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A GENERAL MULTIPARAMETER VERSION OF GNEDENKO'S TRANSFER THEOREM

Предельные теоремы для случайного числа независимых случайных величин часто называют теоремами о переносе. Начало исследованиям в этом направлении для сумм случайных величин с независимым случайным объемом выборки было положено Б.В.Гнеденко. Мы приводим допускающую широкое применение теорему о переносе для случайных величин на общем метрическом пространстве со случайными многомерными параметрами вместо случайных объемов выборки. Это обобщает существенный принцип, лежащий в основе известных в литературе результатов типа теорем о переносе.

Ключевые слова и фразы: теорема о переносе, случайная остановка, случайная замена времени, независимые случайные индексы, случайные поля, отрицательная ассоциированность, инфинитезимальная схема серий, полустойчивая область притяжения, случайные размещения.

1. Introduction. In 1969 Gnedenko and Fahim [15] proved the following limit theorem for sums of a random number of real-valued random variables; see also [14] for the precise assumptions. For every $n \in \mathbb{N}$ let $(X_{n,k})_{k \in \mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) random variables, and let $(T_n)_{n \in \mathbb{N}}$ be a sequence of positive integer-valued random variables such that T_n and $(X_{n,k})_{k \in \mathbb{N}}$ are independent for every $n \in \mathbb{N}$. If there exists a sequence $k_n \rightarrow \infty$ such that as $n \rightarrow \infty$

$$\sum_{k=1}^{k_n} X_{n,k} \Rightarrow \mu \quad \text{and} \quad \frac{T_n}{k_n} \Rightarrow \rho \quad (1.1)$$

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for some probability distributions μ on $\mathbb{R}$, respectively, ρ on $(0, \infty)$, then the random sum converges to a mixture distribution

$$\sum_{k=1}^{T_n} X_{n,k} \Rightarrow \int_0^\infty \mu^t d\rho(t), \quad (1.2)$$

where $\Rightarrow$ denotes convergence in distribution, i.e., the distribution of the random variable on the left-hand side converges weakly to the measure on the right-hand side, and μ^t denotes the t -fold convolution power of the necessarily infinitely divisible distribution μ , well defined by the Lévy–Khinchin formula. The result is frequently called the *transfer theorem of Gnedenko*, since randomization of the sample size transfers the limits in (1.1) to a randomized limit in (1.2). There is an ongoing considerable interest in random limit theorems of this kind from numerous applications, and today there exists a vast literature on transfer theorems generalizing the above. Without demanding to give a complete list, we refer to [31], [34], [27], [13], [25], [26], [11] and the literature cited therein, to mention just a few direct generalizations. Comprehensive studies of random limit theorems of various kind are given in the monographs [17], [16], [7], [33]. Although there is an intrinsic principle behind all the transfer type results, a new proof is given in every new situation. Our aim is to capture this principle and to prove a widely applicable version of the transfer theorem for randomly indexed sequences of random elements on a general metric space. For this reason we replace the random number T_n by a random multiparameter in $\mathbb{N}^d$. We will introduce a natural homeomorphism on the discrete multiparameters that serves as a control mapping for the transfer mechanism. We also aim to consider general state spaces for the random variables $X_{n,k}$. Since we are concerned with weak convergence of probability measures, a requirement on our state space E is that a probability measure μ on E is uniquely determined by the values $\int_E f d\mu$ for bounded and continuous functions $f: E \rightarrow \mathbb{R}$, i.e., the Riesz theorem applies. Thus we may consider E to be a general metric space; see [30], [20] for details. In Section 2 we present our general transfer theorem and briefly show how it covers classical transfer results appearing in the literature. In Section 3 we focus on three specific examples that give a glance on possible applications of our general transfer theorem.

