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BEIBEL M.* , LERCHE H. R.*

A NOTE ON OPTIMAL STOPPING OF REGULAR DIFFUSIONS UNDER RANDOM DISCOUNTING

Пусть X — одномерная диффузия, удовлетворяющая некоторым условиям регулярности, A — положительный непрерывный аддитивный функционал от X и h — измеримая вещественнозначная функция. Предложен метод нахождения правила остановки T^* , которое максимизирует $E\{e^{-A_T} h(X_T) 1_{\{T < \infty\}}\}$ по всем моментам остановки T процесса X . Обсуждается ряд примеров, некоторые из них из области финансовой математики.

Ключевые слова и фразы: диффузия, обобщенная задача о выборе стоянки, оптимальная остановка, случайные потери.

1. Introduction

Let B denote standard Brownian motion and let r be a non-negative measurable function on the real numbers $\mathbf{R}$. Let us consider the problem of maximizing the gain

$$E\left(\exp\left\{-\int_0^T r(B_s) ds\right\} (B_T^+)^{\alpha} 1_{\{T < \infty\}}\right)$$

for some $\alpha > 0$ over all stopping times T of B . To find an optimal stopping time for this and related problems we extend the approach of Beibel and Lerche [1]. In the present case that means to represent the payoff $\exp\{-\int_0^t r(B_s) ds\} (B_t^+)^{\alpha}$ as

$$\exp\left\{-\int_0^t r(B_s) ds\right\} \psi(B_t) \frac{(B_t^+)^{\alpha}}{\psi(B_t)},$$

where ψ is chosen such that $\exp\{-\int_0^t r(B_s) ds\} \psi(B_t)$ becomes a positive local martingale and $(x^+)^{\alpha}/\psi(x)$ is a function which assumes its maximum

*Institut für Mathematische Stochastik, Universität Freiburg, Eckerstr., 1, 79104 Freiburg, Germany; e-mail: beibel@stochastik.uni-freiburg.de; lerche@stochastik.uni-freiburg.de

over $\mathbf{R}$ at some point $x^* > 0$. Then we obtain for all stopping times T of B that

$$\begin{aligned} & \mathbf{E} \left(\exp \left\{ - \int_0^T r(B_s) ds \right\} (B_T^+)^{\alpha} 1_{\{T < \infty\}} \right) \\ & \leq \left(\sup_{x \in \mathbf{R}} \frac{(x^+)^{\alpha}}{\psi(x)} \right) \mathbf{E} \left(\exp \left\{ - \int_0^T r(B_s) ds \right\} \psi(B_T) 1_{\{T < \infty\}} \right) \\ & \leq \psi(0) \frac{(x^*)^{\alpha}}{\psi(x^*)}. \end{aligned}$$

In order to prove the optimality of $T^* = \inf \{t \geq 0 \mid B_t = x^*\}$, it is left to show that

$$\mathbf{E} \left(\exp \left\{ - \int_0^{T^*} r(B_s) ds \right\} \psi(B_{T^*}) 1_{\{T^* < \infty\}} \right) = \psi(0). \quad (1)$$

To be more explicit, let us consider the case $r(x) = r 1_{[0, \infty)}(x)$ for some positive constant r . Let

$$\psi(x) = \begin{cases} \frac{1}{2} \left(\exp(\sqrt{2rx}) + \exp(-\sqrt{2rx}) \right) & \text{for } x \geq 0, \\ 1 & \text{for } x < 0. \end{cases} \quad (2)$$

Then it holds that $\sup_{x \in \mathbf{R}} [(x^+)^{\alpha} / \psi(x)] < \infty$ and the supremum is attained at the unique positive solution x^* of the implicit equation

$$\sqrt{2rx} (e^{\sqrt{2rx}} - e^{-\sqrt{2rx}}) = \alpha (e^{\sqrt{2rx}} + e^{-\sqrt{2rx}}).$$

The generalized Itô formula (see [6, IV.45.2]) yields that

$$\exp \left\{ - r \int_0^t 1_{(0, \infty)}(B_s) ds \right\} \psi(B_t)$$

is indeed a positive local martingale. Moreover ψ is nondecreasing. Therefore the process $\exp \{ - \int_0^t r(B_s) ds \} \psi(B_t)$ is bounded by $\psi(x^*)$ for $0 \leq t \leq T^*$. Since $\mathbf{P}\{T^* < \infty\} = 1$, this yields (1). Thus T^* is an optimal stopping rule.

