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SOME RESULTS OF STABLE CONVERGENCE FOR
EXCHANGEABLE RANDOM VARIABLES
IN HILBERT SPACES

Доказаны результаты об устойчивой сходимости в схеме серий конечно перестановочных случайных величин со значениями в гильбертовом пространстве. Эти результаты, полученные без каких-либо предположений об интегрируемости, дополняют результаты типа классической центральной предельной теоремы. Приведена оценка типа Берри–Эссеена скорости сходимости.

Ключевые слова и фразы: перестановочные случайные величины, устойчивая сходимость, центральная предельная теорема в гильбертовых пространствах, скорость сходимости, оценки типа Берри–Эссеена.

1. Introduction and preliminaries. The literature concerning central limit theorems in the case of independent random variables, with values in a separable Hilbert space, is rather large; here we refer only to [3], [10], [20], [21], [24]. On the other hand, the theorem of de Finetti provides a tool to obtain central limit theorems for triangular arrays of infinite exchangeable (not necessarily independent) sequences of random variables. However, we note that not every finite sequence of exchangeable random variables can be embedded in an infinite sequence. It follows that many asymptotic techniques, employed in central limit results for infinite exchangeable sequences, have no more meaning in the case of a finite exchangeable sequence. In this paper we study the asymptotic behaviour of triangular arrays of finitely exchangeable random variables, with values in a separable Hilbert space H .

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For this purpose, let $(X_i)_{i \in I}$ be a family of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and $\mathcal{G}$ be a sub σ -algebra of $\mathcal{A}$. The family $(X_i)_{i \in I}$ is said to be *$\mathcal{G}$ -exchangeable* if, for any permutation τ on I and for any event K of $\mathcal{G}$, with $\mathbf{P}(K) \neq 0$, the law of $[X_{\tau(i)}]_{i \in I}$ with respect to the probability measure $\mathbf{P}(\cdot | K)$ is equal to $[X_i]_{i \in I}(\mathbf{P})$. Obviously, when $\mathcal{G}$ is the degenerate σ -algebra $\{\emptyset, \Omega\}$, we have the usual notion of exchangeability. In particular, if I is a finite set, the random variables X_i are said to be *$\mathcal{G}$ -finitely exchangeable*.

Denoting by $\langle \cdot, \cdot \rangle$ the scalar product in H , we recall that a random variable X , with values in H , is called *Gaussian* (or *normal*) if, for any element $h \in H$, the real random variable $\langle h, X \rangle$ has a normal distribution. If the expectations $\mathbf{E}[\langle h, X \rangle]$ are zero (or, equivalently, X and $-X$ have the same distribution), we say that X is *centered*. In the following we will suppose that the Gaussian random variables are centered. The operator V on H , defined by the relation

$$\langle Vh, h' \rangle = \text{Cov}[\langle h, X \rangle, \langle h', X \rangle] \quad (1)$$

is called *covariance (operator)* of a Gaussian random variable X , where $h, h' \in H$ and Vh , for simplicity, indicates the value of V on h . It is not difficult to show that the covariance is a positive, symmetric and nuclear operator; namely, there exists a complete orthonormal system $(e_n)_{n \geq 1}$ in H such that V has the representation

$$V = \sum_{n \geq 1} \lambda_n (e_n \otimes e_n), \quad (2)$$

where λ_n denotes $\mathbf{D}[\langle e_n, X \rangle]$ and $e_n \otimes e_n$ is the operator $h \mapsto \langle h, e_n \rangle e_n$. From (2) a Gaussian random variable X , with covariance V , has the same distribution as that of the random series $\sum_{n \geq 1} Y_n \sqrt{\lambda_n} e_n$, where $(Y_n)_n$ is a sequence of independent $\mathcal{N}(0, 1)$ random variables. Therefore the distribution of a Gaussian random variable depends only on its covariance and is denoted by $\mathcal{N}(0, V)$.

In order to characterize in terms of covariance the notion of a measurable family of normal distributions on H , it is necessary to give a σ -algebra on the space $\mathcal{L}$ of symmetric, positive nuclear operators. $\mathcal{L}$ is provided with the smallest σ -algebra which makes the functionals $x \mapsto \langle xh, h' \rangle$ measurable. So a family $(\mathcal{N}(0, V_\omega))_{\omega \in \Omega}$ is a *Gaussian kernel* (or a *measurable family* of symmetric normal distributions on H) if and only if the mapping $\omega \mapsto V_\omega$ of Ω into $\mathcal{L}$ is a random variable, that is $\omega \mapsto \langle V_\omega h, h \rangle$ is a (positive) random variable for any $h \in H$. Given a random variable V with values in $\mathcal{L}$, we indicate by $\mathcal{N}(0, V)$ the Gaussian kernel $(\mathcal{N}(0, V(\omega)))_{\omega \in \Omega}$. Now let $\mathcal{V}$ be a sub σ -algebra of $\mathcal{A}$ for which V is measurable. A sequence $(Y_n)_n$ of H -valued random variables converges *$\mathcal{V}$ -stably* to the Gaussian kernel $\mathcal{N}(0, V)$ if, for any event $K \in \mathcal{V}$, with $\mathbf{P}(K) \neq 0$, the random variables Y_n converge in distribution with respect to $\mathbf{P}_K = \mathbf{P}(\cdot | K)$ to the mixture of normal distributions