2. Main result. Let

$$\mathcal{F} = \{Y_{\mathbf{n}}: \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d\}$$

be a family of E -valued random variables, where E is a metric space equipped with its Borel σ -algebra. We use boldface notation for multiparameters. Let $\phi: \mathbb{N}^d \rightarrow D \subseteq \mathbb{R}^\ell$ be a homeomorphism. Note that we consider $\mathbb{N}^d$ and D as subsets of $\mathbb{R}^d$ and $\mathbb{R}^\ell$, respectively, with the induced standard topology. Hence any subset of $\mathbb{N}^d$ is a Borel set and any mapping $\phi: \mathbb{N}^d \rightarrow \mathbb{R}^\ell$ is continuous. The only assumptions we have to make to ensure that ϕ is a homeomorphism are that ϕ is injective and that the image $D = \text{Im } \phi$ is discrete in the sense that intersections of D with sufficiently small open neighborhoods of any $\mathbf{x} \in D$ coincide with the single point set $\{\mathbf{x}\}$. Now let $\Delta \subseteq \overline{D} \setminus D$ be a nonempty Borel set of limit points of D , where $\overline{D}$ denotes the closure of D in $\mathbb{R}^\ell$. We assume that for any $\mathbf{t} = (t_1, \dots, t_\ell) \in \Delta$ and any sequence $(\mathbf{N}_n)_{n \in \mathbb{N}}$ in $\mathbb{N}^d$ we have

$$Y_{\mathbf{N}_n} \Rightarrow \mu_{\mathbf{t}} \quad \text{whenever} \quad \phi(\mathbf{N}_n) \rightarrow \mathbf{t}, \quad (2.1)$$

where $\{\mu_{\mathbf{t}}: \mathbf{t} \in \Delta\}$ is a weakly continuous family of probability measures on E . Thus the homeomorphism ϕ serves as a control mapping for weak convergence in (2.1). For randomizations of the multiparameter $\mathbf{n}$ of $\mathcal{F}$ we are able to give the following general version of a transfer theorem.

Theorem 1. *With the above assumptions and notation let $(\mathbf{T}_n)_{n \in \mathbb{N}}$ be a sequence of random variables with values in $\mathbb{N}^d$ such that $\mathbf{T}_n$ and $\mathcal{F}$ are independent for every $n \in \mathbb{N}$.*

Suppose that $\phi(\mathbf{T}_n) \Rightarrow \rho$ for some probability distribution ρ with $\rho(\Delta) = 1$. Then we have as $n \rightarrow \infty$

$$Y_{\mathbf{T}_n} \Rightarrow \int_{\Delta} \mu_{\mathbf{t}} d\rho(\mathbf{t}). \quad (2.2)$$

P r o o f. For a probability measure μ and a fixed, bounded, and continuous function $f: E \rightarrow \mathbb{R}$ we use the abbreviation

$$\langle \mu, f \rangle = \int_E f(x) d\mu(x).$$

Let $\nu(\mathbf{n}) = P_{Y_{\mathbf{n}}}$ denote the distribution of $Y_{\mathbf{n}}$, then $\varphi: D \rightarrow \mathbb{R}$ defined by

$$\varphi(\mathbf{t}) = \langle \nu(\phi^{-1}(\mathbf{t})), f \rangle \quad \text{for } \mathbf{t} \in D$$

is a bounded continuous function, where continuity is due to the fact that D is discrete. We will now extend φ to a bounded continuous function $\psi: D \cup \Delta \rightarrow \mathbb{R}$ by

$$\psi(\mathbf{t}) = \begin{cases} \varphi(\mathbf{t}) & \text{if } \mathbf{t} \in D, \\ \langle \mu_{\mathbf{t}}, f \rangle & \text{if } \mathbf{t} \in \Delta. \end{cases}$$

Clearly, ψ is bounded on $D \cup \Delta$, and continuous on D and on Δ . The latter is due to our assumption on weak continuity of $\mathbf{t} \mapsto \mu_{\mathbf{t}}$ on Δ . Now let $\mathbf{t} \in \Delta$ be arbitrary and let $(\mathbf{t}_n)_{n \in \mathbb{N}}$ be a sequence in D such that $\mathbf{t}_n \rightarrow \mathbf{t}$. Then $\mathbf{N}_n = \phi^{-1}(\mathbf{t}_n)$ fulfills the right-hand side of (2.1) and thus we get

$$\psi(\mathbf{t}_n) = \varphi(\mathbf{t}_n) = \langle \nu(\phi^{-1}(\mathbf{t}_n)), f \rangle = \langle \nu(\mathbf{N}_n), f \rangle \rightarrow \langle \mu_{\mathbf{t}}, f \rangle = \psi(\mathbf{t})$$

showing continuity of ψ .

