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NEW DEVELOPMENTS ON THE MODIGLIANI–MILLER THEOREM

Знаковая система Модильяни–Миллера (см. [22]) является краеугольным камнем теории корпоративных финансов. Она обеспечивает условия, при которых изменения в структуре капитала предприятия не влияют на его основную стоимость. В связи с финансовым кризисом 2008 года возобновилась дискуссия вокруг уместности применения теоремы Модильяни–Миллера в банковском секторе. В этой статье мы приводим обзор этой теоремы с учетом последних исследований, в которых рассматривались расширения исходной модели. В статье предложена другая формулировка теоремы Модильяни–Миллера с учетом разработок Марковица. При таком подходе, мы рассматриваем случай неявных государственных гарантий, предоставляемых банкам. Наш основной результат показывает, что в этом случае банк не удовлетворяет условиям теоремы Модильяни–Миллера, а именно: банки предпочитают использовать систему финансовых рычагов вместо справедливости.

Ключевые слова и фразы: теорема Модильяни–Миллера, отсутствие арбитража, банки, рычаги, регулирование.

1. Introduction. The capital structure of banks has become the focus of an extended debate among policy-makers, regulators and academics since the 2008 financial crisis. It has become central to the debate on banking regulation since the regulator relies on it to justify his policy of requiring more equity. Indeed, if the theorem holds, banks' overall cost of financing is invariant to leverage. This raises the question on to what extent does Modigliani–Miller theorem hold for banks. The reason comes from the stiff combat carried out by the banking sector to mitigate the Basel regulation consequences on the banks profitability. The pros supports the theory applicability on banks while the cons argue that banks cannot be considered as classical firms.

The supporters of this theorem believe that bank leverage is excessive, and should be curtailed by regulations requiring much larger equity issuance.

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We send the readers to the recent papers [14], [2], [16], [6], [3], and [1] for details. The critics of this theorem believe that the main reason which breaks the Modigliani–Miller theorem is the government interventions (tax shield, deposit insurance, implicit government guarantee) that reward leverage, see [27], [11], [24], [7], [30], [25], [8], [18], [29], and [1]. What can we learn from this on-going debate? It is that the costs and benefits of higher capital requirements for banks still remain uncertain when facing the arguments of the pros vs. the cons (see [16] vs. [8]). Hence, the irrelevancy nature of the theorem applied to the banking sector is not trivial despite the common wisdom that it is an expected or even obvious result. Even if the effect of government’s guarantees to bank’s value has been extensively discussed ([15], [12], [26], etc.), surprisingly few or even no references were made to [22]. Most of these papers are empirical studies and none of them has brought a formal proof likely to close the debate. We propose to fill this gap despite the large number of published works on this issue. As in [1], we analyse the relevancy of this theorem using a theoretical approach based on nonarbitrage reasoning inspired by the seminal article of Modigliani and Miller [22].

Contrary to empirical studies subject to various biases (data, period, economic context, estimation method, etc.), nonarbitrage reasoning encompasses all-at-once the possible trading strategies within the Modigliani–Miller paradigm, where only arbitrageurs can modify the firm’s market value equilibrium. Actually, Modigliani and Miller [22] suppose that «nothing is involved in the way of specific assumptions about investor attitudes or behaviour other than that investors behave consistently and prefer more income to less income, *ceteris paribus*». Nevertheless, it is assumed that arbitrage opportunities may only occur in presence of two firms having distinct values but same income distribution. The natural question is why such investors dismiss opportunities between firms of same value but distinct capital structures? To answer this question, we reformulate the Modigliani–Miller theorem by considering risk-neutral arbitrageurs who do not care of their own risk. The implication of their presence is that firms’ values depend on their capital structures.

In the contrary, in the classical Modigliani–Miller model, the same value is given to two firms having distinct capital structures but the same distribution income and the same risk. In this setting, the risk of a firm is not explicitly defined but it is tacitly assumed that every agent acting on the market measures the risk of a firm in the same manner [22]. We follow this general approach by supposing that each agent of the market can compare two firms through a measure m . For instance, this could be a monetary measure of risk, see [10]. Moreover, we suppose that each agent uses his own indicator ρ to measure his investment risk. This assumption means that agents take care of their investment risk and agrees with the Modigliani–Miller hypoth-

esis even if not explicitly stated. Indeed, otherwise, the Modigliani–Miller theorem is not satisfied anymore as we shall see. The conclusion of our study is that agents acting on the market defined by Modigliani–Miller are indifferent to two firms having same value but different capital structures because they are Markowitz [19] rational in the sense that they obtain the same Sharpe [28] ratio (with respect to ρ).

