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ROUGH BOUNDARY TRACE FOR SOLUTIONS OF $Lu = \psi(u)^1$

Dedicated to A. V. Skorokhod on his 70th birthday

Пусть L — эллиптический дифференциальный оператор второго порядка на $\mathbf{R}^d$, и пусть E — ограниченная область в $\mathbf{R}^d$ с гладкой границей ∂E . С каждым положительным решением полулинейного дифференциального уравнения $Lu = \psi(u)$ в E связана пара (Γ, ν) , где Γ — замкнутое подмножество ∂E и ν есть мера Радона на $O = \partial E \setminus \Gamma$. Мы называем эту пару *грубым (rough) следом решения на ∂E* . (В [6] был введен тонкий (thin) след, который позволяет различать решения с одинаковым грубым следом.)

Случай $\psi(u) = u^\alpha$ с $\alpha > 1$ был исследован с помощью различных методов Легаллом (Le Gall), Дынкиным и Кузнецовым, а также Маркусом и Вероном. В настоящей статье мы рассматриваем широкий класс функций ψ и существенно упрощаем доказательства, содержащиеся в наших предыдущих работах.

Ключевые слова и фразы: след решения на границе, умеренные решения, выметание, устранимые и тонкие подмножества границы, стохастические граничные значения, диффузия, ранг супердиффузии.

1. Conditions on L and ψ . Suppose that E is a bounded domain of class $C^{2,\lambda}$ in $\mathbf{R}^d$ and

$$Lu = \sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_i b_i \frac{\partial u}{\partial x^i}$$

is a uniformly elliptic operator with bounded smooth coefficients. We assume that a function $\psi(x, u)$ on $\mathbf{R}^d \times \mathbf{R}_+$ has the following properties.

1.A. For every x , $\psi(x, \cdot)$ is convex and $\psi(x, 0) = 0$, $\psi(x, u) > 0$ for $u > 0$.

1.B. $\psi(x, u)$ is continuously differentiable.

1.C. ψ is locally Lipschitz continuous in u uniformly in x , i.e., for every $c > 0$, there exists a constant $q(c)$ such that

$$|\psi(x; u_1) - \psi(x; u_2)| \leq q(c)|u_1 - u_2| \quad \text{for all } x \in E, u_1, u_2 \in [0, c]. \quad (1.1)$$

1.D. There is a constant a such that $\psi(x, 2u) \leq a\psi(x, u)$ for all x and u .

We study the set $\mathcal{U}$ of all positive functions of class C^2 on E such that

$$Lu = \psi(u) \quad \text{in } E. \quad (1.2)$$

In addition to 1.A–1.C, we suppose that all positive bounded solutions of (1.2) in any subdomain D of E are uniformly bounded. Keller [8] and Osseman [9] have shown that

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this property holds if a certain integral involving ψ is finite. All conditions listed in this section hold for $\psi(x, u) = k(x)u^\alpha$ under mild conditions on $k(x)$ if $\alpha > 1$.

We define a boundary trace of $u \in \mathcal{U}$ by using two tools: (a) trace of moderate solutions; (b) sweeping of solutions.

2. Trace of moderate solutions. Two linear operators play a key role in our investigation: the Green operator G and the Poisson operator K . They can be defined by probabilistic formulae

$$Gf(x) = \Pi_x \int_0^\zeta f(\xi_t) dt, \quad Kf(x) = \Pi_x f(\xi_\zeta),$$

where $\xi = (\xi_t, \Pi_x)$ is the diffusion in E with the generator L and ζ is the first exit time of ξ from E .

Denote by $\mathcal{H}$ the set of all positive solutions of the linear equation $Lu = 0$ in E . We call elements of $\mathcal{H}$ harmonic functions.

A function $u \in \mathcal{U}$ is called a *moderate solution* if $u \leq h$ for some $h \in \mathcal{H}$. Denote by $\mathcal{U}_1$ the set of all moderate solutions. We have proved in [4] that the formula

$$h = u + G\psi(u) \quad (2.1)$$

defines a one-to-one mapping i from $\mathcal{U}_1$ onto a subset $\mathcal{H}_1$ of $\mathcal{H}$. We denote the inverse mapping by j . For every $u \in \mathcal{U}_1$, $h = j(u)$ is the minimal harmonic majorant of u , and, for every $h \in \mathcal{H}_1$, $u = i(h)$ is the maximal element of $\mathcal{U}$ dominated by h . Clearly, mappings i and j are monotone. Moreover, $i(h_1 + h_2) \leq i(h_1) + i(h_2)$ (see Section 4.3 in [6]).

