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EXACT MAXIMAL INEQUALITIES FOR
EXCHANGEABLE SYSTEMS OF RANDOM VARIABLES¹⁾

Для конечной системы $(\xi_1, \dots, \xi_n)$ перестановочных случайных величин со значениями в банаховом пространстве таких, что $\sum_1^n \xi_i = 0$, доказывается эквивалентность в смысле (2) величин $E\Phi(\max_{k \leq n} \|\xi_1 + \dots + \xi_k\|)$ и $E\Phi(\|\sum_1^n \xi_i r_i\|)$ для любой возрастающей выпуклой функции $\Phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\Phi(0) = 0$, где $(r_1, \dots, r_n)$ — система случайных величин Радемахера, не зависящих от $(\xi_1, \dots, \xi_n)$. Устанавливается также эквивалентность хвостов соответствующих распределений. Эти результаты, по-видимому, являются новыми и для скалярных случайных величин. В качестве следствия найдены наилучшие оценки среднего для $\max_{k \leq n} \|a_{\pi(1)} + \dots + a_{\pi(k)}\|$ относительно всех перестановок π неслучайных векторов $a_1, \dots, a_n$ из нормированного пространства.

Ключевые слова и фразы: перестановочные случайные величины, банахово пространство, максимальное неравенство, перестановки.

Introduction. The aim of this paper is to give various versions of maximal inequalities for *finite* systems of exchangeable random variables, scalar or with values in a Banach space. It is well known that infinite exchangeable systems of random variables enjoy nice properties due to the fact that they are conditionally independent with respect to the tail σ -algebra. This fact fails to be true for general finite systems.

In 1974 Maurey and Pisier have shown that for exchangeable systems of random variables $\xi_1, \dots, \xi_n$ with $\sum_1^n \xi_j = 0$ the value $E \max_{k \leq n} \|\xi_1 + \dots + \xi_k\|^p$ is equivalent (can be estimated both from below and above) to $E \|\sum_1^n \xi_j r_j\|^p$, where $r_1, \dots, r_n$ are Rademacher random variables independent of $(\xi_1, \dots, \xi_n)$.

It is interesting to note that some consequences of the result for a long time have not been noticed by analysts. One such consequence is the gener-

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alization of the Garsia result estimating the expression

$$\frac{1}{n!} \sum_{\pi} \max_{k \leq n} \left| \sum_1^k a_{\pi(i)} \right|^p, \quad 1 \leq p < \infty, \quad (*)$$

for given numbers $a_1, \dots, a_n$ with $\sum_1^n a_j = 0$, where π runs over all possible permutations of $\{1, \dots, n\}$. A consequence of Maurey and Pisier result permits to establish exact estimation of $(*)$ for vectors $a_1, \dots, a_n$ of a Banach space playing a crucial role in the theory of rearranged functional series (see [5], [2], [9]).

In Section 1 we show that the Maurey and Pisier result can be generalized to the case of general increasing and convex functions Φ . Our presentation uses a special trick (see Lemma) that allows one to get the desired inequalities immediately in an elementary way.

Section 2 deals with applications. Besides generalizations of the Garsia result we point out the following curious fact. Let $\xi_1, \dots, \xi_n$ be i.i.d. random variables with a continuous distribution and let $\zeta_i = (\xi_i - \bar{\xi})/s$, $i = 1, \dots, n$, where $\bar{\xi}$ is the sample mean, $\bar{\xi} = (\xi_1 + \dots + \xi_n)/n$, while s is the sample standard deviation, $s^2 = \sum_1^n (\xi_j - \bar{\xi})^2/n$. Then for each p , $1 \leq p < \infty$, there are absolute constants $c_p > 0$ and $C_p > 0$ such that $c_p n^{p/2} \leq \mathbf{E} \max_{k \leq n} |\zeta_1 + \dots + \zeta_k|^p \leq C_p n^{p/2}$. The result remains valid for exchangeable systems in a Banach space after specifying a proper analog of s .

Section 3 contains the main result of the paper. Here we find bilateral inequalities for $\mathbf{P}\{\max_{k \leq n} \|\xi_1 + \dots + \xi_k\| > t\}$ in the normed space case.

In Section 4 there are given moment maximal inequalities with multipliers. We find upper estimations for $\mathbf{E} \max_{k \leq n} \|A_1 \xi_1 + \dots + A_k \xi_k\|^p$, $1 \leq p < \infty$, where $A_1, \dots, A_k$ are linear bounded operators in the Banach space, where the random variables of an exchangeable system take their values. We follow the line suggested in [8] reducing the problem to the case of identity operators, $A_1 = \dots = A_n = I$, that is considered in Section 1.

1. Moment maximal inequality. Let X be a real or complex Banach space. By an X -valued random variable we mean a Bochner measurable mapping $\xi: \Omega \rightarrow X$ defined on underlying probability space $(\Omega, \mathcal{A}, \mathbf{P})$. A finite system $\xi_1, \dots, \xi_n$ of X -valued random variables is called *exchangeable*, if for any permutation π of $\{1, \dots, n\}$ the joint distribution of $(\xi_{\pi(1)}, \dots, \xi_{\pi(n)})$ coincides with that of $(\xi_1, \dots, \xi_n)$. By a (finite or countable) family (r_i) of Rademacher random variables we mean independent random variables taking values ± 1 with probabilities $\frac{1}{2}$.