By the independence assumption together with $\phi(\mathbf{T}_n) \Rightarrow \rho$ we finally obtain

$$\begin{aligned} \langle P_{Y_{\mathbf{T}_n}}, f \rangle &= \int_{\mathbb{N}^d} \langle \nu(\mathbf{n}), f \rangle dP_{\mathbf{T}_n}(\mathbf{n}) = \int_D \langle \nu(\phi^{-1}(\mathbf{t})), f \rangle dP_{\phi(\mathbf{T}_n)}(\mathbf{t}) \\ &= \int_{D \cup \Delta} \psi(\mathbf{t}) dP_{\phi(\mathbf{T}_n)}(\mathbf{t}) \rightarrow \int_{D \cup \Delta} \psi(\mathbf{t}) d\rho(\mathbf{t}) \\ &= \int_{\Delta} \langle \mu_{\mathbf{t}}, f \rangle d\rho(\mathbf{t}) = \left\langle \int_{\Delta} \mu_{\mathbf{t}} d\rho(\mathbf{t}), f \right\rangle. \end{aligned}$$

This proves our assertion, since the bounded continuous function f is arbitrary. Theorem 1 is proved.

We will now apply Theorem 1 to the classical convergence scheme of row-sums of triangular arrays of random variables, extending versions of the transfer theorem on second countable locally compact groups by Hazod [18], [19] and on separable Banach spaces by Siegel [32]. We have to assume second countability to ensure that products of Borel measurable random variables are again Borel measurable; cf. [12, Proposition 4.1.7]. For a metric space E second countability is equivalent to separability; cf. [12, Proposition 2.1.4]. Our multiparameter setting allows us to extend considerations to random fields as follows. For notational convenience we will write the group multiplication as summation. Note that in case of non-Abelian groups this operation has to be taken in a certain fixed order. In the sequel, relations and operations on multiparameters in $\mathbb{R}^d$ are always meant componentwise.

Let E be a separable metrizable group. For $n \in \mathbb{N}$, let $\mathcal{F}_n = \{X_{n,\mathbf{k}}: \mathbf{k} \in \mathbb{N}^d\}$ be a random field, i.e., a family of E -valued random variables, and define for any $n \in \mathbb{N}$ and $\mathbf{N} \in \mathbb{N}^d$

$$Y_{n,\mathbf{N}} = \sum_{\mathbf{k} \leq \mathbf{N}} X_{n,\mathbf{k}}.$$

Suppose the existence of a (componentwise) strictly increasing sequence $(\mathbf{k}_n)_{n \in \mathbb{N}}$ in $\mathbb{N}^d$ such that for some $\emptyset \neq T \subseteq [0, \infty)^d$ and some family $\{\mu_{\mathbf{t}}: \mathbf{t} \in T\}$ of probability distributions on E we have as $n \rightarrow \infty$

$$Y_{n, \lfloor \mathbf{k}_n \mathbf{t} \rfloor} \Rightarrow \mu_{\mathbf{t}}, \quad (2.3)$$

uniformly on compact subsets of $\mathbf{t} \in T$, i.e., for an arbitrary sequence $(\mathbf{t}_n)_{n \in \mathbb{N}}$ in T with $\mathbf{t}_n \rightarrow \mathbf{t} \in T$ we have $\langle P_{Y_{n, [\mathbf{k}_n \mathbf{t}_n]}}, f \rangle \rightarrow \langle \mu_{\mathbf{t}}, f \rangle$ for any bounded continuous function $f: E \rightarrow \mathbb{R}$.