The paper is organised as follows. We first recall the Modigliani–Miller theorem and formulate a generalised version. We then reformulate the Modigliani–Miller theorem under a generalised Markowitz approach beyond mean-variance framework. This allows us to address the policy-oriented issue of the implicit government guarantees’ impact on the Modigliani–Miller equilibrium.

2. The Modigliani–Miller model.

2.1. Basic notation and definitions. In this section, we rederive the Modigliani–Miller setting [22]. Consider a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Let us define a financial market as a set of firms we call *classical firms* (CF). Let S be the market value of the common shares of a company and let D be the market value of its debt at time $t = 0$. The value V of the firm is defined as $V := S + D$. We denote by X the random income, for each period of the infinite horizon setting on the assets owned by the company before deduction of the debt interest. We assume that $X \geq 0$ and $X \neq 0$. The net income received by the shareholders is given by $Y := X - rD$, where r is the interest rate of the debt D fixed at time 0. Therefore, a firm is characterized at time $t = 0$ by the quadruplet (S, D, X, r) . We say that the portfolio structure of agent’s investment is (S, D) at time $t = 0$ if the initial endowment of the agent’s investment is S and D is the debt the agent faces. This means that the agent must pay the amount rD to the lender of his debt for the next period. Each capital structure (S, D) is associated with a net income, we also denote it by Y . For example, when an investor owns the proportion $\alpha \in [0, 1]$ of a company (S, D, X, r) , his capital structure is $(\alpha S, \alpha D)$ and the associated net income is $Y = \alpha(X - rD)$. In the following, when the firms (respectively, the investments) are designated by the names A, B , etc., the associated quadruplets are denoted by (S_A, D_A, X_A, r_A) , (S_B, D_B, X_B, r_B) , etc. (respectively, the couples (S_A, D_A) , (S_B, D_B) , ...) and the net incomes Y_A, Y_B , etc. Given a random variable x , we denote by $\mathbf{E}x$ the expectation of x under $\mathbf{P}$.

In the classical arbitrage theory, market’s equilibrium is reached when there is no arbitrage opportunity. In the following, we recall the type of arbitrage opportunity which is considered by Modigliani and Miller [22]. It is supposed that an agent (called arbitrageur) may make an arbitrage opportunity between two firms if they have the same profile (i.e., the expected incomes are the same), the risk is the same but they do not have the same

value. To do so, the agent switches his investment from the overvalued firm to the cheapest one so that he obtains a positive net gain.

Definition 2.1. Let us consider a market composed of firms identified by their quadruplet (S, D, X, r) . We say that an agent realizes a *classical* arbitrage opportunity if the following conditions holds:

(1) There exists a firm (S_A, D_A, X_A, r) such that $V_A = S_A + D_A$ is overvalued, i.e., there exists an other firm (S_B, D_B, X_B, r) such that $\mathbf{E} X_A = \mathbf{E} X_B$, the risk of X_B is the same than the risk of X_A but $V_B = S_B + D_B < V_A$.

(2) The agent may partially replicate the firm (S_A, D_A, X_A, r) by investing in the firm (S_B, D_B, X_B, r) and a bond of interest rate r . He obtains a larger expected net income, i.e., he may construct an investment of associated quadruplet $(\alpha S_A, \alpha D_A, \tilde{X}_A, r)$, $\alpha \in [0, 1]$, such that the net income $\tilde{Y}_A$ satisfies $\mathbf{E} \tilde{Y}_A > \alpha \mathbf{E} Y_A$.

The concept of risk does not need to be clearly identified. We only suppose that an agent acting on the market can measure and compare the risk of two companies. In [22], the income X characterizes the risk. In the following, we suppose that the risk of X is measured by each agent by an individual risk indicator $m(X)$.