The approach just presented can be applied to a general class of optimal stopping problems. Let X denote a one-dimensional regular diffusion in the sense of Feller, Itô, and McKean on some probability space $(\Omega, \mathcal{F})$. «Regular» here means that $\mathbf{P}\{T_y < \infty \mid X_0 = x\} > 0$ holds for all $y \in I$ and $x \in \text{int}(I)$, with $T_y = \inf \{t \geq 0 \mid X_t = y\}$. In the sequel we assume that the structure of $(\Omega, \mathcal{F})$ is such that there exist measurable shift operators θ_s for $s \geq 0$ with $X_{t+s}(\omega) = X_t(\theta_s(\omega))$ for all $\omega \in \Omega$ and $s, t \geq 0$. Let $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$. Let h be a measurable function and r a positive constant. Let $(A_s; s \geq 0)$ be a non-negative continuous additive functional of X , which means that A satisfies the following conditions:

- i) $A_t \in [0, \infty)$ for $0 \leq t < \infty$;
- ii) A_t is measurable with respect to $\mathcal{F}_t^X$;

iii) $A_{s+t} = A_s + A_t \circ \theta_s$;

iv) A_t is continuous in t .

We will consider the following general optimal stopping problem.

Problem 1. Find a stopping time T^* of X that maximizes

$$\mathbf{E}\left(e^{-A_T} h(X_T) 1_{\{T < \infty\}} \mid X_0 = x\right)$$

among all stopping times T of X .

Note that the stopping time $T \equiv +\infty$ yields the expected payoff 0. Hence we will never stop whenever $h(X_t) < 0$. This means we can always replace h by $\max\{h, 0\}$.

The case of deterministic discounting $A_t = rt$ is treated by Mucci [5] and Salminen [7]. Their methods and conditions on h differ somewhat from ours. Under additional technical conditions on the drift and diffusion coefficient of X , Bensoussans and Lions [2] treat the case, where h is bounded and $A_t = \int_0^t r(X_s) ds$ with a positive bounded measurable function r with $r \geq \beta$ for some strictly positive constant β (see [2, Theorem 3.19, p. 387]).

In the following the state space of X is an interval I on the real line. The interior of I will be denoted by $\text{int}(I)$. An endpoint z of I with $z \in I$ will be called closed endpoint. We will sometimes write $\mathbf{P}_x$ and $\mathbf{E}_x$ instead of $\mathbf{P}\{\cdot \mid X_0 = x\}$ and $\mathbf{E}(\cdot \mid X_0 = x)$ respectively. For later use we introduce the following classification of closed boundary points. A closed endpoint z of I is called a reflecting boundary point if for some $y \in \text{int}(I)$ (and hence for all $y \in I$) it holds that $\mathbf{P}\{T_y < \infty \mid X_0 = z\} > 0$ and it is called an absorbing boundary point if $\mathbf{P}\{T_y < \infty \mid X_0 = z\} = 0$ holds for all $y \in \text{int}(I)$.

This paper has the following structure. In Section 2 we use the approach of Itô and McKean [3, Section 4.6]) and construct suitable functions ψ for which $e^{-A_t} \psi(X_t)$ are local martingales. Section 3 contains our main results. In Section 4 we present further examples. The proofs of the results in Section 3 are given in Section 5, and Section 6 covers briefly some extensions.

2. The functions ψ_+ and ψ_-

In this section we follow the ideas of Section 4.6 in [3]. The proofs are similar to the arguments there and in Section V.46 of [6]. We therefore omit them. Let x_0 denote some point in $\text{int}(I)$. We then define the functions ψ_+ and ψ_- on $\text{int}(I)$ by (compare [3, p. 128–129, equations 2a) and 2b)])

$$\psi_+(x) = \begin{cases} \mathbf{E}_x \left(e^{-A(T_{x_0})} 1_{\{T_{x_0} < \infty\}} \right) & \text{for } x \leq x_0, \\ \left[\mathbf{E}_{x_0} \left(e^{-A(T_x)} 1_{\{T_x < \infty\}} \right) \right]^{-1} & \text{for } x \geq x_0, \end{cases} \quad (3)$$

and

$$\psi_-(x) = \begin{cases} \left[\mathbf{E}_{x_0} \left(e^{-A(T_x)} 1_{\{T_x < \infty\}} \right) \right]^{-1} & \text{for } x \leq x_0, \\ \mathbf{E}_x \left(e^{-A(T_{x_0})} 1_{\{T_{x_0} < \infty\}} \right) & \text{for } x \geq x_0. \end{cases} \quad (4)$$

The values at closed endpoints of I can be computed by using the continuity properties of ψ_+ and ψ_- . Let

$$M_t^{(+)} = e^{-A(t)} \psi_+(X_t) \quad \text{and} \quad M_t^{(-)} = e^{-A(t)} \psi_-(X_t).$$

Lemma 1. $M_t^{(+)}$ is a uniformly integrable $\mathbf{P}_x$ -martingale on $0 \leq t \leq T_b$ for all $b \geq x$ and $M_t^{(-)}$ is a uniformly integrable $\mathbf{P}_x$ -martingale on $0 \leq t \leq T_a$ for all $a \leq x$. For any $x \leq b$ we have

$$\mathbf{E}_x \left(M_{T_b}^{(+)} 1_{\{T_b < \infty\}} \right) = \psi_+(x)$$

and for any $x \geq a$ we have

$$\mathbf{E}_x \left(M_{T_a}^{(-)} 1_{\{T_a < \infty\}} \right) = \psi_-(x).$$

Clearly $\psi_+(a) > 0$ if a is a reflecting left boundary point and $\psi_+(a) = 0$ if a is an absorbing left boundary point. Also $\psi_-(b) > 0$ if b is a reflecting right boundary point and $\psi_-(b) = 0$ if b is an absorbing right boundary point. Moreover $\psi_+(b) > 0$ for any closed right endpoint b and $\psi_-(a) > 0$ for any closed left endpoint. Moreover:

Lemma 2. The functions ψ_+ and ψ_- are continuous and strictly positive on $\text{int}(I)$. If a is a closed left endpoint of I , then

$$\lim_{x \downarrow a} \psi_+(x) = \psi_+(a) \quad \text{and} \quad \lim_{x \downarrow a} \psi_-(x) = \psi_-(a).$$

If b is a closed right endpoint of I , then

$$\lim_{x \uparrow b} \psi_+(x) = \psi_+(b) \quad \text{and} \quad \lim_{x \uparrow b} \psi_-(x) = \psi_-(b).$$

The function ψ_+ is nondecreasing on I and ψ_- is nonincreasing on I .

When $A_t = \int_0^t r(X_s) ds$ is given with some non-negative measurable function r on I , we may use the Feynman-Kac formula to determine the functions ψ_+ and ψ_- (see [11, Section 12.3]). When $\mathcal{G}$ denotes the infinitesimal operator of X , then one has to solve the equation $\mathcal{G}\psi(x) = r(x)\psi(x)$ subject to the appropriate boundary conditions.

3. Optimal stopping

Let us first assume that for all $x \in I$ holds $\psi_+(x) > 0$ and $\psi_-(x) > 0$. This means that we have no absorbing boundary points. We will extend our results to absorbing boundaries in Section 6. Theorem 1 below corresponds to the case, where the optimal payoff is infinite. Theorem 2 essentially covers the case, where the optimal stopping rule is «one-sided» whereas Theorem 3 treats the «two-sided» case.

We fix the starting point X_0 of the process X at some point $x_0 \in I$. Without loss of generality we may assume that the functions ψ_+ and ψ_- are

standardized in such a way that $\psi_+(x_0) = \psi_-(x_0) = 1$. We may write the payoff as

$$e^{-At} h(X_t) = e^{-At} (p\psi_+(X_t) + (1-p)\psi_-(X_t)) \frac{h(X_t)}{p\psi_+(X_t) + (1-p)\psi_-(X_t)} \quad (5)$$

for any $p \in [0, 1]$ and $0 \leq t < \infty$. The process $e^{-At} (p\psi_+(X_t) + (1-p)\psi_-(X_t))$ is a positive local martingale and hence a supermartingale. This yields for any stopping time T of X

$$\mathbf{E}_{x_0} \left(e^{-AT} (p\psi_+(X_T) + (1-p)\psi_-(X_T)) 1_{\{T < \infty\}} \right) \leq 1. \quad (6)$$

The problem of maximizing

$$\mathbf{E}_{x_0} \left(e^{-AT} h(X_T) 1_{\{T < \infty\}} \right)$$

over all stopping times T is equivalent to the problem of maximizing

$$\frac{h(x)}{p\psi_+(x) + (1-p)\psi_-(x)}$$

over all $x \in I$ for a proper choice of p . We have to consider five different cases. It is easy to see that our cases cover all possible functions h . However, they are not exclusive.

i) $\sup_{x \geq x_0, x \in I} [h(x)/\psi_+(x)] = +\infty$. Then $p = 1$ is a proper choice and Theorem 1 applies.

ii) $\sup_{x \leq x_0, x \in I} [h(x)/\psi_-(x)] = +\infty$. Then $p = 0$ is a proper choice and Theorem 1 applies.

iii) $0 < C^* = \sup_{x \in I} [h(x)/\psi_+(x)] = \sup_{x \geq x_0, x \in I} [h(x)/\psi_+(x)] < \infty$. Then $p = 1$ is a proper choice and Theorem 2 i) applies.

iv) $0 < C^* = \sup_{x \in I} [h(x)/\psi_-(x)] = \sup_{x \leq x_0, x \in I} [h(x)/\psi_-(x)] < \infty$. Then $p = 0$ is a proper choice and Theorem 2 ii) applies.

v) $0 < \sup_{x \geq x_0, x \in I} [h(x)/\psi_+(x)] < \infty$ and $0 < \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)} < \infty$, and at the same time

$$\sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_+(x)} > \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} \quad \text{and} \quad \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_-(x)} > \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)}.$$

Similar arguments as in [1] then provide the existence of a $p^* \in (0, 1)$ such that

$$\begin{aligned} & \sup_{x \geq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)} \\ &= \sup_{x \leq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)}. \end{aligned} \quad (7)$$

Now $p = p^*$ is a proper choice and Theorem 3 applies.

The following elementary example illustrates all possibilities. Let X denote the three-dimensional Bessel process and put $X_0 = 1$. We have

for $x, y \in (0, \infty)$ that $\mathbf{P}_x\{T_y < \infty\} = \min\{(y/x), 1\}$. Let $A \equiv 0$. Then $\psi_+(x) = 1$ and $\psi_-(x) = 1/x$.