$$\int \mathcal{N}(0, V(\omega)) \mathbf{P}_K(d\omega). \quad (3)$$

Thanks to the characterization of convergence in distribution in Banach spaces [17], a *tight* sequence $(Y_n)_n$ converges $\mathcal{V}$ -stably to the Gaussian kernel $\mathcal{N}(0, V)$ if and only if, for any element $h \in H$ and for any event $K \in \mathcal{V}$, we have

$$\lim_n \int_K \exp(\langle h, Y_n \rangle) d\mathbf{P} = \int_K \exp\left(-\frac{\langle h, Vh \rangle}{2}\right) d\mathbf{P}.$$

For further properties of the $\mathcal{V}$ -stable convergence, see [15] or [12]. Finally, a finite sequence of H -valued random variables $(X_j)_{1 \leq j \leq k}$ is said to be a *suite centrée* (or *martingale differences*) with respect to a filtration $(\mathcal{F}_j)_{0 \leq j \leq k}$ if, for any integer $j = 1, \dots, k$, the random variable X_j is $\mathcal{F}_j$ -measurable and has zero conditional mean with respect to $\mathcal{F}_{j-1}$, that is $\mathbf{E}[\langle h, X_j \rangle | \mathcal{F}_{j-1}] = 0$, $\forall h \in H$. We recall a very useful result on the $\mathcal{V}$ -stable convergence of suites centrées (see [16, Theorem 3.1]).

Theorem 1. Let $(X_{n,j})_{n \geq 1, 1 \leq j \leq k_n}$ be a triangular array of real random variables, such that, for any n , the finite sequence $(X_{n,j})_{1 \leq j \leq k_n}$ is a *suite centrée* with respect to a filtration $(\mathcal{F}_{n,j})_{0 \leq j \leq k_n}$ (which we consider to be defined on integers by $\mathcal{F}_{n,j} = \mathcal{F}_{n,k_n}$ for $j \geq k_n$). Put

$$S_n = \sum_j X_{n,j}, \quad X_n^* = \sup_j |X_{n,j}|, \quad U_n = \sum_j X_{n,j}^2$$

and denote the algebra $\liminf_n \mathcal{F}_{n,j}$ by $\mathcal{H}_j$. Let $\mathcal{U}$ be the completion of the σ -algebra generated by the algebra $\cup_j \mathcal{H}_j$. Suppose (X_n^*) converges to 0 in L^1 and (U_n) converges in probability to a real positive random variable U , measurable with respect to $\mathcal{U}$. Then (S_n) converges $\mathcal{U}$ -stably to the Gaussian kernel $\mathcal{N}(0, U)$.

2. The main result. With the notation of the previous section, for any nuclear operator $L \in \mathcal{L}$, the trace of L can be defined as $\text{Tr } L = \sum_{n \geq 1} \langle L e_n, e_n \rangle$. Note that $\text{Tr } L$ doesn't depend on the complete orthonormal system $(e_n)_n$ chosen. Now we can prove the following result.

Theorem 2. Let $(X_{n,j})_{1 \leq j \leq k_n, n \geq 1}$ be a triangular array of H -valued random variables such that, for any fixed n , the finite sequence $(X_{n,j})_{1 \leq j \leq k_n}$ is $\mathcal{G}_n$ -exchangeable. Denote by $\mathcal{S}_n$ the symmetric σ -algebra generated by the random variables $X_{n,j}$, with $j = 1, \dots, k_n$. Let $\mathcal{V}$ be the σ -algebra generated by the algebra $\liminf_n \mathcal{S}_n \vee \mathcal{G}_n$ and $(a_{n,j})_{1 \leq j \leq k_n}$ be a finite sequence of continuous operators on H . Put

$$a_n^* = \sup_{1 \leq j \leq k_n} \|a_{n,j}\|, \quad M_{n,j} = \frac{1}{j} \sum_{h=1}^j X_{n,h}, \quad 1 \leq j \leq k_n,$$

where $\|a_{n,j}\|$ is the norm of $a_{n,j}$ in the space of continuous operators on H . Suppose the following conditions are satisfied:

- (i) $a_n^* \sup_{1 \leq j \leq k_n} \|X_{n,j}\|$ converges in probability to 0;
- (ii) There exists a random variable $\mathbf{V}$, with values in $\mathcal{L}$ and measurable with respect to $\mathcal{V}$, such that, for any $h \in H$, the sequence $(\sum_{j=1}^{k_n} \langle h, a_{n,j}(X_{n,j} - M_{n,j}) \rangle^2)$ converges in probability to the random variable $\omega \mapsto \langle h, \mathbf{V}(\omega)h \rangle$;
- (iii) $\sum_{j=1}^{k_n} \|a_{n,j}(X_{n,j} - M_{n,j})\|^2$ converges in probability to $\text{Tr } \mathbf{V}$.