2.2. The generalization of the Modigliani–Miller theorem. In this section, we propose a generalisation of the Modigliani–Miller theorem [22] by considering the possibility of firms having different leverage ratios or different borrowing costs. Recall that $V = \mathbf{E} X/w$, where $w \geq r$ is the weighted average cost of capital as supposed in [22]. Hence $V \leq \mathbf{E} X/r$, i.e., $\mathbf{E} X \geq rV$. We may recover this property otherwise. Indeed, suppose that $\mathbf{E} X < rV$. With this information, the agent could sell his shares αS , borrow the capital αD and invest the total amount in the risk-free bond of interest rate r . The corresponding portfolio structure is also $(\alpha S, \alpha D)$. The generated net income is $\tilde{Y} = \alpha S r$. Assuming that there is no arbitrage opportunity, the inequality $\mathbf{E} \tilde{Y} > \mathbf{E} Y$ does not hold hence

$$\mathbf{E} X \geq rV. \quad (2.1)$$

Let us first recall the Modigliani–Miller theorem (see proof in [22]).

Theorem 2.1. *Suppose that the financial market is defined by the Modigliani–Miller setting. Suppose that each firm of the market may borrow money at the same riskless interest rate r . Assume that the market does not admit any classical arbitrage opportunity. Then, the value V of a firm (S, D, X, r) is independent of the capital structure (S, D) .*

In practice, we may think that a firm cannot borrow money at the risk-free interest rate r but an interest rate $r_A \geq r$ depending on the firm's capital structure itself. It yields the following proposition (see [22]).

Theorem 2.2. *In the Modigliani–Miller framework, let us consider two firms (S_A, D_A, X_A, r_A) and (S_B, D_B, X_B, r_B) such that X_A and X_B have the same expectation $\mathbf{E} X_A = \mathbf{E} X_B$ and represent the same risk $m(X_A) = m(X_B)$. Suppose that there is no classical arbitrage opportunity. Then, $r_A \geq r_B$ implies that $V_A \leq V_B$. Moreover, if*

$$(V_B - S_A)\mathbf{E} X_A - r_A V_B D_A + (r_B - r_A) D_B S_A > 0,$$

then $V_A = V_B$.

In clear, if r_A differs from r_B , then V_A diverges from V_B . In general, we cannot claim whether $V_A < V_B$ or $V_A = V_B$ since the Modigliani–Miller framework does not make any assumption about the behavior of an agent who may choose to invest between two firms having the same value.

2.3. Effect of risk-neutral agents on the Modigliani–Miller equilibrium. In this section, we generalize the Modigliani–Miller theorem [22] by considering arbitragers who do not take care about their own risk but the expected gain, i.e., risk-neutral agents in portfolio choice theory. By a contradictory argument, we actually see that in the Modigliani–Miller model, risk neutral arbitrageurs cannot exist. Indeed, as shown below, in that case the Modigliani–Miller theorem is no longer valid.

Definition 2.2. We say that two firms (S_A, D_A, X_A, r) and (S_B, D_B, X_B, r) have the same «profile», when X_A and X_B have the same expectation and the same risk¹⁾.

Definition 2.3. In the Modigliani–Miller model, we say that an investor realizes a «risk-neutral arbitrage opportunity», when switching his investment from a firm to another one of same profile and same value but offering a better net income Y whatever the investment's risk is.

Remark 2.1. Recall that in the Modigliani–Miller model such a risk-neutral arbitrage opportunity is forbidden as if agents could not expect better profits when investing in one of the firms rather than the others of same value. Actually, this restrictive assumption might be explained by an agent's behaviour of Markowitz type as developed in the next section. Introducing such a possibility of risk-neutral arbitrage opportunity is another way to say that risk-neutral agents do not give the same equilibrium's value to two firms of same profile if one of the firms offers a better net income, which is rather natural.

Theorem 2.3. *Suppose that, in the Modigliani–Miller setting, the firms (S, D, X, r) of the market which take the same risk may borrow money at the same interest rate $r = r(X)$ depending on their risk such that $\mathbf{E} Y/S > r$*

¹⁾ The risk of a distribution income does not need to be explicitly defined as already mentioned, see [22].

(i.e., $V > \mathbf{E} X/r^2$). Assume that the market does not admit any classical arbitrage opportunity nor risk-neutral arbitrage opportunity. Consider two firms A and B such that $\mathbf{E} X_A = \mathbf{E} X_B$ and $m(X_A) = m(X_B)$. Then, they have the same value V and the same capital structure.