Corollary 1. *With the above assumptions and notation let $(\mathbf{T}_n)_{n \in \mathbb{N}}$ be a sequence of random variables with values in $\mathbb{N}^d$ such that $\mathbf{T}_n$ and $\mathcal{F}_n$ are independent for every $n \in \mathbb{N}$. Suppose $\mathbf{T}_n/\mathbf{k}_n \Rightarrow \rho$ for some probability distribution ρ with $\rho(T) = 1$. Then we have as $n \rightarrow \infty$*

$$Y_{n, \mathbf{T}_n} \Rightarrow \int_T \mu_{\mathbf{t}} d\rho(\mathbf{t}). \quad (2.4)$$

P r o o f. Let $\mathcal{F} = \{Y_{n, \mathbf{N}}: n \in \mathbb{N}, \mathbf{N} \in \mathbb{N}^d\}$ and note that without loss of generality we may assume that $\mathbf{T}_n$ and $\mathcal{F}$ are independent for every $n \in \mathbb{N}$. Otherwise we may change to random variables $\mathbf{T}'_n$ independent of $\mathcal{F}$ and with the same distribution as $\mathbf{T}_n$ which leave the assertion in (2.4) unchanged. One may easily check that $\phi: \mathbb{N}^{1+d} \rightarrow D$ defined by $\phi(n, \mathbf{N}) = (1/n, \mathbf{N}/\mathbf{k}_n)$ with $D = \text{Im } \phi$ is a homeomorphism. For $\Delta = \{0\} \times T \subseteq \overline{D} \setminus D$ and any sequence $(n_k, \mathbf{N}_k) \in \mathbb{N}^{1+d}$ with $\phi(n_k, \mathbf{N}_k) \rightarrow (0, \mathbf{t}) \in \Delta$ we have $n_k \rightarrow \infty$ and $\mathbf{m}_k = \mathbf{N}_k/\mathbf{k}_{n_k} \rightarrow \mathbf{t}$ so that by (2.3) we have

$$Y_{n_k, \mathbf{N}_k} = Y_{n_k, [\mathbf{k}_{n_k} \mathbf{m}_k]} \Rightarrow \mu_{\mathbf{t}}.$$

Note that the family $\{\mu_{0, \mathbf{t}} = \mu_{\mathbf{t}}: (0, \mathbf{t}) \in \Delta\}$ is weakly continuous due to the uniform compact convergence in (2.3). Hence (2.1) is fulfilled and we further have $\phi(n, \mathbf{T}_n) = (1/n, \mathbf{T}_n/\mathbf{k}_n) \Rightarrow \delta_0 \otimes \rho$. An application of Theorem 1 now gives

$$Y_{n, \mathbf{T}_n} \Rightarrow \int_{\Delta} \mu_{0, \mathbf{t}} d(\delta_0 \otimes \rho)(s, \mathbf{t}) = \int_T \mu_{\mathbf{t}} d\rho(\mathbf{t})$$

concluding the proof of Corollary 1.

In case $d = 1$ and $E = \mathbb{R}^m$ the classical transfer theorem of Gnedenko [14], [15] is a special case of Corollary 1 as follows. Suppose $(X_{n, k})_{k \in \mathbb{N}}$ is a sequence of i.i.d. random variables for every $n \in \mathbb{N}$ such that for some strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$ we have

$$Y_{n, k_n} = \sum_{k=1}^{k_n} X_{n, k} \Rightarrow \mu,$$

where μ is a necessarily infinitely divisible probability distribution on $\mathbb{R}^m$. Then (2.3) is fulfilled, where $(\mu_t)_{t \geq 0}$ is the continuous convolution semigroup generated by μ . Thus in this situation Corollary 1 restates the transfer theorem of Gnedenko as presented in Introduction. Note that the proof in [14], [15] is given in terms of convergence of Fourier transforms. Further note that instead of the above partial sums it is also possible to consider partial (componentwise) maxima of random vectors [8], [1], [2] or partial products of random variables [28].

R e m a r k 1. Our random fields approach in Corollary 1 can be further applied to the convergence of finite dimensional marginal distributions of certain randomly stopped partial sum processes on E (e.g., [6, Theorem 4.1] for $E = \mathbb{R}^m$). It is also possible to apply Theorem 1 to obtain a transfer type result for randomly stopped stochastic processes on the path space $E = D([0, \infty), S)$ of càdlàg functions on a complete separable metric space S , since $D([0, \infty), S)$ can itself be seen as a complete separable metric space; see [9], [23] for details.