Then $Y_n = \sum_{j=1}^{k_n} a_{n,j}(X_{n,j} - M_{n,j})$ converges $\mathcal{V}$ -stably to the Gaussian kernel $\mathcal{N}(0, \mathbf{V})$.

P r o o f. It is sufficient (and necessary) to prove the following two conditions:

(α) For every $h \in H$, $\langle h, Y_n \rangle$ converges $\mathcal{V}$ -stably to the real Gaussian kernel $\mathcal{N}(0, \langle h, \mathbf{V}h \rangle)$;

(β) (Y_n) is a tight sequence.

From now on, we denote by $(\mathcal{F}_{n,j})_{0 \leq j \leq k_n}$ the filtration of the $\mathcal{G}_n$ -symmetric σ -algebras with respect to the sequence $(X_{n,j})_{1 \leq j \leq k_n}$; i.e., $\mathcal{F}_{n,j}$ is the σ -algebra generated by the random variables $I_{A_n} f(X_{n,1}, \dots, X_{n,k_n})$, where A_n is an event of $\mathcal{G}_n$ and f is a Borel function on $\mathbf{R}^{k_n}$, symmetric with respect to the first $k_n - j$ arguments. To prove (α), fixed $h \in H$, put

$$X_n^* = \sup_{1 \leq j \leq k_n} \|X_{n,j}\|, \quad U_n = \sum_{j=1}^{k_n} \langle h, a_{n,j}(X_{n,j} - M_{n,j}) \rangle^2, \quad J_n = I_{\{a_n^* X_n^* \leq 1\}}.$$

It then follows that

$$\sup_j \left| \langle h, a_{n,j}(X_{n,j} - M_{n,j}) \rangle \right| \leq 2 \|h\| a_n^* X_n^*. \quad (4)$$

Under the hypothesis (i), (ii), we have the following statements:

- J_n is $\mathcal{F}_{n,0}$ -measurable and converges in probability to 1;
- $(J_n U_n)$ converges in probability to $\langle h, \mathbf{V}h \rangle$;
- The finite sequence

$$\left(J_n \langle h, a_{n,k_n-j+1}(X_{n,k_n-j+1} - M_{n,k_n-j+1}) \rangle \right)_{1 \leq j \leq k_n} \quad (5)$$

is a suite centrée with respect to the filtration $(\mathcal{F}_{n,j})_{0 \leq j \leq k_n}$;

— $\sup_j |J_n \langle h, a_{n,j}(X_{n,j} - M_{n,j}) \rangle|$ converges in probability to 0 and, because of (4), is dominated by $2 \|h\|$.

It follows, from Theorem 1, that $(J_n Y_n)$ converges $\mathcal{V}$ -stably to the Gaussian kernel $\mathcal{N}(0, \langle h, \mathbf{V}h \rangle)$ and therefore (α) holds. (Actually, there is stable convergence with respect to the σ -algebra generated by the algebras $\liminf_j \mathcal{F}_{n,j}$.) We refer now to condition (β).

In view of (α) , it suffices to prove that, for any $\varepsilon > 0$, there exists a finite dimensional space M such that

$$\limsup_n \mathbf{P} \left\{ \|Y_n - \pi_M Y_n\|^2 \geq \varepsilon \right\} < \varepsilon, \quad (6)$$

where π_M denotes the projection onto M . Let $(e_n)_n$ be a complete orthonormal system in H and denote by T_m the random variable $\sum_{i=1}^m \langle V e_i, e_i \rangle$. It is immediate to note that T_m converges increasing to $\text{Tr} V$. By considering $J_n X_{n,j}$, it is not restrictive to suppose $a_n^* X_n^*$ is dominated by 1. Given a positive constant c , let $\tau_n = \inf\{j: 1 \leq j \leq k_n, V_{n,j} > c\}$, where $V_{n,j} = \sum_{l=k_n-j+1}^{k_n} \|a_{n,l}(X_{n,l} - M_{n,l})\|^2$. Moreover, put

$$\tilde{X}_{n,j} = a_{n,k_n-j+1}(X_{n,k_n-j+1} - M_{n,k_n-j+1}) I_{\{j \leq \tau_n\}}, \quad \tilde{Y}_n = \sum_{j=1}^{k_n} \tilde{X}_{n,j}; \quad (7)$$