P r o o f. A direct application of Theorem 2.2 implies that firms A and B have the same value V . Moreover, if an agent owns the fraction α of the common shares of a firm A , he obtains the return $Y_A(\alpha) := \alpha(X_A - rD_A)$. If the same agent invests the capital αS_A in the firm B , he obtains the return

$$\hat{Y}_B(\alpha) := \frac{\alpha S_A}{S_B} (X_B - rD_B).$$

Under absence of risk-neutral opportunity, we have $\mathbf{E} Y_A(\alpha) = \mathbf{E} \hat{Y}_B(\alpha)$. We deduce that

$$\frac{S_A}{S_B} (\mathbf{E} X_B - rD_B) = \mathbf{E} X_A - rD_A. \quad (2.2)$$

Since $V = V_A = V_B$, we deduce that $(D_B - D_A)(\mathbf{E} X_A - rV) = 0$ and the conclusion follows. Theorem 2.3 is proved.

This result shows the impact of the presence of risk-neutral arbitrageurs on firms' values. Precisely, contrary to arbitrageurs, risk-neutral arbitrageurs can take advantage of capital structure difference between firms equally valued by arbitrageurs to make profit. Therefore, when risk-neutral arbitrageurs opportunities are realized, two firms of same profile have the same capital structure.

3. A generalized Markowitz approach of the Modigliani–Miller theorem. We suppose that each agent of the market can compare two firms, as supposed in [22], through a measure m . For instance, this could be a monetary measure of risk, see [10]. In the following, we use the notation $X_A \equiv X_B$, when the incomes X_A and X_B are such that $\mathbf{E} X_A = \mathbf{E} X_B$ and $m(X_A) = m(X_B)$, i.e., two firms (S_A, D_A, X_A, r_A) and (S_B, D_B, X_B, r_B) have the same profile. Moreover, when $X_A \equiv X_B$, we suppose that firms benefit from the same interest rate $r_A = r_B$, which depends, in general, on X_A and X_B .

By Theorem 2.3, we may suppose that arbitrageurs in the classical Modigliani–Miller model take care of their own risk when investing in a firm. Indeed, otherwise, two firms having same characteristics would have the same capital structure, which contradicts the Modigliani–Miller theorem. Therefore, we suppose that each agent i of the market may measure the risk of his own investment's return $R = R_i$ by a *risk indicator* function

²⁾ We have already justified this assumption, see also [22].

$\rho(R) = \rho_i(R) \geq 0$. Precisely, we suppose that the risk indicator ρ is defined on the set L^0 of all random variables and satisfies the following properties:

C o n d i t i o n (ρ).

- (i) $\mathbf{E} R \geq 0$ implies that $\rho(R) \geq \rho(R^+)$,
- (ii) $X_A \equiv X_B$ implies that $\rho(X_A + k) = \rho(X_B + k)$ for all $k \in \mathbf{R}$, and $\rho(kX_A) = \rho(kX_B)$ if $k \geq 0$,
- (iii) (a) $\rho(X + k) = \rho(X)$, $k \in \mathbf{R}$ and $\rho(kX) = c(k)\rho(X)$, $k \in \mathbf{R}_+$, or
 (b) $X \leq Y$ implies $\rho(X) \geq \rho(Y)$, where c is a deterministic function.

R e m a r k 3.1. These conditions hold with (iii) (a) and $c(k) = k^2$, when $\rho(X) = \text{var}(X)$ or $\rho(X) = \mathbf{P}(X \leq \mathbf{E} X)$ and $c(k) = 1$, see Lemma 6.1, which implies that (iii) (a) holds. These assumptions with (iii) (b) are satisfied when $\rho(X)$ is a monetary measure of risk which is cash invariant, positively homogeneous and $\rho(X)$ is the positive amount of cash we need to add so that $\rho(X) + X$ is considered as acceptable (see [10]). In this case, observe that (iii) (b) implies (i).