The aim of this section is to prove the following

Theorem 1.1. *Let $(\xi_1, \dots, \xi_n)$ be a system of exchangeable X -valued random variables with $\sum_1^n \xi_j = 0$, and let $\Phi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$, $\Phi(0) = 0$, be any continuous increasing convex function. Then*

a) For any collection $(\theta_1, \dots, \theta_n)$ of signs $\theta_j = \pm 1$

$$\mathbf{E}\Phi\left(\max_{k \leq n} \left\| \sum_1^k \xi_j \right\| \right) \leq \mathbf{E}\Phi\left(\max_{k \leq n} \left\| \sum_1^k \xi_j \theta_j \right\| \right); \quad (1)$$

b) The following two-sided inequality holds

$$\mathbf{E}\Phi\left(\frac{1}{2} \left\| \sum_1^n \xi_j r_j \right\| \right) \leq \mathbf{E}\Phi\left(\max_{k \leq n} \left\| \sum_1^k \xi_j \right\| \right) \leq 2\mathbf{E}\Phi\left(\left\| \sum_1^n \xi_j r_j \right\| \right), \quad (2)$$

where (r_j) is a family of Rademacher random variables which is independent of $(\xi_1, \dots, \xi_n)$.

An elementary lemma below plays a crucial role in the proof of the theorem. Let $(b_1, \dots, b_n)$ be a collection of elements of X . Denote

$$|b| = |(b_1, \dots, b_n)| = \max_{k \leq n} \|b_1 + \dots + b_k\|;$$

$|\cdot|$ represents a norm in the space of ordered collections $(b_1, \dots, b_n)$.

Lemma. If $\sum_1^n b_j = 0$, then

a) For each k , $1 \leq k \leq n$,

$$\max\left(|(b_1, \dots, b_k)|, |(b_{k+1}, \dots, b_n)|\right) = |(b_1, \dots, b_k, b_n, b_{n-1}, \dots, b_{k+1})|;$$

b) For any collection of signs $\theta = (\theta_1, \dots, \theta_n)$ there exists a permutation $\pi(\theta)$ of $\{1, \dots, n\}$ such that

$$|(b)| + |(b\theta)| \geq 2|(b_{\pi(\theta)})|,$$

where $(b\theta) = (b_1\theta_1, \dots, b_n\theta_n)$.

P r o o f. a) is obvious, since $\sum_1^k b_i + \sum_n^{l-1} b_i = -\sum_{k+1}^l b_i$ (due to the fact that $\sum_1^n b_i = 0$) for any l , $k < l \leq n$.

To prove b) observe that for any θ we have by the triangle inequality $|(b)| + |(b\theta)| \geq 2|(b^+)|$ and $|(b)| + |(b\theta)| \geq 2|(b^-)|$, where (b^+) is the part of (b) corresponding to θ_i 's which are $+1$'s, while (b^-) corresponds to -1 's. So we can write according to a): $|(b)| + |(b\theta)| \geq 2|(b_{\pi(\theta)})|$, where $\pi(\theta)$ is the permutation (dependent on θ) so that indices of (b^+) go first, and then those of (b^-) in the reverse order.

P r o o f o f T h e o r e m 1.1. Making use of Lemma we obtain for any collection $(\theta_1, \dots, \theta_n)$

$$\mathbf{E}\Phi(|(\xi)|) = \mathbf{E}\Phi(|(\xi)| + |(\xi\theta)| - |(\xi\theta)|) \geq \mathbf{E}\Phi(2|(\xi_{\pi(\theta)})| - |(\xi\theta)|)$$

(due to convexity of Φ)

$$\geq 2\mathbf{E}\Phi(|(\xi_{\pi(\theta)})|) - \mathbf{E}\Phi(|(\xi\theta)|),$$

where π is a permutation of $\{1, \dots, n\}$ dependent on the collection $(\theta_1, \dots, \theta_n)$. Now the exchangeability of $(\xi_1, \dots, \xi_n)$ immediately leads to a).

To prove b) we take average of (1) with respect to all collections $(\theta_1, \dots, \theta_n)$. Then the Fubini theorem along with the Lévy inequality yields the right-hand side part of (2). The left-hand side part is the consequence of Lemma and the exchangeability of $(\xi_1, \dots, \xi_n)$:

$$\begin{aligned} \mathbf{E}\Phi\left(\left\|\sum_1^n \xi_j r_j\right\|\right) &\leq \mathbf{E}\Phi(|(\xi r)|) \leq \mathbf{E}\Phi(2 \max(|(\xi_j^+)|, |(\xi_j^-)|)) \\ &= \mathbf{E}\Phi(2 |(\xi_{\pi(r)})|) = \mathbf{E}\Phi(2 |(\xi)|). \end{aligned}$$

Theorem is proved.

Remark. The condition $\sum_1^n \xi_j = 0$ in Theorem 1.1 is imposed for simplicity of formulation. The case of general exchangeable system $(\xi_1, \dots, \xi_n)$ can easily be reduced to the case of exchangeable system with vanishing sum by the substitution $\xi'_i = \xi_i - (\sum_1^n \xi_j)/n$, $i = 1, \dots, n$.