The cases i) and iii) are trivial.

Case ii). If $h_2(x) = 1/x^2$, then $\sup_{0 < x \leq 1} [h_2(x)/\psi_-(x)] = +\infty$.

Case iv). If $h_4(x) = (1-x)^+$, then $\sup_{x > 0} [h_4(x)/\psi_-(x)] = \sup_{0 < x \leq 1} [h_4(x)/\psi_-(x)] = \frac{1}{4}$. The supremum is assumed at $x = \frac{1}{2}$.

Case v). If

$$h_5(x) = \begin{cases} 1 - x^2 & \text{for } 0 < x \leq 1, \\ \frac{1}{x} - \frac{1}{x^3} & \text{for } 1 < x < \infty, \end{cases}$$

then

$$0 < \sup_{x \geq 1} \frac{h_5(x)}{\psi_+(x)} = \frac{2}{3\sqrt{3}} = \sup_{0 < x \leq 1} \frac{h_5(x)}{\psi_-(x)},$$

and at the same time

$$\sup_{0 < x \leq 1} \frac{h_5(x)}{\psi_+(x)} = 1 = \sup_{x \geq 1} \frac{h_5(x)}{\psi_-(x)}.$$

Moreover $xh_5(x)/[1+x] = x(x-1)$ for $x \in (0, 1]$ and $xh_5(x)/[1+x] = (1/x)(1-1/x)$ for $x \in (1, \infty)$. Therefore

$$\sup_{x \geq 1} \frac{h_5(x)}{0.5\psi_+(x) + 0.5\psi_-(x)} = 0.5 = \sup_{0 < x \leq 1} \frac{h_5(x)}{0.5\psi_+(x) + 0.5\psi_-(x)}.$$

So, $p^* = 0.5$. Note that the supremum is attained at $x = 0.5$ and $x = 2$ respectively.

Theorem 1. If

$$\sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} = +\infty \quad \text{or} \quad \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)} = +\infty,$$

then

$$\sup_T \mathbf{E}_{x_0} \left(e^{-AT} h(X_T) 1_{\{T < \infty\}} \right) = +\infty.$$

Theorem 2. i) If

$$0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_+(x)} = \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} < \infty,$$

then

$$\sup_T \mathbf{E}_{x_0} \left\{ e^{-rT} h(X_T) 1_{\{T < \infty\}} \right\} = C^*. \quad (8)$$

If there exists a point $x^* \geq x_0$ with $C^* = h(x^*)/\psi_+(x^*)$, then the supremum in (8) is attained for T^* with $T^* = \inf\{t \geq 0 \mid X_t = x^*\}$.

ii) If

$$0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)} < \infty,$$

then

$$\sup_T \mathbf{E}_{x_0} \left\{ e^{-rT} h(X_T) 1_{\{T < \infty\}} \right\} = C^*. \quad (9)$$

If there exists a point $x^* \leq x_0$ with $C^* = h(x^*)/\psi_-(x^*)$, then the supremum in (9) is attained for T^* with $T^* = \inf\{t \geq 0 \mid X_t = x^*\}$.

Theorem 3. Let p^* be such that

$$\begin{aligned} 0 &< \sup_{x \geq x_0, x \in I} \frac{h(x)}{p^* \psi_+(x) + (1 - p^*) \psi_-(x)} \\ &= \sup_{x \leq x_0, x \in I} \frac{h(x)}{p^* \psi_+(x) + (1 - p^*) \psi_-(x)} = C^* < \infty, \end{aligned}$$

then

$$\sup_T \left\{ \mathbf{E}_{x_0} e^{-rT} h(X_T) 1_{\{T < \infty\}} \right\} = C^*. \quad (10)$$

If there exist points $x_1 > x_0$ and $x_2 < x_0$ such that

$$\frac{h(x_1)}{p^* \psi_+(x_1) + (1 - p^*) \psi_-(x_1)} = \frac{h(x_2)}{p^* \psi_+(x_2) + (1 - p^*) \psi_-(x_2)} = C^*,$$

then the supremum in (10) is attained for $T^* = \inf\{t > 0 \mid X_t = x_1 \text{ or } X_t = x_2\}$.

4. Further examples

Let B denote standard Brownian motion throughout this section.