D e f i n i t i o n 3.1. Let R be the random return of an investment and $\rho(R) \geq 0$ be the associated risk. Consider an interest rate r . The r -Sharpe ratio³⁾ is defined as

$$\Theta^r := \frac{\mathbf{E} R - r}{\sqrt{\rho(R)}} 1_{\rho(R) > 0} + \infty 1_{\mathbf{E} R \geq r, \rho(R) = 0} - \infty 1_{\mathbf{E} R < r, \rho(R) = 0}. \quad (3.1)$$

When there is no ambiguity, we note $\theta^r = \theta$.

D e f i n i t i o n 3.2. Let us consider a market composed of firms identified by their quadruplet (S, D, X, r) . We say that an agent realizes an arbitrage opportunity of Markowitz type [19] if the following conditions hold:

1. There exists a firm (S_A, D_A, X_A, r_A) such that $V_A = S_A + D_A$ is overvalued, i.e., there exists an other firm (S_B, D_B, X_B, r_B) such that $X_A \equiv X_B$ but $V_B = S_B + D_B < V_A$.
2. If an agent invests in the firm (S_B, D_B, X_B, r_B) and a bond of interest rate $r = r_A = r_B$, by selling his shares of firm A , he obtains a strictly larger Sharpe ratio $\Theta_B^r > \Theta_A^r$.

This definition is analog to the definition of a classical arbitrage opportunity considered by Modigliani and Miller, where an agent switches his investment from firm A to firm B under condition 1 of Definition 3.2 above provided that he obtains a larger net income instead of a larger Sharpe ratio as stated in condition 2. Our main result of this section is the following.

³⁾ In the Markowitz theory (see [19], [28]) r is the riskless interest rate, $\rho(X) = \text{var}(X) = \sigma^2(X)$, and an efficient portfolio position $(\mathbf{E} R, \sigma(R))$ satisfies $\mathbf{E} R = r + \sqrt{\text{Sh}} \sigma(R)$, where Sh is the so-called Sharpe ratio depending on the market's characteristics. This implies that $\Theta^r = \sqrt{\text{Sh}}$.

Theorem 3.1. *Suppose that, in the Modigliani–Miller setting, the firms (S, D, X, r) of the market may borrow and lend money at the same interest rate $r = r(\mathbf{E} X, m(X))$. Moreover, suppose that firms' values satisfy $V < \mathbf{E} X/r$. Assume that the market does not admit any arbitrage opportunity of Markowitz type. Then, two firms A and B such that $X_A \equiv X_B$ have the same value.*

P r o o f. Consider two firms (S_A, D_A, X_A, r_A) and (S_B, D_B, X_B, r_B) such that $X_A \equiv X_B$. It follows that $r = r_A = r_B$. In the case where $V_A \neq V_B$, suppose without loss of generality that $V_A > V_B$. An agent who invests the part αS_A in firm A obtains the net income $Y_A(\alpha) = \alpha(X_A - rD_A)$. The return is

$$R_A = \frac{Y_A(\alpha)}{\alpha S_A} = \frac{1}{S_A} (X_A - rD_A).$$

First assume that condition (iii) (a) holds. Its risk is given by $\sigma_A = \sqrt{\rho(R_A)} = \sigma \sqrt{c(1/S_A)}$, where $\sigma^2 = \rho(X_A) = \rho(X_B)$ by condition (ii). If the agent sells his shares αS_A , borrows or lends $\alpha(S_B - S_A)$ and invests the total amount αS_B in firm B , the associated net income is $\bar{Y}_B(\alpha) := \alpha(X_B - rD_B) - r\alpha(S_B - S_A)$. The return is

$$R_B = \frac{\bar{Y}_B(\alpha)}{\alpha S_A} = \frac{1}{S_A} (X_B - rD_B - r(S_B - S_A))$$

and its risk is given by $\sigma_B = \sigma_A$. Moreover, $\mathbf{E} \bar{Y}_B(\alpha) - \mathbf{E} Y_A(\alpha) = \alpha r(V_A - V_B) > 0$, i.e., there exists an arbitrage opportunity of Markowitz type, i.e., the Sharpe ratio when investing in firm B , as proposed here, is strictly larger than the one we obtain when investing in firm A . We get a contradiction and finally deduce that $V_A = V_B$.