for any n and for any $h \in H$, the sequence $(\langle h, \tilde{X}_{n,k_n-j+1} \rangle)_{1 \leq j \leq k_n}$ is still a suite centre with respect to the filtration $(\mathcal{F}_{n,j})_j$. For every m , let M_m be the vector space generated by $e_1, \dots, e_m$. We have

$$\mathbf{E} [\|\tilde{Y}_n - \pi_{M_m} \tilde{Y}_n\|^2] = \sum_{j=1}^{k_n} \mathbf{E} [\|\tilde{X}_{n,j} - \pi_{M_m} \tilde{X}_{n,j}\|^2] = \mathbf{E} \left[\sum_{j=1}^{k_n} \|\tilde{X}_{n,j} - \pi_{M_m} \tilde{X}_{n,j}\|^2 \right]. \quad (8)$$

Thanks to (ii), (iii), the above equality implies that $\limsup_n \mathbf{E} [\|\tilde{Y}_n - \pi_{M_m} \tilde{Y}_n\|^2] \leq \mathbf{E}[(\text{Tr} V - T_m) \wedge c]$. Finally, for any $\varepsilon > 0$, we have

$$\begin{aligned} \mathbf{P} \left\{ \|Y_n - \pi_{M_m} Y_n\|^2 \geq \varepsilon \right\} &\leq \mathbf{P} \left\{ \|\tilde{Y}_n - \pi_{M_m} \tilde{Y}_n\|^2 \geq \varepsilon \right\} + \mathbf{P}\{V_{n,k_n} > c\} \\ &\leq \varepsilon^{-1} \mathbf{E} [\|\tilde{Y}_n - \pi_{M_m} \tilde{Y}_n\|^2] + \mathbf{P}\{V_{n,k_n} > c\}, \end{aligned}$$

and hence

$$\limsup_n \mathbf{P} \left\{ \|Y_n - \pi_{M_m} Y_n\|^2 \geq \varepsilon \right\} \leq \varepsilon^{-1} \mathbf{E}[(\text{Tr} V - T_m) \wedge c] + \mathbf{P}\{\text{Tr} V \geq c\}.$$

Because c and m are arbitrary, the estimate (6) follows and this completes the proof.

R e m a r k 1. Note that the random variables $\|X_{n,k}\|$ are not necessarily square integrable and there are no conditions on the ratio n/k_n , as it happens in most of the central limit results ([8], [23]). If H is a finite dimensional space, condition (iii) is obviously implied by (ii). In particular, the real case was studied in [18] and similar results can be obtained also in the infinite dimensional case. Moreover, if the random variables $\|X_{n,j}\|^2$ are integrable and hypothesis (ii) ensures the L^1 -convergence as well as the convergence in probability, condition (iii) can be replaced by

$$\lim_n \sum_{j=1}^{k_n} \mathbf{E} [\|a_{n,j}(X_{n,j} - M_{n,j})\|^2] = \mathbf{E}[\text{Tr} V].$$

3. Comparison. Let $(X_{n,j})_{n \geq 1, 1 \leq j \leq k_n}$ be a triangular array of real random variables and $(\mathcal{G}_n)_n$ be a sequence of sub σ -algebras of $\mathcal{A}$. For any fixed n , we assume from now on that $(X_{n,j})_{1 \leq j \leq k_n}$ can be embedded in an infinite sequence of $\mathcal{G}_n$ -exchangeable random variables.

In this section we prove that $\sum_{j=1}^{k_n} a_{n,j} M_{n,j}$ cannot be omitted (because it is not asymptotically negligible!) even if $X_{n,j}$, $j = 1, \dots, k_n$, are i.i.d. random variables, with zero mean. So Theorem 2 cannot be obtained from the classical results on triangular arrays of exchangeable sequences [9]. Let $\mathcal{T}_n$ be the tail σ -algebra relative to $(X_{n,j})_j$ and $\mathcal{R}_n$ be the σ -algebra generated by $\mathcal{G}_n$ and $\mathcal{T}_n$. Denote by $(a_{n,j})_{1 \leq j \leq k_n}$ a finite family of real numbers and put

$$\mathcal{R} = \bigcap_n \mathcal{R}_n, \quad Z_n = \sum_{j=1}^{k_n} a_{n,j}(X_{n,j} - \bar{X}_{n,j}),$$

where $\bar{X}_{n,j}$ is a version of $\mathbf{E}[X_{n,j} \mid \mathcal{R}_n]$. It follows from de Finetti's theorem that, for any n , $X_{n,1}, X_{n,2}, \dots$ are $\mathcal{R}_n$ -conditional independent and pairwise $\mathcal{R}_n$ -equivalent.

$\bar{X}_{n,j}$ is obviously equal to $\bar{X}_{n,1}$. Under a hypothesis of Lindeberg type, some results of convergence in distribution of Z_n to mixtures of normal distributions are well known [8]. We prove, under the same hypothesis, a result on $\mathcal{R}$ -stable convergence for $[Y_n, Z_n]$, where Y_n is defined as in the statement of Theorem 2.