4.1. Standard Brownian motion — state-dependent discounting. Put $X = B$ and $x_0 = 0$. Let $r(\cdot)$ be a non-negative measurable function on $\mathbf{R}$. The function $\phi(x, y) = \mathbf{E}_x \exp\{-\int_0^{T_y} r(X_s) ds\}$ is closely related to the Feynman-Kac functional $\exp(-\int_0^t r(X_s) ds)$ and the stationary Schrödinger equation $\frac{1}{2} \psi'' = r\psi$.

a) We have already discussed the particular example $r(x) = r 1_{[0, \infty)}(x)$ for some positive constant r . The solution we have presented in the introduction can be viewed as a special case of our general theorems.

b) Let $r(x) = rx^2$ for some constant $r > 0$. Put

$$\psi(x) = e^{-x^2/4} \frac{2^{5/4}}{\Gamma(1/2)} \int_0^{+\infty} e^{xt-t^2/2} \frac{1}{\sqrt{t}} dt.$$

In the notation of [4, p. 91–94] $\psi(x)$ equals $2^{5/4} D_{-1/2}(-x)$; the function $D_{-1/2}$ is usually called Weber function (see also [8, Section 3]). The function ψ solves $\psi'' = \frac{1}{4} x^2 \psi$ and we have $\psi(0) = 1$. Hence $\psi(\sqrt{8/r}x)$ is a solution of $\frac{1}{2} \psi'' = rx^2 \psi$. We have $\psi \geq 0$ and $\lim_{x \rightarrow -\infty} \psi(x) = 0$. This yields that ψ is bounded on the interval $(-\infty, y]$ for any $x \in \mathbf{R}$.

If we apply the optional stopping theorem to the local martingale $\exp\{-r \int_0^t X_s^2 ds\} \psi(\sqrt{8/r} X_t)$, we obtain for $y \geq 0$

$$\begin{aligned} \mathbf{E}_0 \exp \left\{ -r \int_0^{T_y} X_s^2 ds \right\} &= \frac{1}{\psi(\sqrt{8/r} y)} \quad \text{and} \\ \mathbf{E}_x \exp \left\{ -r \int_0^{T_0} X_s^2 ds \right\} &= \psi \left(\sqrt{\frac{8}{r}} x \right) \end{aligned}$$

for $x \leq 0$. According to (4) this provides $\psi_+(x) = \psi((8/r)^{1/2} x)$. Suppose now we are interested in maximizing

$$\mathbf{E}_0 \left(\exp \left\{ -r \int_0^T X_s^2 ds \right\} (X_T^+)^{\alpha} 1_{\{T < \infty\}} \right)$$

for some $\alpha > 0$. We have $\lim_{x \rightarrow +\infty} [\sqrt{x} \psi(x)/e^{x^2/4}] = K$ for some strictly positive constant K . This provides $\sup_{x \in \mathbf{R}} [(x^+)^{\alpha}/\psi_+(x)] = \sup_{x \geq 0} [(x^+)^{\alpha}/\psi_+(x)] < \infty$. Theorem 2 i) yields that $T^* = \inf\{t \geq 0 \mid X_t = x^*\}$, where x^* is the positive solution of the transcendental equation $\alpha \psi_+(x) = x \psi'_+(x)$.

4.2. Standard Brownian motion — discounting with local time. Put $X = B$ and $x_0 = 0$. Let L denote the local time of B at zero. We will now discuss the problem of maximizing $\mathbf{E}_0(e^{-rL_T} (X_T^+)^{\alpha} 1_{\{T < \infty\}})$ for some $\alpha, r > 0$. Let

$$\psi(x) = \begin{cases} 1 + rx & \text{for } x \geq 0, \\ 1 & \text{for } x < 0. \end{cases}$$

The generalized Itô formula (see [6, IV.45.2]) yields $d[e^{-rL_t} \psi(X_t)] = re^{-rL_t} dX_t$. Therefore $e^{-rL_t}(1 + rX_t^+)$ is a positive local martingale. We have $|e^{-rL_t}(1 + rX_t^+)| \leq 1 + rx$ on $0 \leq t \leq T_x$ for any $x \geq 0$ and hence we obtain $\mathbf{E}_0 \exp\{-rL_{T_x}\} = 1/(1 + rx)$ for $x \geq 0$. For $x < 0$ it holds that $\mathbf{P}_x\{L_{T_0} = 0\} = 1$. This yields together $\psi_+(x) = \psi(x)$.

a) If $\alpha > 1$, we obtain $\sup_{x \geq 0} [(x^+)^{\alpha}/\psi_+(x)] = +\infty$, so Theorem 1 yields

$$\sup_T \mathbf{E}_0(e^{-rL_T} (X_T^+)^{\alpha} 1_{\{T < \infty\}}) = +\infty.$$

b) If $\alpha = 1$, we obtain $\sup_{x \geq 0} [(x^+)^{\alpha}/\psi_+(x)] = 1/r$ and therefore

$$\sup_T \mathbf{E}_0(e^{-rL_T} (X_T^+)^{\alpha} 1_{\{T < \infty\}}) = \frac{1}{r}.$$

The function $x/(1 + rx)$ does not attain its supremum over $[0, +\infty)$ at some finite point and so there exists no optimal stopping rule.

c) If $0 < \alpha < 1$, we obtain

$$\sup_{x \geq 0} \frac{(x^+)^{\alpha}}{\psi_+(x)} = (1 - \alpha) \left(\frac{\alpha}{1 - \alpha} \right)^{\alpha} \left(\frac{1}{r} \right)^{\alpha} = C^*$$

and this supremum is attained at $x^* = \alpha/[r(1 - \alpha)]$. Theorem 2 i) implies

$$\sup_T \mathbf{E}_0 \left(e^{-rL_T} (X_T^+)^{\alpha} 1_{\{T < \infty\}} \right) = C^* = \mathbf{E}_0 \left(e^{-rL_{T^*}} (X_{T^*}^+)^{\alpha} 1_{\{T^* < \infty\}} \right),$$

where $T^* = \inf \{t > 0 \mid X_t = \alpha/[(1 - \alpha)r]\}$.