Part (b) of Theorem 1.1 for the case $\Phi(t) = t^p$, $1 \leq p < \infty$, has been proved in [8] (may be with a greater constant).

2. Corollaries and applications. *Bounds for centered and normalized system of exchangeable random variables.* Let $(\xi_1, \dots, \xi_n)$ be a general exchangeable system of random variables with values in a Banach space X (the condition $\sum_1^n \xi_j = 0$ is not imposed). Consider the following associated exchangeable system defined by

$$\eta_k = \frac{\xi_k - \bar{\xi}}{(\mathbf{E}_r \|\sum_1^n (\xi_j - \bar{\xi}) r_j\|^2)^{1/2}}, \quad k = 1, \dots, n,$$

(we put $\eta_k = 0$, if $\xi_1 = \xi_2 = \dots = \xi_n$), where $\bar{\xi} = (\xi_1 + \dots + \xi_n)/n$, while $\mathbf{E}_r$ denotes the expectation with respect to the argument of r 's (without loss of generality we can assume that ξ 's and r 's are defined on different probability spaces). Then we have $\sum_1^n \eta_j = 0$, $(\eta_1, \dots, \eta_n)$ still being an exchangeable system of X -valued random variables, and Theorem 1.1b) implies

Corollary 2.1. *For each p , $1 \leq p < \infty$, there exist absolute constants $c_p > 0$ and $C_p > 0$ such that for any exchangeable system $(\xi_1, \dots, \xi_n)$ with values in a Banach space satisfying the condition*

$$\mathbf{P}\{\xi_1 = \xi_2 = \dots = \xi_n\} = 0, \quad (3)$$

the following inequalities hold true

$$c_p \leq \mathbf{E} \max_{k \leq n} \left\| \sum_1^k \eta_j \right\|^p \leq C_p. \quad (4)$$

The right-hand side inequality is true without the restriction (3).

P r o o f. We have by Theorem 1.1b)

$$\begin{aligned} \mathbf{E} \max_{k \leq n} \left\| \sum_1^k \eta_j \right\|^p &\leq 2 \mathbf{E} \left\| \sum_1^n \eta_j r_j \right\|^p \\ &= 2 \mathbf{E}_\xi \mathbf{E}_r \frac{\left\| \sum_1^n (\xi_j - \bar{\xi}) r_j \right\|^p}{(\mathbf{E}_r \left\| \sum_1^n (\xi_j - \bar{\xi}) r_j \right\|^2)^{p/2}} \leq 2 K_{p,2}^p, \end{aligned}$$

where $K_{p,q}$ is Kahane's constant involved in the Kahane inequality ($1 \leq q$, $p < \infty$):

$$\left(\mathbf{E} \left\| \sum_1^n a_j r_j \right\|^p \right)^{1/p} \leq K_{p,q} \left(\mathbf{E} \left\| \sum_1^n a_j r_j \right\|^q \right)^{1/q}.$$

This proves the right-hand side of (4). To prove the left-hand side we note that due to condition (3) the denominator in the definition of η_k does not vanish a.s. By Theorem 1.1b)

$$\begin{aligned} \mathbf{E} \max_{k \leq n} \left\| \sum_1^k \eta_j \right\|^p &\geq \frac{1}{2^p} \mathbf{E} \left\| \sum_1^n \eta_j r_j \right\|^p \\ &= \frac{1}{2^p} \mathbf{E}_\xi \mathbf{E}_r \frac{\left\| \sum_1^n (\xi_j - \bar{\xi}) r_j \right\|^p}{(\mathbf{E}_r \left\| \sum_1^n (\xi_j - \bar{\xi}) r_j \right\|^2)^{p/2}} \geq \frac{1}{2^p K_{2,p}^p}. \end{aligned}$$

It is worth to point out a particular case of Corollary 2.1. Let $\xi_1, \dots, \xi_n$ be i.i.d. real-valued random variables, and let $\zeta_j = (\xi_j - \bar{\xi})/s$ (as before we put $\zeta_j = 0$, if $\xi_1 = \xi_2 = \dots = \xi_n$), $j = 1, \dots, n$, where s denotes the sample standard deviation, $s^2 = \sum_1^n (\xi_j - \bar{\xi})^2/n$.

Corollary 2.2. For any p , $1 \leq p < \infty$, there exist absolute constants d_p and D_p such that for any i.i.d. random variables $\xi_1, \dots, \xi_n$ with a continuous distribution

$$d_p n^{p/2} \leq \mathbf{E} \max_{k \leq n} (|\zeta_1 + \dots + \zeta_k|)^p \leq D_p n^{p/2}.$$

The right-hand side inequality is true without any assumption on the distribution of ξ .

Note that d_p and D_p (as well as c_p and C_p in Corollary 2.1) are not dependent on the (continuous) distribution of ξ .

Corollary 2.1 also implies the following Kahane type inequality between various moments.