Every $h \in \mathcal{H}$ has a representation

$$h(x) = \int_{\partial E} k(x, y) \nu(dy), \quad (2.2)$$

where $k(x, y)$ is the Poisson kernel and ν is a finite measure on ∂E . Formula (2.2) establishes a one-to-one correspondence between the set of all finite measures on ∂E and $\mathcal{H}$. If ν and h are related by formula (2.2), then we write $h = K\nu$. Put $\nu \in \mathcal{N}_1$ if $K\nu \in \mathcal{H}_1$.

We say that ν is the *trace of a moderate solution* u if

$$u + G\psi(u) = K\nu. \quad (2.3)$$

Formula (2.3) defines a one-to-one correspondence between $\mathcal{N}_1$ and $\mathcal{U}_1$.

3. Sweeping. It is proved in [6] that $\mathcal{U}$ with the partial order $\leq$ is a complete lattice that is, for every subset C of $\mathcal{U}$ there exist $\text{Sup } C$ and $\text{Inf } C$ in $\mathcal{U}$ (writing $v = \text{Sup } C$ means that: (i) $v \in \mathcal{U}$; (ii) $u \leq v$ for all $u \in C$; (iii) if $w \in \mathcal{U}$ and $u \leq w$ for all $u \in C$, then $v \leq w$; writing $v = \text{Inf } C$ has a similar meaning). For every pair $u, v \in \mathcal{U}$, we put $u \vee v = \text{Sup}\{u, v\}$, $u \wedge v = \text{Inf}\{u, v\}$. We proved in [6] that, if $C \subset \mathcal{U}$ is closed under $\vee$, then there exists a sequence $u_n \in C$ such that $u_n(x) \uparrow v(x)$ for all $x \in E$. We have $v(x) = \sup_C u(x)$ for all $x \in E$.

We use this result to introduce a *sweeping* $Q_B(u)$ of $u \in \mathcal{U}$ to a closed subset B of ∂E . We put

$$Q_B(u) = \text{Sup}\{v \in \mathcal{U} : v \leq u, v = 0 \text{ on } \partial E \setminus B\}. \quad (3.1)$$

Operators Q_B have the following properties (see [7, Section 3]).

3.A. $Q_B(u_1) \leq Q_B(u_2)$ for $u_1 \leq u_2$.

3.B. $Q_B(u) \leq u$.

3.C. If $B_1 \supset B_2$, then $Q_{B_1}(u) \geq Q_{B_2}(u)$.

3.D. For every B_1, B_2 , $Q_{B_1 \cup B_2}(u) \leq Q_{B_1}(u) + Q_{B_2}(u)$.

3.E. $Q_{\partial E}(u) = u$.

3.F. If $u \leq u_1 + u_2$, then $Q_B(u) \leq Q_B(u_1) + Q_B(u_2)$.

3.G. If u is a moderate solution with the trace ν , then $v = Q_B(u)$ is a moderate solution with the trace equal to the restriction ν_B of ν to B .

Indeed, 3.A–3.C and 3.E follow easily from the definition. 3.F and 3.G could be established by the arguments similar to those in [7]. Finally, 3.D can be derived from [6, Lemma 5.2].

4. Rough trace. Now we are prepared to define a rough trace of an arbitrary $u \in \mathcal{U}$. We say that a compact set $B \subset \partial E$ is *moderate* for u if the solution $u_B = Q_B(u)$ is moderate. Let ν_B stand for the trace of u_B . By 3.D, union of two moderate sets is moderate. Suppose that B is moderate and let $\tilde{B} \subset B$. By 3.G, $\tilde{B}$ is moderate and $\nu_{\tilde{B}}$ is the restriction of ν_B to $\tilde{B}$.

A relatively open subset A of E' is called *moderate* if all compact subsets of A are moderate. The union O of all moderate open sets is moderate. Clearly, there exists a unique measure ν on O such that its restriction to an arbitrary compact subset B coincides with ν_B .

The measure ν has the property: for every compact $B \subset O$ the restriction of ν to B belongs to the class $\mathcal{M}_1$. We denote by $\mathcal{M}_1(O)$ the class of measures with this property. We call closed set $\Gamma = \partial E \setminus O$ the *special set of the solution u* and we call pair (Γ, ν) the *rough trace of u on ∂E* .