4.3. Russian options. Let $\mu \in \mathbf{R}$ and $\sigma > 0$. Let

$$S_t = \exp \left\{ \sigma B_t + \left(\mu - \frac{\sigma^2}{2} \right) t \right\}$$

and $M_t = \max_{0 \leq s \leq t} S_s$. Then $dS_t = \mu S_t dt + \sigma S_t dB_t$. Let $r > \mu$. We now consider the problem of maximizing $\mathbf{E}(e^{-rT} M_T 1_{\{T < \infty\}})$ over all stopping times T of S . Shepp and Shiryaev [9] proved the following result.

Let

$$\gamma_{1,2} = - \left(\frac{\mu}{\sigma^2} - \frac{1}{2} \right) \mp \sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2} \quad \text{and} \quad \alpha = \left(\frac{1 - 1/\gamma_1}{1 - 1/\gamma_2} \right)^{1/(\gamma_2 - \gamma_1)}.$$

Then

$$\sup_T \mathbf{E}(e^{-rT} M_T) = \mathbf{E}(e^{-rT^*} M_{T^*}),$$

where $T^* = \inf \{t > 0 \mid M_t/S_t = \alpha\}$.

To obtain this result we first follow the approach of Shepp and Shiryaev [10] and rewrite the problem appropriately. Let $X_t = M_t/S_t$. Let $\tilde{\mathbf{P}}$ denote the probability measure given by

$$\left. \frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} \right|_{\sigma(B_s, 0 \leq s \leq t)} = e^{\sigma B_t - (\sigma^2/2) B_t} = e^{-\mu t} S_t.$$

Let $\tilde{\mathbf{E}}$ denote the expectation with respect to $\tilde{\mathbf{P}}$. Then for all stopping times T of S

$$\begin{aligned} \mathbf{E} \left(e^{-rT} M_T 1_{\{T < \infty\}} \right) &= \mathbf{E} \left(e^{-(r-\mu)T} e^{-\mu T} S_T X_T 1_{\{T < \infty\}} \right) \\ &= \tilde{\mathbf{E}} \left(e^{-(r-\mu)T} X_T 1_{\{T < \infty\}} \right). \end{aligned}$$

The process X is a regular diffusion under $\tilde{\mathbf{P}}$ with state space $[1, \infty)$ and instantaneous reflection at 1. The infinitesimal generator of X in $(1, \infty)$ is equal to

$$\frac{\sigma^2}{2} x^2 \frac{d}{dx^2} - \mu x \frac{d}{dx}.$$

This means that X behaves like an exponential Brownian motion with drift $-\mu$, diffusion coefficient σ and instantaneous reflection at 1. To maximize $\tilde{\mathbf{E}}(e^{-(r-\mu)T} X_T 1_{\{T < \infty\}})$ over all stopping times T of X , let

$$H(x) = \frac{\eta_2}{\eta_2 - \eta_1} x^{\eta_1} - \frac{\eta_1}{\eta_2 - \eta_1} x^{\eta_2}.$$

Here $\eta_1 < 0$ and $\eta_2 > 1$ are the two roots of

$$\frac{\sigma^2}{2} x^2 - \left(\mu + \frac{\sigma^2}{2} \right) x = (r - \mu).$$

Note that $\eta_{1,2} = 1 - \gamma_{2,1}$. Then

$$\frac{\sigma^2}{2} x^2 H''(x) - \left(\mu + \frac{\sigma^2}{2} \right) H'(x) = (r - \mu) H(x)$$

and $H'(1) = 0$. Moreover $H(x) \geq 0$ for $x \geq 1$ and $H(1) = 1$. So, $\exp\{-(r - \mu)t\} H(X_t)$ is a positive locale martingale. This yields for $a \geq 1$ that

$$\tilde{\mathbf{E}}\left(e^{-(r-\mu)T_a} 1_{\{T_a < \infty\}}\right) = \tilde{\mathbf{E}}\left(e^{-(r-\mu)T_a} H(X_{T_a}) \frac{1}{H(X_{T_a})} 1_{\{T_a < \infty\}}\right) = \frac{1}{H(a)}.$$

Note that $\tilde{\mathbf{P}}\{T_a < \infty\} = 1$ for all $a > 1$ (see [10]) and $0 \leq H(X_t) \leq \sup_{1 \leq x \leq a} H(x) < \infty$ for $0 \leq t \leq T_a$. The function $x/H(x)$ assumes its maximum over $[1, \infty)$ uniquely at $x^* = \alpha$. Therefore $T^* = T_{x^*}$. Theorem 2 i) now yields the assertion with $\psi_+(x) = H(x)$.