Theorem 3. Put

$$b_{n,j} = \sum_{h=j}^{k_n} a_{n,h} h^{-1}, \quad \alpha_n = \sum_{j=1}^{k_n} a_{n,j}^2, \quad \beta_n = \sum_{j=1}^{k_n} b_{n,j}^2$$

and suppose the following conditions hold:

(i) $V_n = \mathbf{E}[(X_{n,1} - \bar{X}_{n,1})^2 | \mathcal{R}_n] \sum_{j=1}^{k_n} a_{n,j}^2$ converges in probability to a random variable V ;

(ii) For every $t > 0$,

$$\lim_n \sum_{j=1}^{k_n} a_{n,j}^2 \mathbf{E} \left[(X_{n,j} - \bar{X}_{n,j})^2 I_{\{|a_{n,j}(X_{n,j} - \bar{X}_{n,j})| > t\}} \right] = 0;$$

(iii) There exists $\lim_n \beta_n \alpha_n^{-1} = \rho \in \mathbf{R}$.

Then $[Y_n, Z_n]$ converges $\mathcal{R}$ -stably to the Gaussian kernel $\mathcal{N}(0, \Gamma)$, with

$$\Gamma = \begin{pmatrix} 1 & 1 - \frac{\rho}{2} \\ 1 - \frac{\rho}{2} & 1 \end{pmatrix} V.$$

In particular,

$$Z_n - Y_n = \sum_{j=1}^{k_n} a_{n,j} \left(\frac{X_{n,1} + \cdots + X_{n,j}}{j} - \bar{X}_{n,j} \right)$$

converges $\mathcal{R}$ -stably to the (real) Gaussian kernel $\mathcal{N}(0, \rho V)$.

P r o o f. Observe first that $Z_n - Y_n = \sum_{j=1}^{k_n} b_{n,j}(X_{n,j} - \bar{X}_{n,j})$. For any pair of real numbers t, u , consider the random variables

$$tY_n + uZ_n = (u+t)Z_n + t(Y_n - Z_n) = \sum_{j=1}^{k_n} (a_{n,j}(u+t) - tb_{n,j})(X_{n,j} - \bar{X}_{n,j}).$$

Let $\Gamma_{i,j}$ be the elements of the matrix Γ ; it suffices to prove that $tY_n + uZ_n$ converges $\mathcal{R}$ -stably to the Gaussian kernel $\mathcal{N}(0, \gamma_{t,u})$, where $\gamma_{t,u} = t^2 \Gamma_{1,1} + 2tu \Gamma_{1,2} + u^2 \Gamma_{2,2}$. It follows from de Finetti's theorem that $(X'_{n,j})_{1 \leq j \leq k_n}$ is a suite centre with respect to the filtration $(\mathcal{F}_{n,j})_j$, where $X'_{n,j} = (a_{n,j}(u+t) - tb_{n,j})(X_{n,j} - \bar{X}_{n,j})$ and $\mathcal{F}_{n,j}$ denotes $\mathcal{R}_n \vee \sigma(X_{n,1}, \dots, X_{n,j})$. By Theorem 1, applied to $(X'_{n,j})$, it suffices to prove that $\sup_j |X'_{n,j}|$ converges to 0 in L^1 and $\sum_{j=1}^{k_n} (X'_{n,j})^2$ converges in probability to $\gamma_{t,u} V$. Thanks to condition (ii) of Lindeberg and Theorem (2.23) in [11], it is enough to show that $V'_n = \sum_{j=1}^{k_n} \mathbf{E}[(X'_{n,j})^2 | \mathcal{F}_{n,j-1}]$ converges in probability to $\gamma_{t,u} V$. Except for considering a subsequence, it is not restrictive to suppose

$$\lim_n \frac{\sum_{j=1}^{k_n} a_{n,j} b_{n,j}}{\alpha_n} = \gamma. \quad (9)$$

Because

$$V'_n = \left[\sum_{j=1}^{k_n} a_{n,j}^2 (u+t)^2 + b_{n,j}^2 t^2 - 2a_{n,j} b_{n,j} t(u+t) \right] \left(\sum_{j=1}^{k_n} a_{n,j}^2 \right)^{-1} V_n,$$

it follows that V'_n converges in probability to $[(u+t)^2 + \rho t^2 - 2t(u+t)\gamma] V$, that is,

$$[t^2(1 + \rho - 2\gamma) + 2ut(1 - \gamma) + u^2] V. \quad (10)$$

In particular, the convergence of $Z_n - Y_n$ is obtained by taking $t = -1$ and $u = 1$. To complete the proof, we need to verify $\rho = 2\gamma$, namely Y_n converges $\mathcal{R}$ -stably