5. Solutions u_ν and w_B . The class $\mathcal{U}$ is closed under pointwise convergence. Moreover, the class of solutions vanishing on a relatively open subset O of ∂E is also closed under pointwise convergence (see, for instance [2, Theorem 1.2]). Suppose that B_n are closed and $B_n \uparrow O$. If $\nu \in \mathcal{M}_1(O)$ and if ν_n is the restriction of ν to B_n , then $\nu_n \in \mathcal{M}_1$ and u_{ν_n} is an increasing sequence. We denote its limit by u_ν . It is easy to see that it does not depend on the choice of B_n and that $u_\mu \leq u_\nu$ if and only if $\mu \leq \nu$.

For every closed $B \subset \partial E$, we put

$$w_B = \sup\{u \in \mathcal{U}: u = 0 \text{ on } \partial E \setminus B\}. \quad (5.1)$$

Note that, for every $x \in E$, $w_B(x)$ is equal to the supremum of $u(x)$ over all $u \in \mathcal{U}$. We have:

5.A. $w_{B_1} \leq w_{B_2}$ if $B_1 \subset B_2$.

5.B. $Q_B(u) \leq w_B$ for all $u \in \mathcal{U}$.

5.C. $Q_B(w_B) = w_B$.

5.D. $w_{B_1 \cup B_2} \leq w_{B_1} + w_{B_2}$.

5.E. If $B_n \downarrow B$, then $w_{B_n} \downarrow w_B$.

5.A and 5.B follow from (3.1). By 5.B, $Q_B(w_B) \leq w_B$. On the other hand, $v = w_B$ satisfies conditions $v \leq w_B$, $v = 0$ on $\partial E \setminus B$ and $Q_B(w_B)$ is a maximal solution with these properties. Hence, $w_B \leq Q_B(w_B)$ which proves 5.C. To prove 5.D, we put $B = B_1 \cup B_2$ and we note that, by 5.C, 3.D and 5.B,

$$w_B = Q_B(w_B) \leq Q_{B_1}(w_B) + Q_{B_2}(w_B) \leq w_{B_1} + w_{B_2}.$$

Let us prove 5.E. Function $w_n = w_{B_n}$ is a maximal element of $\mathcal{U}$ vanishing on $O_n = \partial E \setminus B_n$. By 5.A, $w_n \downarrow v \geq w_B$. The limit v is a solution equal to 0 on $O = \partial E \setminus B$. Hence, $v \leq w_B$.

6. Removable and thin boundary sets. We say that a compact set $B \subset \partial E$ is *removable* if 0 is the only element of $\mathcal{U}$ vanishing on $\partial E \setminus B$. (In the literature, such sets are called removable boundary singularities for solutions of (2.1).) Clearly, B is removable if and only if $w_B = 0$. A set A is called *thin* if all its compact subsets are removable.²⁾ It follows from 5.A and 5.D that:

6.A. All closed subsets of a removable set are removable and all subsets of a thin set are thin.

6.B. The class of all removable sets and the class of all thin sets are closed under the finite unions.

We also have:

6.C. All thin Borel sets are not charged by any $\nu \in \mathcal{M}_1$.

P r o o f. Since $\mathcal{M}_1$ contains, with every ν its restriction to any B , it is sufficient to show that, if $\nu \in \mathcal{M}_1$ is concentrated on a removable compact set B , then $\nu = 0$. The Poisson kernel $k(x, y)$ is bounded on every set $\{|x - y| > \varepsilon\}$, $\varepsilon > 0$, and it tends to 0 as

²⁾ If X is a (L, ψ) -superdiffusion, then, by Theorem 1.2 in [5], a compact set B is removable if and only if $P_x\{\mathcal{R} \cap B \neq \emptyset\} = 0$, where $\mathcal{R}$ is the range of X .

$y \rightarrow \tilde{y} \in \partial E, \tilde{y} \neq y$. Therefore $h = K\nu = 0$ on $\partial E \setminus B$. The solution $i(h)$ satisfies the same condition because $i(h) \leq h$. Therefore $i(h) \leq w_B$. If $w_B = 0$, then $i(h) = 0$. Thus $h = 0$ and $\nu = 0$.

6.D. If compact $B \subset \Gamma$ and if $Q_B(w_\Gamma)$ is moderate, then B is removable.

P r o o f. By 5.A, $w_B \leq w_\Gamma$ and, by 5.C and 3.A, $w_B = Q_B(w_B) \leq Q_B(w_\Gamma)$. If $Q_B(w_\Gamma)$ is moderate, then so is w_B . Hence, $w_B = i(h)$ for some $h \in \mathcal{N}_1$. By Theorem 4.3 in [6], $2h \in \mathcal{N}_1$.³⁾ Since $i(2h) \leq 2i(h)$, we have $v \leq 2w_B$. Hence, $v = 0$ on $\partial E \setminus B$ which implies that $v \leq w_B$. By the monotonicity of j , $2h = j(v) \leq j(w_B) = h$. Therefore $h = 0$ and $w_B = i(h) = 0$.