3. Examples. To illustrate some possible applications of our Theorem 1 we aim to give further examples in which different limits under different regimes arise with a mixture distribution of the transfer type limit.

3.1. A transfer theorem for negatively associated random fields. Our first example is a generalization of Gnedenko's transfer theorem for partial sums in a multiparameter and non-i.i.d. setting. Let $\{X_{\mathbf{n}}: \mathbf{n} \in \mathbb{N}^d\}$ be a field of stationary, centered and

negatively associated random variables with positive and finite second moment. Stationarity means that $\{X_{\mathbf{n}+\mathbf{m}}: \mathbf{n} \in \mathbb{N}^d\}$ is distributed as $\{X_{\mathbf{n}}: \mathbf{n} \in \mathbb{N}^d\}$ for every $\mathbf{m} \in \mathbb{N}^d$. Negative association was introduced in [22] and claims that

$$\text{Cov} \left(f(X_{\mathbf{n}}: \mathbf{n} \in I), g(X_{\mathbf{m}}: \mathbf{m} \in J) \right) \leq 0$$

for every pair I, J of disjoint subsets of $\mathbb{N}^d$ and any coordinatewise increasing functions f, g with $\mathbb{E}[f^2(X_{\mathbf{n}}: \mathbf{n} \in I)] < \infty$ and $\mathbb{E}[g^2(X_{\mathbf{m}}: \mathbf{m} \in J)] < \infty$. Let us define

$$Y_{\mathbf{n}, \mathbf{N}} = \frac{1}{\sqrt{|\mathbf{n}|}} \sum_{\mathbf{k} \leq \mathbf{N}} X_{\mathbf{k}}$$

for $\mathbf{n}, \mathbf{N} \in \mathbb{N}^d$, where $|\mathbf{n}| = \prod_{i=1}^d n_i$. Introduce a homeomorphism $\phi: \mathbb{N}^{2d} \rightarrow D$ by

$$\phi(\mathbf{n}, \mathbf{N}) = \left(\frac{1}{\mathbf{n}}, \frac{\mathbf{N}}{\mathbf{n}} \right)$$

with $D = \text{Im } \phi \subseteq (0, \infty)^{2d}$. Let $\Delta = \{\mathbf{0}\} \times [0, \infty)^d \subseteq \overline{D} \setminus D$, then for any sequence $(\mathbf{n}_k, \mathbf{N}_k) \in \mathbb{N}^{2d}$ with $\phi(\mathbf{n}_k, \mathbf{N}_k) \rightarrow (\mathbf{0}, \mathbf{t}) \in \Delta$ we necessarily have $\mathbf{n}_k \rightarrow \infty$ (component-wise) and $\mathbf{m}_k = \mathbf{N}_k / \mathbf{n}_k \rightarrow \mathbf{t}$. It follows from Theorem 1 in [35] that

$$Y_{\mathbf{n}_k, \mathbf{N}_k} = \frac{1}{\sqrt{|\mathbf{n}_k|}} \sum_{\mathbf{m} \leq \lfloor \mathbf{n}_k \mathbf{m}_k \rfloor} X_{\mathbf{m}} \Rightarrow \mu_{\mathbf{t}},$$

where $\mu_{\mathbf{t}} = \mathcal{N}_{0, \sigma^2|\mathbf{t}|}$ has a Gaussian distribution for some $\sigma^2 > 0$ depending on the covariance structure of the random field $\{X_{\mathbf{n}}: \mathbf{n} \in \mathbb{N}^d\}$. Clearly, the family of probability distributions $\{\mu_{\mathbf{0}, \mathbf{t}} = \mu_{\mathbf{t}}: (\mathbf{0}, \mathbf{t}) \in \Delta\}$ is weakly continuous. An application of our Theorem 1 directly leads to the following result, the same way we have proven Corollary 2.