5. Proofs

P r o o f o f T h e o r e m 1. Suppose $\sup_{x \geq x_0, x \in I} [h(x)/\psi_+(x)] = +\infty$. Then there exists a sequence $x_n \geq x_0$ with $\lim_{n \rightarrow \infty} [h(x_n)/\psi_+(x_n)] = +\infty$. Let $S_n = T_{x_n}$. Lemma 1 and (5) yield

$$\mathbf{E}_{x_0}\left(e^{-A(S_n)} h(X_{S_n}) 1_{\{S_n < \infty\}}\right) = \frac{h(x_n)}{\psi_+(x_n)}.$$

Hence

$$\sup_T \mathbf{E}_{x_0}\left(e^{-AT} h(X_T) 1_{\{T < \infty\}}\right) \geq \sup_n \frac{h(x_n)}{\psi_+(x_n)} = +\infty.$$

A similar argument applies to the case $\sup_{x \leq x_0, x \in I} [h(x)/\psi_-(x)] = +\infty$.

P r o o f o f T h e o r e m 2 i). There exists a sequence $x_n \geq x_0$ with

$$\lim_{n \rightarrow \infty} \frac{h(x_n)}{\psi_+(x_n)} = \sup_{x \in I} \frac{h(x)}{\psi_+(x)}.$$

Let $S_n = T_{x_n}$. Lemma 1 and (5) yield

$$\mathbf{E}_{x_0}\left(e^{-A(S_n)} h(X_{S_n}) 1_{\{S_n < \infty\}}\right) = \frac{h(x_n)}{\psi_+(x_n)}.$$

Hence

$$\sup_T \mathbf{E}_{x_0}\left(e^{-AT} h(X_T) 1_{\{T < \infty\}}\right) \geq \sup_n \frac{h(x_n)}{\psi_+(x_n)} = \sup_{x \in I} \frac{h(x)}{\psi_+(x)}.$$

On the other hand (6) yields for any stopping time T of X that

$$\mathbf{E}_{x_0}\left(e^{-AT} h(X_T) 1_{\{T < \infty\}}\right) = \mathbf{E}_{x_0}\left(M_T^{(+)} \frac{h(X_T)}{\psi_+(X_T)} 1_{\{T < \infty\}}\right) \leq \sup_{x \in I} \frac{h(x)}{\psi_+(x)}.$$

Now suppose $C^* = h(x^*)/\psi_+(x^*)$ for some $x^* \geq x_0$. Lemma 1 yields for the stopping time $T^* = \inf\{t > 0 \mid X_t = x^*\}$ that

$$\mathbf{E}_{x_0} \left(e^{-\tau T^*} h(X_{T^*}) 1_{\{T^* < \infty\}} \right) = C^* \mathbf{E}_{x_0} M_{T^*}^{(+)} = C^*.$$

The proof of Theorem 2 ii) is similar to the proof of Theorem 2 i).

P r o o f o f T h e o r e m 3. There exist sequences $x_n^{(+)} \geq x_0$ and $x_n^{(-)} \leq x_0$ such that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{h(x_n^{(+)})}{p^* \psi_+(x_n^{(+)}) + (1 - p^*) \psi_-(x_n^{(+)})} \\ &= C^* = \lim_{n \rightarrow \infty} \frac{h(x_n^{(-)})}{p^* \psi_+(x_n^{(-)}) + (1 - p^*) \psi_-(x_n^{(-)})}. \end{aligned}$$

Let S_n denote the stopping time $S_n = \inf\{t \geq 0 \mid X_t = x_n^{(+)} \text{ or } X_t = x_n^{(-)}\}$. Obviously we have $S_n \leq \inf\{t \geq 0 \mid X_t = x_n^{(+)}\}$ and $S_n \leq \inf\{t \geq 0 \mid X_t = x_n^{(-)}\}$ and so Lemma 1 yields $\mathbf{E}_{x_0}[(p^* M_{S_n}^{(+)} + (1 - p^*) M_{S_n}^{(-)}) 1_{\{S_n < \infty\}}] = 1$. Hence

$$\begin{aligned} & \mathbf{E}_{x_0} \left(e^{-A(S_n)} h(X_{S_n}) 1_{\{S_n < \infty\}} \right) \\ & \geq \min \left\{ \frac{h(x_n^{(+)})}{p^* \psi_+(x_n^{(+)}) + (1 - p^*) \psi_-(x_n^{(+)})}, \right. \\ & \quad \left. \frac{h(x_n^{(-)})}{p^* \psi_+(x_n^{(-)}) + (1 - p^*) \psi_-(x_n^{(-)})} \right\} \end{aligned}$$

and consequently $\sup_T \mathbf{E}_{x_0} (e^{-A(T)} h(X_T) 1_{\{T < \infty\}}) \geq C^*$. On the other hand (6) yields

$$\sup_T \mathbf{E}_{x_0} \left(e^{-A(T)} h(X_T) 1_{\{T < \infty\}} \right) \leq \sup_{x \in I} \frac{h(x)}{p^* \psi_+(x) + (1 - p^*) \psi_-(x)} = C^*.$$

6. Extensions

6.1. Absorbing boundaries. We can also treat the case of absorbing boundaries. Let us suppose that a is a left absorbing boundary point and that we have no right absorbing boundary point. The cases

- of a right absorbing boundary point but no left absorbing boundary point,
- of a right absorbing boundary point and a left absorbing boundary point

can be treated in a similar fashion.