6.E. Suppose Γ is removable and $B_n = \{x \in \partial E: d(x, \Gamma) \geq \varepsilon_n\}$. If $\varepsilon_n \downarrow 0$, then $Q_{B_n}(u) \uparrow u$ for every $u \in \mathcal{U}$.

P r o o f. Put $\Gamma_n = \{y \in \partial E: d(y, \Gamma) \leq 2\varepsilon_n\}$. Note that $\Gamma_n \cup B_n = \partial E$ and $\Gamma_n \downarrow \Gamma$. By 3.E, 3.D and 5.D, $u = Q_{\partial E}(u) \leq Q_{\Gamma_n}(u) + Q_{B_n}(u)$. By 5.B and 5.E, $Q_{\Gamma_n}(u) \leq w_{\Gamma_n} \downarrow w_\Gamma = 0$.

7. Principal results. We say that x is an *explosion point of a measure ν* and we write $x \in \text{Ex}(\nu)$ if $\nu(U) = \infty$ for every neighborhood U of x . If $B \cap \text{Ex}(\nu) = \emptyset$ and if B is compact, then $\nu(B) < \infty$. Note that $O \cap \text{Ex}(\nu) = \emptyset$ for every measure $\nu \in \mathcal{M}_1(O)$.

We say that (Γ, ν) is a *normal pair* if:

- (a) Γ is a closed subset of ∂E ;
- (b) $\nu \in \mathcal{M}_1(O)$, where $O = \partial E \setminus \Gamma$;
- (c) the conditions

$$\begin{aligned} \Lambda \subset \Gamma & \text{ is thin and contains no explosion points of } \nu, \\ \Gamma \setminus \Lambda & \text{ is closed imply that } \Lambda = \emptyset. \end{aligned} \quad (7.1)$$

We demonstrate that these conditions hold for the trace of an arbitrary solution u . First, we prove a few auxiliary propositions.

7.A. If $u \in \mathcal{U}$ vanishes on a compact set $B \in \partial E$, then $Q_B(u) = 0$.

Indeed, by (3.1), $v = Q_B(u) = 0$ on $\partial E \setminus B$ and $v \leq u$. Hence, $v = 0$ on ∂E , and $v = 0$ by the comparison principle (see, e.g., [1, p. 113]).

7.B. Suppose that $\text{tr}(u) = (\Gamma, \nu)$. If $u = 0$ on an open subset O_1 , then $O_1 \cap \Gamma = \emptyset$ and $\nu(O_1) = 0$.

Indeed, by 7.A, $Q_B(u) = 0$ for all compact subsets of O_1 .

7.C. Let $\text{tr}(u) = (\Gamma, \nu)$. If Γ is removable and ν is finite, then u is moderate.

To prove this, we apply 6.E. Note that $h = K\nu \in \mathcal{H}$ and $u_n = Q_{B_n}(u) \leq K\nu_n \leq h$. By 6.E, $u_n \uparrow u$. Hence, $u \leq h$.

Theorem 7.1. *The trace of an arbitrary solution u is a normal pair.*

P r o o f. Properties (a) and (b) follow immediately from the definition of the trace. Let us prove (c). Suppose that Λ satisfies the conditions (7.1) and let $\Gamma_0 = \Gamma \setminus \Lambda$. Theorem will be proved if we show that $v = Q_{B_1}(u)$ is moderate for every closed subset B_1 of $O_0 = \partial E \setminus \Gamma_0$. Indeed, this implies $O_0 \subset O$ and therefore $\Gamma_0 \supset \Gamma$, $\Lambda = \emptyset$.

Let (Γ_1, ν_1) be the trace of v . By 7.C, it is sufficient to prove that Γ_1 is removable and ν_1 is finite.

By 3.B, $v \leq u$ and, by 3.A, all moderate sets for u are also moderate for v . Hence $\Gamma_1 \subset \Gamma$. By 7.B, $\partial E \setminus B_1 \subset \partial E \setminus \Gamma_1$ because $v = 0$ on $\partial E \setminus B_1$. Hence, $\partial E \setminus B_1$ is moderate for v and it is contained in $O_1 = \partial E \setminus \Gamma_1$. We conclude that $\Gamma_1 \subset B_1 \cap \Gamma \subset \Lambda$. Hence, Γ_1 is removable.