Corollary 2. *Let $(\mathbf{T}_k)_{k \in \mathbb{N}}$ be a sequence of $\mathbb{N}^d$ -valued random variables such that $\mathbf{T}_k / \mathbf{n}_k \Rightarrow \rho$ for some sequence $\mathbf{n}_k \rightarrow \infty$ in $\mathbb{N}^d$ and some probability distribution ρ on $[0, \infty)^d$. Then we have*

$$\frac{1}{\sqrt{|\mathbf{n}_k|}} \sum_{\mathbf{m} \leq \lfloor \mathbf{n}_k \mathbf{T}_k \rfloor} X_{\mathbf{m}} \Rightarrow \int_{[0, \infty)^d} \mathcal{N}_{0, \sigma^2|\mathbf{t}|} d\rho(\mathbf{t}).$$

3.2. A transfer theorem for semistable domains of attraction. In case $d = 1$ and $E = \mathbb{R}^m$ we consider the special case of $X_{n,k} = A_n(X_k - a_n)$ for a sequence $(X_k)_{k \in \mathbb{N}}$ of i.i.d. random vectors on $\mathbb{R}^m$, invertible linear operators $A_n \in \text{GL}(\mathbb{R}^m)$ and shifts $a_n \in \mathbb{R}^m$. Suppose there exists a strictly increasing sequence $k_n \uparrow \infty$ in $\mathbb{N}$ such that $k_{n+1}/k_n \rightarrow c \geq 1$

$$A_n \sum_{k=1}^{k_n} (X_k - a_n) \Rightarrow \mu \quad (3.1)$$

for some full probability distribution μ on $\mathbb{R}^m$, where *full* means *not supported on any proper hyperplane* of $\mathbb{R}^m$. Then μ is infinitely divisible (in particular it is operator semistable) and it is known that the normalizing operators A_n can be chosen such that there exist $E \in \text{GL}(\mathbb{R}^m)$ and $B_n \in \text{GL}(\mathbb{R}^m)$ with $B_{k_n} = A_n$ fulfilling

$$B_{\lfloor \lambda n \rfloor} B_n^{-1} \rightarrow \lambda^{-E} = \sum_{k=0}^{\infty} \frac{(-\ln \lambda)^k}{k!} E^k$$

uniformly on compact subsets of $\{\lambda > 0\}$. Hence the sequence $(B_n)_{n \in \mathbb{N}}$ is regularly varying with exponent $-E$, see [29] for the details. The case $c = 1$ corresponds to operator stability $\mu^t = t^E \mu * \delta_{a(t)}$ for all $t > 0$ and some continuous function $t \mapsto a(t)$. In case $c > 1$ operator semistability is given by

$$\mu^c = c^E \mu * \delta_a \quad \text{for some } a \in \mathbb{R}^m; \quad (3.2)$$

see [29] for details. Moreover, the shifts $(a_n)_{n \in \mathbb{N}}$ can be chosen to be embeddable into a sequence $(b_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^m$ with $b_{k_n} = a_n$ such that $B_n \sum_{k=1}^n (X_k - b_n)$ is weakly relatively compact with weak limit points

$$\left\{ \mu_t = t^{-E}(\mu^t * \delta_{-a(t)}): t \in [1, c] \right\},$$

where $t \mapsto a(t)$ is continuous with $a(1) = 0$ and $a(t) \rightarrow a$ as $t \uparrow c$. Especially, $\mu_1 = \mu$ and $\mu_t \rightarrow \mu =: \mu_c$ weakly as $t \uparrow c$ by (3.2). A limit μ_t arises for subsequences of the form $n = k_{p_n} t_n$ with $t_n \rightarrow t \in [1, c]$, where $p_n \in \mathbb{N}$ is given by $k_{p_n} \leq n < k_{p_n+1}$; see [3, Section 4] for details. An univariate version ($m = 1$) of the above results is given in [10], where centering shifts are constructed via the quantile function.

Although in general $B_n \sum_{k=1}^n (X_k - b_n)$ does not converge in distribution, it is possible to achieve a limit distribution for random sums by a transfer type theorem as shown in [5]. For $n \in \mathbb{N}$ and the increasing sampling sequence $(k_n)_{n \in \mathbb{N}}$ define $k_0 = 1$ and

$$\psi(n) = t_n \quad \text{if } n = k_{p_n} t_n \text{ with } p_n \in \mathbb{N}_0 \text{ given by } k_{p_n} \leq n < k_{p_n+1}. \quad (3.3)$$

