in $\mathfrak{X}$ and introduce

$$f(\theta) := \|x_\theta\|^t - \|x\|^t = Q(x_\theta)^{t/2} - \|x\|^t,$$

where $\theta \in [0, 1]$ and $x_\theta := x + \theta y$, so that $f(0) = 0$, $f'(0) = (t/2) \|x\|^{t-2} Q'(x)(y)$, and $f(1) = \|x + y\|^t - \|x\|^t$. Moreover, since $\mathfrak{X}$ is $(2, D)$ -smooth and Q is twice differentiable on $\mathfrak{X}$, for all θ one has $Q''(x_\theta)(y, y) \leq 2D^2 \|y\|^2$ and $|Q'(x_\theta)(y)| \leq 2\|x_\theta\| \|y\|$ (cf. [17, Lemma 2.2 and Remark 2.4]); so,

$$\begin{aligned} f''(\theta) &= \frac{t}{2} \left(\frac{t}{2} - 1 \right) Q(x_\theta)^{t/2-2} \left(Q'(x_\theta)(y) \right)^2 + \frac{t}{2} Q(x_\theta)^{t/2-1} Q''(x_\theta)(y, y) \\ &\leq t(t-2+D^2) \|x_\theta\|^{t-2} \|y\|^2 \\ &\leq t(t-2+D^2) \left(p(t) \|x\|^{t-2} \|y\|^2 + q(t) \theta^{t-2} \|y\|^t \right), \end{aligned}$$

since $(\alpha + \beta)^{t-2} \leq p(t) \alpha^{t-2} + q(t) \beta^{t-2}$ for all α and β in $[0, \infty)$. It remains to use the identities

$$\|x + y\|^t - \|x\|^t = f(1) = f(0) + f'(0) + \int_0^1 (1 - \theta) f''(\theta) d\theta.$$

Lemma 1 is proved.

Proof of Corollary 1. For any $j \in 0, \dots, m-1$ and $\lambda_j > 0$

$$\begin{aligned} j! \sum_{J \in \mathcal{J}_{n,j}} A_{\mu_n(J)-1}^{(t-2j)} \prod_{i \in J} b_i^2 &\leq A_n^{(t-2j)} B_n^{2j} \leq (A_n^{(t)})^{(t-2j-2)/(t-2)} (A_n^{(2)} B_n^{t-2})^{2j/(t-2)} \\ &\leq (A_n^{(t)})^{(t-2j-2)/(t-2)} (B_n^t)^{2j/(t-2)} \\ &\leq \frac{t-2j-2}{t-2} \lambda_j^{-2j} A_n^{(t)} + \frac{2j}{t-2} \lambda_j^{t-2j-2} B_n^t; \end{aligned} \tag{14}$$

the second inequality here follows by the log-convexity of $A_n^{(s)}$ in s , and the fourth one follows by Young's inequality $x^{1/p} y^{1/q} \leq x/p + y/q$ for any x, y, p , and q such that $x \geq 0$, $y \geq 0$, $p > 0$, $q > 0$, and $1/p + 1/q = 1$. Further,

$$\begin{aligned} \sum_{J \in \mathcal{J}_{n,m}} \left(A_{\mu_n(J)-1}^{(2)} \right)^{t/2-m} \prod_{i \in J} b_i^2 &\leq \sum_{J \in \mathcal{J}_{n,m}} B_{\mu_n(J)-1}^{t-2m} \prod_{i \in J} b_i^2 \\ &= \sum_{i_m \in 1, \dots, n} b_{i_m}^2 \sum_{i_{m-1} \in 1, \dots, i_m-1} b_{i_{m-1}}^2 \cdots \sum_{i_1 \in 1, \dots, i_2-1} b_{i_1}^{t-2m} B_{i_1-1}^{t-2m} \\ &\leq \left(\prod_{j \in 1, \dots, m} \frac{1}{t/2-m+j} \right) B_n^t; \end{aligned} \tag{15}$$

the iterated sum here is bounded by induction, using the inequality

$$\sum_{i \in 1, \dots, k} b_i^2 B_{i-1}^s \leq \frac{1}{s/2 + 1} B_k^{s+2} \quad (16)$$

for $s \geq 0$ and $k \in 1, \dots, \infty$; in turn, inequality (16) follows because its left-hand side is a left (and hence lower) Riemann sum for the integral $\int_0^{B_k^2} x^{s/2} dx$ of the nondecreasing function $(\cdot)^{s/2}$, whereas the right-hand side of (16) is the value of the integral. Now Corollary 1 follows by (3), (14), and (15).

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