to $\mathcal{N}(0, V)$. Thanks to condition (ii) of Lindeberg and Theorem 2, it suffices to show that $\sum_{j=1}^{k_n} a_{n,j}^2 (X_{n,j} - M_{n,j})^2$ converges in probability to V . Therefore we need to prove that $U_n = \sum_{j=1}^{k_n} [(X_{n,1} - \bar{X}_{n,1} + \dots + X_{n,j} - \bar{X}_{n,j})/j]^2$ converges in probability to 0. It is not restrictive by a stopping method to suppose $\lim_n \mathbf{E}[|V_n - V|] = 0$. Then

$$\lim_n \mathbf{E}[U_n] = \lim_n \sum_{j=1}^{k_n} \frac{1}{j} a_{n,j}^2 \mathbf{E}[(X_{n,1} - \bar{X}_{n,1})^2] = 0$$

because $\limsup_n \mathbf{E}[V_n] < \infty$ and $\lim_n a_{n,j}^2 \mathbf{E}[(X_{n,1}^2 - \bar{X}_{n,1})^2] = 0$ (see also [18, Proposition 4.2]). The proof is thus complete.

R e m a r k 2. To obtain the $\mathcal{V}$ -stable convergence of Y_n , it is possible to weaken the hypothesis of Theorem 3 by conditions of Raikov type, as pointed out in [18]. Moreover the convergence of $[Y_n, Z_n]$ is also $\mathcal{U}$ -stable, where $\mathcal{U}$ is the σ -algebra generated by $\cap_j \mathcal{H}_j$, with $\mathcal{H}_j = \liminf_n \mathcal{R}_n \vee \sigma(X_{n,1}, \dots, X_{n,j})$. In particular, if $X_{n,j}$ is deduced from a sequence $(X_n)_n$, namely $X_{n,j}$ coincides with $\alpha'_{n,j} X_j$, the convergence is $\mathcal{A}$ -stable.

Corollary 1. *With the notation of Theorem 3, suppose that $a_{n,j} = 1$, $\lim_n k_n = \infty$ and that $(k_n X_{n,1}^2)_n$ is a uniformly integrable sequence. Moreover, suppose $k_n \mathbf{E}[(X_{n,1} - \bar{X}_{n,1})^2 | \mathcal{R}_n]$ converges in probability to a random variable V . Then $[Y_n, Z_n]$ converges $\mathcal{U}$ -stably to the Gaussian kernel $\mathcal{N}(0, \mathbf{V})$, with*

$$\mathbf{V} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}. \quad (11)$$

P r o o f. Conditions (i) and (ii) of the previous theorem are clearly verified. Then it suffices to prove that

$$\lim_m \frac{\sum_{j=1}^m (\sum_{h=j}^m h^{-1})^2}{m} = 2.$$

Thanks to Cesaro's formula, it is enough to show that

$$\lim_m \frac{\sum_{j=1}^m \sum_{h=j}^m h^{-1}}{m} = 1.$$

The proof follows from the relation $\sum_{j=1}^m \sum_{h=j}^m h^{-1} = \sum_{h=1}^m \sum_{j=1}^h h^{-1} = 1$.

Corollary 2. *Let $(X_n)_n$ be an exchangeable sequence of square integrable random variables and $\mathcal{T}$ the tail σ -algebra relative to (X_n) . Put*

$$Y_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(X_j - \frac{X_1 + \dots + X_j}{j} \right), \quad Z_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n (X_j - \bar{X}_j),$$

where $\bar{X}_j$ is a version of $\mathbf{E}[X_j | \mathcal{T}]$. Then $[Y_n, Z_n]$ converges $\mathcal{T}$ -stably to the Gaussian kernel $\mathcal{N}(0, \mathbf{V})$, where $\mathbf{V}$ is defined by (11) and V is a version of $\mathbf{E}[(X_1 - \bar{X}_1)^2 | \mathcal{T}]$.

P r o o f. This is proved by applying Corollary 2 to $((1/\sqrt{n})X_j)_{1 \leq j \leq n}$. (The convergence is also $\mathcal{A}$ -stable.)

4. Rate of convergence. In the following, we will give, under more restrictive conditions, a Berry-Esseen type estimate on the rate of convergence of $[Y_n, Z_n]$. We recall an important result:

Theorem 4. *Let $\mathcal{C}$ be the class of convex Borel sets in $\mathbf{R}^2$. There exists a constant A such that, for any k -uple of independent random variables $X_1, \dots, X_k$, with values in $\mathbf{R}^2$, with $\mathbf{E}[|X_j|^3] < \infty$ and $\mathbf{E}[X_j] = 0$, $\text{Cov}[X_1 + \dots + X_k] = I$, the following estimate holds*

$$\sup_{C \in \mathcal{C}} \left| \mathbf{P}\{X_1 + \dots + X_k \in C\} - \mathcal{N}(0, I)(C) \right| \leq A\beta, \quad (12)$$

where $\beta = \sum_{j=1}^k \mathbf{E}[|X_j|^3]$, and I is the identity matrix.