Corollary 3. *Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of positive integer-valued random variables independent of $(X_k)_{k \in \mathbb{N}}$ and assume that $\psi(T_n) \Rightarrow \rho$ for some probability distribution ρ on $[1, c]$. Then we have*

$$B_{T_n} \sum_{k=1}^{T_n} (X_k - b_{T_n}) \Rightarrow \int_1^c \mu_t d\rho(t) = \int_1^c t^{-E}(\mu^t * \delta_{-a(t)}) d\rho(t).$$

P r o o f. Let $\mathcal{F} = \{Y_n = B_n \sum_{k=1}^n (X_k - b_n): n \in \mathbb{N}\}$, and let $\phi: \mathbb{N} \rightarrow D$ be given by $\phi(n) = (1/n, \psi(n))$ using (3.3). Then for $\Delta = \{0\} \times [1, c] = \overline{D} \setminus D$ and any sequence $n_k \in \mathbb{N}$ with $\phi(n_k) \rightarrow (0, t) \in \Delta$ we have $n_k \rightarrow \infty$ and $\psi(n_k) \rightarrow t$ so that we get

$$Y_{n_k} = B_{n_k} \sum_{\ell=1}^{n_k} (X_\ell - b_{n_k}) \Rightarrow \mu_t = t^{-E}(\mu^t * \delta_{-a(t)})$$

as described above. Hence (2.1) is fulfilled for the weakly continuous family of distributions $\{\mu_{0,t} = \mu_t = t^{-E}(\mu^t * \delta_{-a(t)}): (0, t) \in \Delta\}$, where $\mu_1 = \mu = \mu_c$. Further the assumptions of Corollary 3 imply $\phi(T_n) \Rightarrow \delta_0 \otimes \rho$ and hence by Theorem 1 the assertion follows. Corollary 3 is proved.

As an example, it is implicitly shown in [5] that for any $c > 1$ the logarithmic summation with

$$\mathbb{P}\{T_n = k\} = \begin{cases} D_n^{-1} k^{-1} & \text{if } 1 \leq k \leq n, \\ 0 & \text{else,} \end{cases} \quad \text{with } D_n = \sum_{k=1}^n \frac{1}{k} \sim \ln n$$

fulfills $\psi(T_n) \Rightarrow \rho$ for the logarithmic distribution ρ on $[1, c]$ with probability density $t \mapsto (t \ln c)^{-1} 1_{[1,c]}(t)$. Hence by Corollary 3 we get for the distributions ν_k of $B_k \sum_{\ell=1}^k (X_\ell - b_k)$

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} \nu_k \rightarrow \frac{1}{\ln c} \int_1^c t^{-E}(\mu^t * \delta_{-a(t)}) \frac{dt}{t} \quad \text{weakly as } n \rightarrow \infty. \quad (3.4)$$

Further examples of random variables T_n (corresponding to summability methods) with $\psi(T_n)$ converging to the logarithmic distribution are given in [5].

3.3. Transfer theorems for random allocations. Let n balls be allocated to N boxes independently and with equal probability. Denote by $\mu_r(n, N)$ the final number of boxes containing exactly $r \geq 0$ balls, and let

$$\mu_r^*(n, N) = \frac{\mu_r(n, N) - \mathbb{E}[\mu_r(n, N)]}{\sqrt{\mathbb{D}^2[\mu_r(n, N)]}}, \quad 0 \leq r \leq n,$$

denote the standardized random variables. The limit behavior of $\mu_r^*(n, N)$ as simultaneously $n, N \rightarrow \infty$ is well known. The possible limit distributions are either the standard

normal distribution $\mathcal{N}_{0,1}$ or a standardized Poisson distribution π_λ^* with parameter $\lambda > 0$. For details we refer to the monograph [24] and the literature cited therein. For random numbers of balls and boxes we can derive the following transfer theorems as applications of Theorem 1. Let $\{(T_n, U_N): n, N \in \mathbb{N}\}$ be random variables in $\mathbb{N}^2$ such that (T_n, U_N) is independent of the allocations and thus of $\mathcal{F}_r = \{\mu_r^*(n, N): n, N \in \mathbb{N}, n \geq r\}$ for every $(n, N) \in \mathbb{N}^2$ and every $r \geq 0$. The following transfer theorem for the number of empty boxes $\mu_0(n, N)$ is already known by [4]. Its method of proof inspired our general version, but the proof of Theorem 1 is even simpler than the proof given for the special result in [4]. In turn Theorem 1 can be applied to derive transfer theorems for $\mu_r(n, N)$ with $r \geq 1$. Partial results can already be found in [21].