Corollary 2.3. For any p and q , $1 \leq q < p < \infty$, there exists an absolute constant $M_{p,q}$ such that for any exchangeable system $(\xi_1, \dots, \xi_n)$ of Banach space valued random variables the following inequality holds

$$\left(\mathbf{E} \max_{k \leq n} \|\eta_1 + \dots + \eta_k\|^p \right)^{1/p} \leq M_{p,q} \left(\mathbf{E} \max_{k \leq n} \|\eta_1 + \dots + \eta_k\|^q \right)^{1/q}.$$

Integral of max is less than max of integrals.

Corollary 2.4. *If $(\xi_1, \dots, \xi_n)$ is an exchangeable system of X -valued random variables with $\sum_1^n \xi_j = 0$, then for any continuous increasing convex function $\Phi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$, $\Phi(0) = 0$,*

$$\mathbf{E}\Phi\left(\max_{k \leq n} \left\| \sum_1^k \xi_j \right\| \right) \leq 2 \max_{k \leq n} \mathbf{E}\Phi\left(2 \left\| \sum_1^k \xi_j \right\| \right).$$

In particular,

$$\mathbf{E} \max_{k \leq n} \left\| \sum_1^k \xi_j \right\|^p \leq 2^{p+1} \max_{k \leq n} \mathbf{E} \left\| \sum_1^k \xi_j \right\|^p.$$

P r o o f. By Theorem 1.1b)

$$\mathbf{E}\Phi\left(\max_{k \leq n} \left\| \sum_1^k \xi_j \right\| \right) \leq 2\mathbf{E}\Phi\left(\left\| \sum_1^n \xi_j r_j \right\| \right) \leq 2\mathbf{E}_r \mathbf{E}_\xi \Phi\left(\left\| \sum \xi_j^+ \right\| + \left\| \sum \xi_j^- \right\| \right),$$

where (ξ_j^+) is subcollection of (ξ_j) corresponding to $r_j = +1$, while (ξ_j^-) corresponds to $r_j = -1$. Then we continue due to convexity of Φ

$$\leq \mathbf{E}_r \mathbf{E}_\xi \Phi\left(2 \left\| \sum \xi_j^+ \right\| \right) + \mathbf{E}_r \mathbf{E}_\xi \Phi\left(2 \left\| \sum \xi_j^- \right\| \right) \leq 2 \max_{k \leq n} \mathbf{E}\Phi\left(2 \left\| \sum_1^k \xi_j \right\| \right),$$

and the corollary is proved.

The case of Banach lattices of some cotype. In the case $X = \mathbf{R}$ the expression $\mathbf{E} \left| \sum_1^n a_j r_j \right|^p$ is equivalent (Khinchine theorem) to $(\sum_1^n a_i^2)^{p/2}$ for $1 \leq p < \infty$. So Theorem 1.1b) implies

Corollary 2.5. *If $(\xi_1, \dots, \xi_n)$ is an exchangeable system of real-valued random variables with $\sum \xi_j = 0$, then for any p , $1 \leq p < \infty$,*

$$c_p \mathbf{E} \left(\sum_1^n \xi_j^2 \right)^{p/2} \leq \mathbf{E} \max_{k \leq n} \left| \sum_1^k \xi_j \right|^p \leq C_p \mathbf{E} \left(\sum_1^n \xi_j^2 \right)^{p/2},$$

where c_p, C_p are absolute constants dependent on p only.

This result can be generalized to the case of Banach lattices having some cotype q , $2 \leq q < \infty$ (the L_p -spaces, $1 \leq p < \infty$, represent usual examples of such lattices). For such lattices there is a generalization of the Khinchine theorem due to Maurey and Pisier [8]:

$$\mathbf{E} \left\| \sum_1^n a_j r_j \right\|^p \text{ is equivalent to } \left\| \left(\sum_1^n |a_j|^2 \right)^{1/2} \right\|^p.$$

This fact along with Theorem 1.1 b) implies

Corollary 2.6. *If $(\xi_1, \dots, \xi_n)$ is an exchangeable system of random variables with values in a Banach lattice of some cotype q , $2 \leq q < \infty$, $\sum_1^n \xi_j = 0$, then for each p , $1 \leq p < \infty$,*

$$c_p \mathbf{E} \left\| \left(\sum_1^n |\xi_j|^2 \right)^{1/2} \right\|^p \leq \mathbf{E} \max_{k \leq n} \left\| \sum_1^k \xi_j \right\|^p \leq C_p \mathbf{E} \left\| \left(\sum_1^n |\xi_j|^2 \right)^{1/2} \right\|^p,$$

where c_p, C_p are constants dependent on p and on the Banach lattice X .