(A proof of Theorem 4 can be found in [3] or in [10].) Under the hypothesis of Theorem 3, let now $\tilde{\Gamma}_n$ be a version of the conditional covariance matrix of $[Y_n, Z_n]$ with respect to $\mathcal{T}_n$. For any n and for any $\omega \in \Omega$, $\Phi_{n,\omega}$ denotes the distribution function of the Gaussian law with zero mean and covariance matrix $\tilde{\Gamma}_n(\omega)$. Put $H_n = \{\omega: \det(\tilde{\Gamma}_n(\omega)) \neq 0\}$ (I_{H_n} is $\mathcal{T}_n$ -measurable) and, for any $\omega \in H_n$, denote by $\tau_n(\omega)$ the trace of $\tilde{\Gamma}_n^{-1}(\omega)$. Moreover, for every n and for every $K_n \in \mathcal{T}_n$, put

$$W_n = I_{H_n} (X_{n,1} - \bar{X}_{n,1}) \sqrt{\tau_n}, \quad F_{n,K}(x_1, x_2) = \mathbf{P}(K \cap \{Y_n \leq x_1, Z_n \leq x_2\}),$$

$$\Phi_{n,K}(x_1, x_2) = \int_K \Phi_{n,\omega}(x_1, x_2) \mathbf{P}(d\omega).$$

In particular, for brevity, F_n , Φ_n will indicate $F_{n,\Omega}$, $\Phi_{n,\Omega}$. Observe that Φ_n is the distribution function of the mixture $\int \mathcal{N}(0, \Gamma_n(\omega)) \mathbf{P}(d\omega)$. We have the following result.

Proposition 1. *There exists a constant A such that, for any n and for any $K_n \in \mathcal{T}_n$, with $K_n \subset H_n$, the following estimate holds*

$$\sup_{(x_1, x_2)} |F_{n,K_n}(x_1, x_2) - \Phi_{n,K_n}(x_1, x_2)| \leq A' \gamma_n \mathbf{E}[|W_n|^3 I_{K_n}], \quad (13)$$

where γ_n is $\sum_{j=1}^{k_n} ((a_{n,j} - b_{n,j})^2 + a_{n,j}^2)^{3/2}$.

P r o o f. Since we are interested in an inequality which is concerned only with the distribution of the random variables $X_{n,1} - \bar{X}_{n,1}, \dots, X_{n,k_n} - \bar{X}_{n,k_n}$, it is not restrictive to suppose that the random variables $X_{n,j}$ are defined on the canonical space $\mathbf{R}^{N^*}$ and coincide with the first k_n canonical projections. From the theorem of de Finetti, the joint distribution of $[X_{n,1} - \bar{X}_{n,1}, \dots, X_{n,k_n} - \bar{X}_{n,k_n}]$ is a mixture of products $\pi_\omega \otimes \dots \otimes \pi_\omega$, where π_ω is a suitable conditional distribution with $\mathbf{E}^{\pi_\omega}[X_{n,j}] = 0$ almost surely. Fixed n , except for a negligible set, H_n is equal to

$$\{\omega: \det(\text{Cov}^{\pi_\omega}[Y_n, Z_n]) \neq 0\} \quad (14)$$

and therefore it is not restrictive to consider K_n to be contained in the set (14). So there exists, for any $\omega \in K_n$, an invertible matrix $B_{n,\omega}$ such that $B_{n,\omega}^t B_{n,\omega} = (\text{Cov}^{\pi_\omega}[Y_n, Z_n])^{-1}$. Observe that $\omega \mapsto I_{K_n}(\omega) B_{n,\omega}$ is a random variable and, for any $\omega \in K_n$, the covariance of $B_{n,\omega}(Y_n, Z_n)^t$ is the identity matrix I . Put on K

$$(Y'_n, Z'_n)^t = B_{n,\omega}(Y_n, Z_n)^t.$$

Here $(Y'_n, Z'_n)^t$ is the sum of k_n independent random vectors with respect to π_ω . It follows from Theorem 4, that

$$\begin{aligned} d_{n,\omega} &= \sup_{(x_1, x_2)} |\pi_\omega(Y_n \leq x_1, Z_n \leq x_2) - \Phi_{n,\omega}(x_1, x_2)| \\ &\leq \sup_{C \in \mathcal{C}} |\pi_\omega([Y'_n, Z'_n] \in C) - \mathcal{N}(0, I)(C)| \\ &\leq A \gamma_n \tau_n^{3/2}(\omega) \int |X_{n,1} - \bar{X}_{n,1}|^3 d\pi_\omega. \end{aligned} \quad (15)$$

Because $\omega \mapsto I_{K_n}(\omega) b_{n,\omega}^{3/2} \int |X_{n,1} - \bar{X}_{n,1}|^3 d\pi_\omega$ is a version of the conditional expectation $\mathbf{E}[I_{K_n}|W_n|^3 | \mathcal{T}_n]$, we obtain the required estimate by integrating (15) with respect to $\mathbf{P}$.