Corollary 4. *Let $\Delta_0 = (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)$, and for any $(g, d) \in \Delta_0$ let*

$$\nu_{g,d} = \begin{cases} \pi_{1/g}^* & \text{if } 0 < g < \infty, d = 0, \\ \mathcal{N}_{0,1} & \text{if } g = d = 0, \\ \pi_{1/d}^* & \text{if } g = 0, 0 < d < \infty, \end{cases}$$

which defines a weakly continuous family $\{\nu_{g,d}: (g, d) \in \Delta_0\}$ of probability distributions. Assume that for some probability distribution ρ_0 on Δ_0

$$\phi_0(T_n, U_N) = \left(\frac{2U_N}{T_n^2}, \frac{e^{T_n/U_N}}{U_N} \right) \Rightarrow \rho_0.$$

Then we have

$$\mu_0^*(T_n, U_N) \Rightarrow \int_{\Delta_0} \nu_{g,d} d\rho_0(g, d).$$

Corollary 5. *Let $\Delta_1 = \{0\} \times \mathbb{R}_+$, and for any $(0, d) \in \Delta_1$ let*

$$\nu_d = \begin{cases} \mathcal{N}_{0,1} & \text{if } d = 0, \\ \pi_{1/d}^* & \text{if } 0 < d < \infty, \end{cases}$$

which defines a weakly continuous family $\{\nu_d: (0, d) \in \Delta_1\}$ of probability distributions. Assume that for some probability distribution ρ_1 on $\mathbb{R}_+$

$$\phi_1(T_n, U_N) = \left(\frac{U_N}{T_n^2}, \frac{e^{T_n/U_N}}{T_n} \right) \Rightarrow \delta_0 \otimes \rho_1.$$

Then we have

$$\mu_1^*(T_n, U_N) \Rightarrow \int_0^\infty \nu_d d\rho_1(d).$$

Corollary 6. *For $r \geq 2$ let $\Delta_r = \{0\} \times \mathbb{R}_+$. For any $(0, d) \in \Delta_r$ again let*

$$\nu_d = \begin{cases} \mathcal{N}_{0,1} & \text{if } d = 0, \\ \pi_{1/d}^* & \text{if } 0 < d < \infty. \end{cases}$$

Assume that for some probability distribution ρ_r on $\mathbb{R}_+$

$$\phi_r(T_n, U_N) = \left(\frac{1}{T_n}, \frac{r! U_N^{r-1}}{T_n^r} e^{T_n/U_N} \right) \Rightarrow \delta_0 \otimes \rho_r.$$

Then we have

$$\mu_r^*(T_n, U_N) \Rightarrow \int_0^\infty \nu_d d\rho_r(d).$$

P r o o f o f C o r o l l a r i e s 4–6. For any sequences $(n_k)_{k \in \mathbb{N}}$ and $(N_k)_{k \in \mathbb{N}}$ of positive integers such that $\phi_r(n_k, N_k) \rightarrow (g, d) \in \Delta_r$ for some $r \geq 0$, it can easily be shown that $n_k \rightarrow \infty$, $N_k \rightarrow \infty$ and that $\mu_r^*(n_k, N_k)$ converges in distribution to the corresponding normal or Poissonian distribution given in Corollaries 4–6. For $r = 0$ this is already shown in the proof of Theorem 2.1 in [4] and for $r \geq 1$ it follows in the same spirit using the limit

theorems given in [24]. Hence for an application of Theorem 1 we simply have to argue that $\phi_r: \mathbb{N}^2 \rightarrow D = \text{Im } \phi_r$ is a homeomorphism for any $r \geq 0$. Again, for $r = 0$ these arguments have been given in the proof of Theorem 2.1 in [4] and for $r \geq 1$ they can also be derived easily.

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