Spaces of type p . If X is a space of type p , $1 \leq p \leq 2$, then

$$\mathbf{E} \left\| \sum_1^n a_j r_j \right\|^p \leq C \sum_1^n \|a_j\|^p,$$

where C is a constant dependent on X . Actually this is one of equivalent definitions of a space of type p . L_r -spaces are of type r , if $1 \leq r \leq 2$, and of type 2, if $r \geq 2$. Theorem 1.1b) for such spaces implies

Corollary 2.7 ([1], [8]). *If $(\xi_1, \dots, \xi_n)$ is an exchangeable system of random variables with values in a Banach space X of type p , $1 \leq p \leq 2$, with $\sum_1^n \xi_j = 0$, then for some constant $C > 0$ dependent on the space X only*

$$\mathbf{E} \max_{k \leq n} \left\| \sum_1^k \xi_j \right\|^p \leq C \mathbf{E} \sum_1^n \|\xi_j\|^p.$$

Permutations of vectors. Let $a_1, \dots, a_n$ be a collection of elements of a normed space with $\sum_1^n a_j = 0$. In a series of problems of analysis of interest is the estimation of

$$\text{Ave}_\pi \Phi \left(\max_{k \leq n} \left\| \sum_1^k a_{\pi(j)} \right\| \right) = \frac{1}{n!} \sum_\pi \Phi \left(\max_{k \leq n} \left\| \sum_1^k a_{\pi(j)} \right\| \right),$$

where π is a permutation of $\{1, \dots, n\}$, while the summation is carried out over all such permutations. Theorem 1.1 permits to find a proper bilateral estimation for this quantity.

Corollary 2.8. *Let $\Phi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ be a continuous increasing convex function, $\Phi(0) = 0$.*

a) *The following bilateral inequality holds true*

$$\mathbf{E} \Phi \left(\frac{1}{2} \left\| \sum_1^n a_j r_j \right\| \right) \leq \text{Ave}_\pi \max_{k \leq n} \Phi \left(\left\| \sum_1^k a_{\pi(j)} \right\| \right) \leq 2 \mathbf{E} \Phi \left(\left\| \sum_1^n a_j r_j \right\| \right).$$

b) *If $X = \mathbf{R}$ and $1 \leq p < \infty$, then*

$$\frac{1}{2^p c_p^p} \left(\sum_1^n a_j^2 \right)^{p/2} \leq \text{Ave}_\pi \max_{k \leq n} \left| \sum_1^k a_{\pi(j)} \right|^p \leq 2 c_p^p \left(\sum_1^n a_j^2 \right)^{p/2},$$

where c_p is the Khinchine constant taking part in the inequality

$$\mathbf{E} \left| \sum a_j r_j \right|^p \leq c_p^p \left(\sum_1^n a_j^2 \right)^{p/2}.$$

Proof. Consider the random variables $\xi_1, \dots, \xi_n$ associated with $a_1, \dots, a_n$. They are defined on the space Π of all permutations of $\{1, \dots, n\}$ with the uniform distribution as follows: $\xi_i(\pi) = a_{\pi(i)}$, $\pi \in \Pi$, $i = 1, \dots, n$. It is easy to see that the system $(\xi_1, \dots, \xi_n)$ is exchangeable and $\sum_1^n \xi_i = 0$. Now application of Theorem 1.1 implies Corollary 2.8.

The inequality in b) of Corollary 2.8 (the most important right-hand side of it) has been established by Garsia [4] in connection with proving a.e. convergence of rearranged orthogonal series. The vector case for the power function $\Phi(t) = t^p$, $1 \leq p < \infty$, has been treated by Maurey and Pisier [8].

Corollary 2.8a) implies the following result playing a crucial role in problems of rearrangements of functional series.

Proposition. Assume that for a series $\sum_1^\infty a_j$ in a normed space X , a subsequence S_{n_k} of the sequence of partial sums converges to an element $s \in X$. Then if $\sum_1^\infty a_j r_j$ converges a.s. in X , then there exists a permutation $\pi: \mathbf{N} \rightarrow \mathbf{N}$ such that the series $\sum_1^\infty a_{\pi(j)}$ converges to s .

The reader is referred to [2] and [3] for a refinement of the last proposition and its applications.

3. Inequalities for tails. Moment inequalities established in Section 1 prompt that there should be a relation between tail probabilities of $\max_{k \leq n} \|\xi_1 + \dots + \xi_k\|$ and $\|\sum_1^n \xi_j r_j\|$. In the next statement we establish such a relation.

Theorem 3.1. Let $(\xi_1, \dots, \xi_n)$ be an exchangeable system of random variables with $\sum_1^n \xi_i = 0$ defined on $(\Omega, \mathcal{A}, \mathbf{P})$ and taking values in a Banach space X . Then for any $t > 0$ the following inequalities hold true

$$\begin{aligned} (\lambda \times \mathbf{P}) \left\{ \left\| \sum_1^n \xi_j r_j \right\| > 2t \right\} &\leq \mathbf{P} \left\{ \max_{k \leq n} \|\xi_1 + \dots + \xi_k\| > t \right\} \\ &\leq 10(\lambda \times \mathbf{P}) \left\{ \left\| \sum_1^n \xi_j r_j \right\| > \frac{t}{4} \right\}, \end{aligned}$$

where $r_1, \dots, r_n$ are Rademacher functions defined on $[0, 1]$ with the Lebesgue measure λ .

We reduce Theorem 3.1 to the following assertion proved in [2]. For the one-dimensional version see also [6].