Note that $B_1 \subset O \cup \Lambda$ does not contain explosion points of ν and therefore $\nu(B_1) < \infty$. Since $Q_B(v) = 0$ for $B \cap B_1 = \emptyset$, measure ν_1 vanishes on $\partial E \setminus B_1$. Since $v \leq u$, $\nu_1 \leq \nu$ on $O \subset O_1$. We have

$$\nu_1(O_1) = \nu_1(O_1 \cap B_1) = \nu_1[(O_1 \setminus O) \cap B_1] + \nu_1(O \cap B_1) \leq \nu(O \cap B_1) < \infty$$

because $\Gamma \cap B_1$ is removable and therefore $\nu_1(\Gamma \cap B_1) = 0$ by 6.C. Theorem 7.1 is proved.

For every $u, v \in \mathcal{U}$ we put $u \oplus v = \text{Sup}\{\tilde{u} \in \mathcal{U}: \tilde{u} \leq u + v\}$. Note that $u_\mu \oplus u_\nu = u_{\mu+\nu}$.

³⁾ Here is the only place we use the property 1.D of ψ .

Theorem 7.2. *If (Γ, ν) is a normal pair, then $u = w_\Gamma \oplus u_\nu$ is a solution with the trace (Γ, ν) . Moreover, u is maximal among such solutions.*

Proof. 1°. Let $B \subset O = \partial E \setminus \Gamma$. We claim that $Q_B(u) = u_{\nu_B}$, where ν_B is the restriction of ν to B . Indeed, B is contained in an open subset O_1 of ∂E such that $\overline{O_1} \subset O$. Let ν_1 and ν_2 be the restrictions of ν to O_1 and to $O \setminus O_1$. Note that $u_\nu = u_{\nu_1} \oplus u_{\nu_2}$. By 3.D, $Q_B(u) \leq Q_B(u_\nu) + Q_B(w_\Gamma)$, $Q_B(u_\nu) \leq Q_B(u_{\nu_1}) + Q_B(u_{\nu_2})$. Since ν_2 does not charge a neighborhood of B , $u_{\nu_2}(B) = 0$. The same is true for w_B . By 7.A, $Q_B(w_\Gamma) = Q_B(u_{\nu_2}) = 0$. Hence, $Q_B(u) \leq Q_B(u_{\nu_1})$ and, by 3.A and 3.G, $Q_B(u) = Q_B(u_{\nu_1}) = u_{\nu_B}$.

2°. Denote the trace of u by (Γ_0, ν_0) . It follows from 1° that $O \subset O_0$ and $\nu = \nu_0$ on O_0 . Since ν is concentrated on O , we have $\nu \leq \nu_0$ and therefore $\text{Ex}(\nu) \subset \text{Ex}(\nu_0) \subset \Gamma_0 \subset \Gamma$. Every compact $B \subset \Lambda = \Gamma \setminus \Gamma_0$ is moderate for u (because $B \subset O_0$) and it is removable by 6.D. Thus, Λ is thin. Since $\Lambda \cap \text{Ex}(\nu) = \emptyset$ and $\Gamma \setminus \Lambda = \Gamma_0$ is closed, $\Lambda = \emptyset$ by the definition of a normal pair.

3°. Let us show that an arbitrary solution v with the trace (Γ, ν) is dominated by $u = w_\Gamma \oplus u_\nu$. Since u is the maximal element of $\mathcal{U}$ dominated by $w_\Gamma + u_\nu$, it is sufficient to show that $v \leq w_\Gamma + u_\nu$. Consider compact sets B_n and Γ_n such that $B_n \uparrow O$, $\Gamma_n \downarrow \Gamma$ and $B_n \cup \Gamma_n = \partial E$ for all n . By 3.E and 3.D,

$$v = Q_{\partial E}(v) \leq Q_{B_n}(v) + Q_{\Gamma_n}(v). \quad (7.2)$$

Let ν_n be the restriction of ν to B_n . Note that $Q_{B_n}(v) = u_{\nu_n} \leq u_\nu$. By 5.B, $Q_{\Gamma_n}(v) \leq w_{\Gamma_n}$ and, by 5.E, $w_{\Gamma_n} \downarrow w_\Gamma$. Therefore (7.2) implies that $v \leq u_\nu + w_\Gamma$. Theorem 7.2 is proved.

Remark. If X is a (L, ψ) -superdiffusion, then

$$(w_\Gamma \oplus u_\nu)(x) = -\ln \mathbf{P}_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-Z_\nu}\}, \quad (7.3)$$

where $\mathcal{R}$ is the range of X and Z_ν is the stochastic boundary value of $h = K\nu$ (see [3]).

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