Now $\psi_+(a) = 0$ and $\psi_-(x) > 0$ for all $x \in I$. The case $x_0 = a$ is trivial. Let $x_0 > a$. The situation, where Theorem 2 ii) applies requires no changes. If $h(a) = 0$, one simply has to replace $\sup_{x \leq x_0, x \in I} [h(x)/\psi_+(x)]$ and $\sup_{x \in I} [h(x)/\psi_+(x)]$ by $\sup_{a < x \leq x_0, x \in I} [h(x)/\psi_+(x)]$ and $\sup_{x \in I, x > a} [h(x)/\psi_+(x)]$ respectively. Then one can apply Theorem 2 i) and Theorem 3. Therefore we assume

$$\sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_-(x)} > \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)} \quad (11)$$

and $h(a) > 0$.

We now have to look at two different cases separately. Let $\tilde{h}(x) = h(x) 1_{\{x > a\}}$. It holds for any stopping time T that

$$\mathbf{E}_{x_0} \left(e^{-A_T} h(X_T) 1_{\{T < \infty\}} \right) \geq \mathbf{E}_{x_0} \left(e^{-A_T} \tilde{h}(X_T) 1_{\{T < \infty\}} \right).$$

Theorem 1 now yields $\sup_T \mathbf{E}_{x_0} (e^{-A_T} h(X_T) 1_{\{T < \infty\}}) = +\infty$ if

$$\sup_{x \geq x_0, x \in I} \frac{\tilde{h}(x)}{\psi_+(x)} = \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} = +\infty.$$

If $\sup_{x \geq x_0, x \in I} [h(x)/\psi_+(x)] < \infty$, we obtain together with (11) the existence of some $p^* \in (0, 1)$ satisfying (7). That means we are now in the situation covered by Theorem 3. It is intuitively clear that Theorem 2 i) cannot apply if $h(a) > 0$ since for any $x > x_0$ we have

$$\mathbf{E}_{x_0} \left(e^{A(T_x)} h(X_{T_x}) 1_{\{T_x < \infty\}} \right) < \mathbf{E}_{x_0} \left(e^{A(T_x \wedge T_a)} h(X_{T_x \wedge T_a}) 1_{\{T_x \wedge T_a < \infty\}} \right).$$

6.2. Terminal times. We can also cover situations, where the process X is eventually killed. Let ξ be a terminal time of X ; that is a stopping time of X with the property that $\xi = s + \xi \circ \theta_s$ on the event $\{\xi > s\}$. After possibly switching to a smaller state space, we may assume that $\mathbf{P}\{T_y < \xi \mid X_0 = x\} > 0$ holds for all $y \in I$ and $x \in \text{int}(I)$. Our method can be adapted to deal with the problem of finding a stopping time T that maximizes

$$\mathbf{E} \left(e^{-A(T)} h(X_T) 1_{\{T < \xi\}} \right).$$

In this case the functions ψ_+ and ψ_- in (3) and (4) have to be replaced by

$$\psi_+(x) = \begin{cases} \mathbf{E}_x \left(e^{-A(T_{x_0})} 1_{\{T_{x_0} < \xi\}} \right) & \text{for } x \leq x_0, \\ \left[\mathbf{E}_{x_0} \left(e^{-A(T_x)} 1_{\{T_x < \xi\}} \right) \right]^{-1} & \text{for } x \geq x_0, \end{cases} \quad (12)$$

and

$$\psi_-(x) = \begin{cases} \mathbf{E}_{x_0} \left[\left(e^{-A(T_x)} 1_{\{T_x < \xi\}} \right) \right]^{-1} & \text{for } x \leq x_0, \\ \mathbf{E}_x \left(e^{-A(T_{x_0})} 1_{\{T_{x_0} < \xi\}} \right) & \text{for } x \geq x_0. \end{cases} \quad (13)$$

6.3. Discounting with functionals which assume negative values. One might also think of applying our approach to problems, where A assumes positive and negative values. For a particular example, where A_t changes sign, see Section 2.6 in [1]. In this situation we can still follow the arguments in Section 2 and construct the functions ψ_+ and ψ_- . Under suitable integrability conditions, like $\mathbf{E}_x(e^{-A(T_y)} 1_{\{T_y < \infty\}}) < \infty$ for all x, y the processes M_t^- and M_t^+ will become local martingales and Lemma 1 will hold. Then we are able to give analogues of Theorem 1 and Theorem 2 i) and ii). Unfortunately Lemma 2 will in general fail to hold and so we are not able to prove the existence of a p^* satisfying (7). Therefore we can not give an analogue of Theorem 3 in this case.

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