Theorem 3.1'. Let $(a_1, \dots, a_n)$ be any collection of elements of a normed space X with $\sum_1^n a_j = 0$. Then the following bilateral inequality holds

$$\begin{aligned} \lambda \left\{ \left\| \sum_1^n a_j r_j \right\| > 2t \right\} &\leq \frac{1}{n!} \text{card} \left\{ \pi: \max_{k \leq n} \|a_{\pi(1)} + \dots + a_{\pi(k)}\| > t \right\} \\ &\leq 10\lambda \left\{ \left\| \sum_1^n a_j r_j \right\| > \frac{t}{4} \right\}. \end{aligned}$$

Proof of Theorem 3.1. Using the exchangeability of the system $(\xi_1, \dots, \xi_n)$ we can write for any permutation $\pi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$

$$\mathbf{P} \left\{ \max_{k \leq n} \|\xi_1 + \dots + \xi_k\| > t \right\} = \mathbf{P} \left\{ \max_{k \leq n} \|\xi_{\pi(1)} + \dots + \xi_{\pi(k)}\| > t \right\}.$$

Integrating both sides with respect to the uniform distribution m on the set Π of all permutations we get

$$\mathbf{P} \left\{ \max_{k \leq n} \|\xi_1 + \dots + \xi_k\| > t \right\} = \int_{\Pi} \mathbf{P} \left\{ \max_{k \leq n} \|\xi_{\pi(1)} + \dots + \xi_{\pi(k)}\| > t \right\} dm(\pi)$$

(by the Fubini theorem)

$$= \int_{\Omega} m \left\{ \max_{k \leq n} \|\xi_{\pi(1)} + \dots + \xi_{\pi(k)}\| > t \right\} d\mathbf{P}\{\omega\}$$

(by Theorem 3.1')

$$\leq 10 \int_{\Omega} \lambda \left\{ \left\| \sum_1^n \xi_j r_j \right\| > \frac{t}{4} \right\} d\mathbf{P} = 10(\lambda \times \mathbf{P}) \left\{ \left\| \sum_1^n \xi_j r_j \right\| > \frac{t}{4} \right\}.$$

This proves the right-hand side inequality. The left-hand side part can be treated in the similar way.

Corollaries. Note first that to any system of random variables $(\xi_1, \dots, \xi_n)$ one can associate an exchangeable system on some extension of the underlying probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Namely, it is easy to see that the system of random variables $\xi_k(\omega, \pi) = \xi_{\pi(k)}(\omega)$, $(\omega, \pi) \in \Omega \times \Pi$, $k = 1, \dots, n$, where Π as before is the set of all permutations of $\{1, \dots, n\}$, is an exchangeable system. Applying Theorem 3.1 to the system we come to the following assertion.

Corollary 3.1. Let $(\xi_1, \dots, \xi_n)$ be any collection of random variables with values in a Banach space X , and let $(\eta_1, \dots, \eta_n)$ be the associated collection of random variables (i.e., centered and normalized as in Section 2). Then for any $t > 0$ the following bilateral inequality holds

$$\begin{aligned} (\lambda \times \mathbf{P}) \left\{ \left\| \sum_1^n \eta_j r_j \right\| > 2t \right\} &\leq \frac{1}{n!} \sum_{\pi} \mathbf{P} \left\{ \max_{k \leq n} \|\eta_{\pi(1)} + \dots + \eta_{\pi(k)}\| > t \right\} \\ &\leq 10(\lambda \times \mathbf{P}) \left\{ \left\| \sum_1^n \eta_j r_j \right\| > \frac{t}{4} \right\}. \end{aligned}$$

Moment maximal inequality for non-convex functions. Theorem 3.1 allows one to extend Theorem 1.1b) to the case of non-convex functions $\Phi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$.

Corollary 3.2. Let $(\xi_1, \dots, \xi_n)$ be a system of exchangeable Banach space valued random variables with $\sum_1^n \xi_j = 0$, and let $\Phi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$, $\Phi(0) = 0$, be any continuous increasing (not necessarily convex) function. Then the following bilateral inequality holds:

$$\mathbf{E} \Phi \left(\frac{1}{2} \left\| \sum_1^n \xi_i r_i \right\| \right) \leq \mathbf{E} \Phi \left(\max_{1 \leq k \leq n} \|\xi_1 + \dots + \xi_k\| \right) \leq 10 \mathbf{E} \Phi \left(4 \left\| \sum_1^n \xi_j r_j \right\| \right).$$

P r o o f. The integration by parts formula yields

$$\mathbf{E} \Phi \left(\max_{1 \leq k \leq n} \|\xi_1 + \dots + \xi_k\| \right) = \int_0^\infty \mathbf{P} \left\{ \max_{1 \leq k \leq n} \|\xi_1 + \dots + \xi_k\| > t \right\} d\Phi(t)$$

(due to Theorem 3.1)

$$\leq 10 \int_0^\infty \mathbf{P} \left\{ \left\| \sum_1^n \xi_k r_k \right\| > \frac{t}{4} \right\} d\Phi(t) = \mathbf{E} \Phi \left(4 \left\| \sum_1^n \xi_k r_k \right\| \right).$$

This proves the right-hand side inequality. The proof of the left-hand side part is similar.