Now we study the behaviour of the distribution functions of $[Y_n, Z_n]$ conditionally on $K'_n \in \mathcal{T}_n$, with $K'_n \subset H'_n = H_n^c \cap \{\omega: \tilde{\Gamma}_n(\omega) \neq 0\}$. For any $\omega \in H'_n$, Z_n is π_ω -almost surely a (nonzero) multiple of Y_n . It then follows that

$$\begin{aligned} d'_{n,\omega} &= \sup_{(x_1, x_2)} |\pi_\omega(Y_n \leq x_1, Z_n \leq x_2) - \Phi_{n,\omega}(x_1, x_2)| \\ &\leq 2 \sup_{x_1} |\pi_\omega(\sigma_{n,\omega}^{-1} Y_n \leq x_1) - \Phi(x_1)|, \end{aligned} \quad (16)$$

where Φ denotes the distribution function of $\mathcal{N}(0, 1)$ and $\sigma_{n,\omega}^{-1}$ is $\mathbf{E}^{\pi_\omega}[Y_n^2]^{1/2}$. In a similar manner, we obtain an inequality which depends on Z_n . It follows from Berry-Esseen

estimate that $d'_{n,\omega} \leq 2\gamma_n [\tau'_n(\omega)]^{3/2} \int |X_{n,1} - \bar{X}_{n,1}|^3 d\pi_\omega$, where $\tau'_n(\omega)$ is $\mathbf{D}^{\pi_\omega}[Y_n]^{-1} \wedge \mathbf{D}^{\pi_\omega}[Z_n]^{-1}$. Integration of the above inequality yields

$$\sup_{(x_1, x_2)} |F_{n, K'_n}(x_1, x_2) - \Phi_{n, K'_n}(x_1, x_2)| \leq 2\gamma_n \mathbf{E}[|W'_n|^3 I_{K'_n}], \quad (17)$$

where W'_n denotes $I_{H'_n}(X_{n,1} - \bar{X}_{n,1})\sqrt{\tau'_n}$. Moreover, proposition 1 and inequality (17) imply the following result.

Proposition 2. *There exists a constant L such that, for any n , the following inequality holds*

$$\sup_{(x_1, x_2)} |F_n(x_1, x_2) - \Phi_n(x_1, x_2)| \leq L\gamma_n \mathbf{E}[|W_n|^3 + |W'_n|^3]. \quad (18)$$

Inequality (18) still holds when the supremum is taken over the class $\mathcal{C}$ of convex Borel sets in $\mathbf{R}^2$. Analogous inequalities, where the right-hand side depends only on $\mathbf{E}[|X_{n,j} - \bar{X}_{n,1}|^3]$ and γ_n , can also be obtained for $\mathbf{E}[f(Y_n, Z_n)]$, with $f \in \mathcal{C}_b^3$. We finally have a particular case of (18).

Corollary 3. *Let $a_{n,j} = k_n^{-1/2}$. There exists a constant L' such that, for any n , the following inequality holds*

$$\sup_{(x_1, x_2)} |F_n(x_1, x_2) - \Phi_n(x_1, x_2)| \leq L' k_n^{-1/2} \mathbf{E}[|W_n|^3 + |W'_n|^3].$$

P r o o f. Thanks to inequality (18), it suffices to prove that $(\gamma_n \sqrt{k_n})_n$ is bounded; that is, there exists a constant c such that $\sum_{j=1}^{k_n} |1 - \sum_{h=j}^{k_n} 1/h|^3 \leq c k_n$. The previous inequality is verified by taking $c = 10$.

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© 2000 г. **RAMACHANDRAN D.***, **RÜSCHENDORF L.****
ON THE MONGE–KANTOROVICH DUALITY THEOREM¹⁾

Теорема двойственности Монжа–Канторовича находит разнообразные применения в теории вероятностей, статистике и математической экономике. В результате большой работы была установлена теорема двойственности для общего случая. В статье, при естественном требовании устойчивости для функционала Монжа–Канторовича, описываются вероятностные пространства (называемые сильно двойственными), гарантирующие выполнение теоремы двойственности. Доказано, что сильная двойственность эквивалентна любому из следующих условий: (i) свойство продолжения, (ii) свойство проецируемости, (iii) свойство продолжения до заряда, (iv) совершенность. Полученная характеристика позволяет вывести много полезных свойств, которыми такие пространства обладают, будучи совершенными.

Ключевые слова и фразы: теорема двойственности, маргинальная мера, совершенная мера, продолжение до заряда, функция Марчевского.

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