4. Inequalities with multipliers. Our next goal is to estimate $\mathbf{E} \max_{k \leq n} \left\| \sum_1^n \mu_j \xi_j \right\|$ with numerical multipliers $\mu_1, \dots, \mu_n$. Although our inequalities are no longer bilateral, in the particular case of $\mu_1 = \mu_2 = \dots = \mu_n$ they give (up to a constant factor) the right-hand side inequality of Theorem 1.1. We follow mainly the method of reducing to the case of equal coefficients suggested by Maurey and Pisier [8]. As we will see the method permits to treat the case when μ_i 's are operators.

Theorem 4.1. Let $(\xi_1, \dots, \xi_n)$ be an exchangeable system of random variables with values in a Banach space X , $\sum_1^n \xi_j = 0$. Then for any bounded linear operators $A_1, \dots, A_n$ in X and any p , $1 < p < \infty$, the following inequality holds

$$\begin{aligned} \left(\mathbf{E} \max_{k \leq n} \left\| \sum_1^k A_j \xi_j \right\|^p \right)^{1/p} &\leq \frac{6p^2}{(p-1)^2} \left(\mathbf{E} \left\| \sum_1^n A_j \xi_j r_j \right\|^p \right)^{1/p} \\ &\quad + \frac{8(4p-1)}{(9p-1)(n-1)} \left(\mathbf{E} \left\| \sum_1^n \xi_j r_j \right\|^p \right)^{1/p} \sum_1^n \|A_j\|, \end{aligned}$$

where (r_j) is a family of Rademacher random variables which is independent of $(\xi_1, \dots, \xi_n)$.

P r o o f. Denote

$$\varphi_k^*(A_1, \dots, A_k) = \left(\mathbf{E} \max_{j \leq k} \left\| \sum_1^j A_i \xi_i \right\|^p \right)^{1/p},$$

$$\varphi_k(A_1, \dots, A_k) = \left(\mathbf{E} \left\| \sum_{j=1}^k A_j \xi_j \right\|^p \right)^{1/p}.$$

It is easy to see that the exchangeability of $(\xi_1, \dots, \xi_n)$ implies that for any $i < n$

$$\mathbf{E}(\xi_{j_1} | \xi_1, \dots, \xi_i) = \mathbf{E}(\xi_{j_2} | \xi_1, \dots, \xi_i) \quad \text{for any } j_1, j_2 > i.$$

Therefore we have for any $i \leq k$ (taking into account that $\sum_{j=1}^n \xi_j = 0$)

$$\begin{aligned} \mathbf{E} \left(\sum_{j=1}^k A_j \xi_j | \xi_1, \dots, \xi_i \right) &= \sum_{j=1}^i A_j \xi_j + \mathbf{E} \left(\sum_{j=i+1}^k A_j \xi_j | \xi_1, \dots, \xi_i \right) \\ &= \sum_{j=1}^i A_j \xi_j + \sum_{j=i+1}^k A_j \frac{1}{n-i} \mathbf{E} \left(\sum_{l=i+1}^n \xi_l | \xi_1, \dots, \xi_i \right) \\ &= \sum_{j=1}^i A_j \xi_j - \frac{1}{n-i} \left(\sum_{j=i+1}^k A_j \right) \left(\sum_{j=1}^i \xi_j \right). \end{aligned}$$

Since $\mathbf{E}(\sum_{j=1}^k A_j \xi_j | \xi_1, \dots, \xi_i)$ is a martingale with respect to i , we have according to the Doob maximal inequality

$$\varphi_k^*(A_1, \dots, A_k) \leq q \varphi_k(A_1, \dots, A_k) + \frac{1}{n-k} \sum_{j=1}^k \|A_j\| \varphi_k^*(I, \dots, I), \quad (5)$$

where $q = p/(p-1)$, while I is the identity operator in X .

Let now $\theta_1, \dots, \theta_k$ be an arbitrary collection of signs, and let σ be a permutation of $\{1, \dots, k\}$ such that in the ordered collection $(\theta_{\sigma(1)}, \dots, \theta_{\sigma(k)})$ beforehand there are pluses in a row followed by minuses. The triangle inequality argument allows one to write $\varphi_k(A_{\sigma(1)}, \dots, A_{\sigma(k)}) \leq \varphi_k^*(A_{\sigma(1)}, \dots, A_{\sigma(k)}) \leq 3 \varphi_k^*(\theta_{\sigma(1)} A_{\sigma(1)}, \dots, \theta_{\sigma(k)} A_{\sigma(k)})$. Using (5) together with the fact that the family (ξ_j) is exchangeable we get from here

$$\begin{aligned} \varphi_k(A_1, \dots, A_k) &= \varphi_k(A_{\sigma(1)}, \dots, A_{\sigma(k)}) \leq 3 q \varphi_k(\theta_{\sigma(1)} A_{\sigma(1)}, \dots, \theta_{\sigma(k)} A_{\sigma(k)}) \\ &\quad + \frac{3}{n-k} \left(\sum_{j=1}^k \|A_j\| \right) \varphi_k^*(I, \dots, I). \end{aligned}$$

Substituting this expression for φ_k in (5) we have

$$\begin{aligned} \varphi_k^*(A_1, \dots, A_k) &\leq 3 q^2 \varphi_k(\theta_1 A_1, \dots, \theta_k A_k) \\ &\quad + \frac{3q+1}{n-k} \sum_{j=1}^k \|A_j\| \varphi_k^*(I, \dots, I). \end{aligned} \quad (6)$$

Analogously we can estimate the quantity $\psi_{n-k}^*(A_{k+1}, \dots, A_n) = (\mathbf{E} \max_{k+1 \leq i \leq n} \|\sum_{j=k+1}^i A_j \xi_j\|^p)^{1/p}$:

$$\begin{aligned} \psi_{n-k}^*(A_{k+1}, \dots, A_n) &\leq 3 q^2 \psi_{n-k}(\theta_{k+1} A_{k+1}, \dots, \theta_n A_n) \\ &\quad + \frac{3q+1}{k} \sum_{k+1}^n \|A_j\| \psi_{n-k}^*(I, \dots, I), \end{aligned} \quad (7)$$

where $\psi_{n-k}(A_{k+1}, \dots, A_n) = (\mathbf{E} \|\sum_{j=k+1}^n A_j \xi_j\|^p)^{1/p}$. Now put $k = [n/2]$. By averaging both sides of (6) and (7) with respect to $\theta = (\theta_1, \dots, \theta_k)$ we obtain by Theorem 1.1

$$\begin{aligned} \varphi_n^*(A_1, \dots, A_n) &\leq \varphi_k^*(A_1, \dots, A_k) + \psi_{n-k}^*(A_{k+1}, \dots, A_n) \\ &\leq 6q^2 \left(\mathbf{E} \left\| \sum_1^n A_j \xi_j r_j \right\|^p \right)^{1/p} \\ &\quad + \frac{8(3q+1)}{n-1} \sum_1^n \|A_j\| \left(\mathbf{E} \left\| \sum_1^n \xi_j r_j \right\|^p \right)^{1/p}, \end{aligned}$$

where (r_j) is a Rademacher system of random variables which is independent of (ξ_j) . Theorem 4.1 is proved.

Theorem 4.1 implies the following

Corollary 4.1. *Let $(\xi_1, \dots, \xi_n)$ be an exchangeable system of X -valued random variables, $\sum_1^n \xi_i = 0$. Then*

a) *If X is of type p , $1 < p \leq 2$, then there exists a constant $c_p(X)$ such that for any bounded linear operators $A_1, \dots, A_n$ in X*

$$\mathbf{E} \max_{k \leq n} \left\| \sum_1^k A_j \xi_j \right\|^p \leq c_p(X) \mathbf{E} \|\xi_1\|^p \sum_1^n \|A_j\|^p;$$

b) *If $X = \mathbf{R}$, while $A_1, \dots, A_n$ are reals, then*

$$\mathbf{E} \max_{k \leq n} \left| \sum_1^k A_j \xi_j \right|^2 \leq 18496 \mathbf{E} \xi_1^2 \sum_1^n A_i^2.$$

Let us give one more consequence of Theorem 4.1.

Corollary 4.2. *If $(\xi_1, \dots, \xi_n)$ is an exchangeable system of orthonormal random variables, then for any collection of reals $\alpha_1, \dots, \alpha_n$ the following inequality holds*

$$\mathbf{E} \max_{k \leq n} \left| \sum_1^k \alpha_j \xi_j \right|^2 \leq c \sum_1^n \alpha_j^2,$$

where c is an absolute constant ($c \leq 18497$).

Corollary 4.2 has been proved by Garsia [4] to show that any orthonormal exchangeable system $(\xi_j)_1^\infty$ is a system of convergence: any series $\sum_1^\infty \alpha_j \xi_j$ with $\sum_1^\infty \alpha_j^2 < \infty$ is a.s. convergent.

Finally we give a result estimating

$$\frac{1}{n!} \sum_\pi \max_{k \leq n} \|\mu_1 a_{\pi(1)} + \dots + \mu_k a_{\pi(k)}\|^2,$$

where as before $a_1, \dots, a_n$ are elements of a normed space X , π runs over the set Π of all permutations of $\{1, \dots, n\}$, while $\mu_1, \dots, \mu_n$ are real numbers with the fixed positions in the sum. As before we consider the exchangeable

system of generated X -valued random variables $\xi_j(\pi) a_{\pi(j)}$, $j = 1, \dots, n$, $\pi \in \Pi$, and apply Corollary 4.1 to obtain the following assertion.

Corollary 4.3. *If $a_1, \dots, a_n$ are elements of a Banach space of type 2, while $\mu_1, \dots, \mu_n$ are real numbers, then*

$$\frac{1}{n!} \sum_{\pi} \max_{k \leq n} \|\mu_1 a_{\pi(1)} + \dots + \mu_k a_{\pi(k)}\|^2 \leq \frac{C \sum_1^n \mu_k^2}{n} \left(\sum_1^n \|a_j\|^2 + \left\| \sum_1^n a_j \right\|^2 \right),$$

where C is a constant dependent on the space X only.

The proof can easily be derived from Corollary 4.1 first for the case when $\sum_1^n a_i = 0$ and then for the general case by use of the Hölder inequality.

The Hilbert space case of Corollary 4.3 has been used in [3] to establish the permutational Kolmogorov SLLN for orthogonal random variables.

In the case $X = \mathbf{R}$ Corollary 4.3 has been established first in [4].

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