In the second case, when condition (iii) (b) holds, since $V_A = S_A + D_A > V_B = S_B + D_B$, we deduce that

$$R_A = \frac{1}{S_A} (X_A - rD_A) \leq \frac{1}{S_A} (X_A - rD_B - r(S_B - S_A)).$$

Using conditions (iii) (b) and (ii), we deduce that $\rho(R_A) \geq \rho(R_B)$ and finally, as previously, the Sharpe ratio when investing in firm B is strictly larger than the one we obtain when investing in firm A . We get a contradiction and finally deduce that $V_A = V_B$. Theorem 3.1 is proved.

Theorem 3.1 implies that the value of a firm is independent of its capital structure as claimed in the Modigliani–Miller theorem. Actually, by excluding classical arbitrage opportunities between two firms having the same value $V_A = V_B$, Modigliani and Miller were tacitly supposing that agents are rational in accordance with Markowitz as shown below. We may legitimately wonder if an arbitrage opportunity could appear when $V_A = V_B$, see the previous subsection. Intuitively, such an arbitrage opportunity should not

exist under the Markowitz approach as two firms should be given the same Sharpe ratio by any investor as soon as they have the same value. This is confirmed by Lemma 3.1 when condition (iii) (a) holds with $c(k) = k^2$, for instance $\rho(X) = \text{var}(X)$. This implies that there cannot exist an arbitrage opportunity of Markowitz type between two firms (S_A, D_A, X_A, r_A) and (S_B, D_B, X_B, r_B) if $V_A = V_B$ and $X_A \equiv X_B$. Indeed, in this case, $r_A = r_B$ hence the Sharpe ratios are the same by (3.2) below.

Lemma 3.1. *Suppose that condition (iii) (a) holds with $c(k) = k^2$. Consider an agent who invests a positive capital in a firm (S, D, X, r) and a bond of interest rate $r = r(\mathbf{E}X, m(X))$. Then, the Sharpe ratio is independent of the invested proportions and is given by*

$$\Theta^r = \frac{\mathbf{E}X - rV}{\sqrt{\rho(X)}}. \quad (3.2)$$

P r o o f. Suppose that an agent invests an amount $S' \in [0, S]$ in the firm (S, D, X, r) and invests the amount D' in a bond of interest rate r . The total investment $S' + D'$ is supposed to be strictly positive. Without loss of generality, we may rewrite $S' = \alpha S$ and $D' = \alpha C'$ for some $C' \geq 0$ and $\alpha = S'/S$. The agent obtains the net income

$$\check{Y} := \alpha(X - rD) + \alpha C' r.$$

The return and the excess return are, respectively, given by

$$\check{R} = \frac{\check{Y}}{S' + D'} = \frac{X - rD + rC'}{S + C'}, \quad \mathbf{E} \check{R} - r = \frac{(\mathbf{E}X - rV)}{S + C'}$$

and its risk is $\check{\sigma} = \sqrt{\rho(X)}/(S' + C')$, when condition (iii) (a) holds with $c(k) = k^2$. The conclusion follows. The lemma is proved.

This means that investors do not initiate such a strategy when considering the arbitrage opportunities in terms of Sharpe ratios: increasing return implies increasing the risk. Therefore, we may conjecture that market's participants of the Modigliani–Miller model are Markowitz rational in the generalised sense, in particular risk-averse. Precisely, they design and implement arbitrage opportunities in terms of Sharpe ratios. This is confirmed by the following theorem, which is an immediate consequence of Theorem 2.1 and Theorem 3.1.

Theorem 3.2. *Suppose that each agent measures his investment risk by an individual measure ρ satisfying condition (ρ) . Then, absence of arbitrage opportunities in accordance with Modigliani–Miller [22] is equivalent to absence of arbitrage opportunities of Markowitz type.*

P r o o f. The equivalence is trivial as $X_A \equiv X_B$ implies that $V_A = V_B$ by Theorem 2.1 but also by Theorem 3.1.

4. The generalization of the Modigliani–Miller theorem to guaranteed firms. In this section, we reformulate the Modigliani–Miller theorem by supposing the existence of so-called *guaranteed firms* (GF) along with classical firms (CF) of the former Modigliani–Miller model. We suppose that the GFs are guaranteed by a government. Precisely, we suppose that debts benefiting from a free governmental implicit guarantee are equivalent to bonds covered by a free put option. Recall that Merton [17] was the first academician to recognise that financial guarantee acts as a put option on the guaranteed underlying assets.

Using the same notation, a GF is by definition a firm such that the net income of (S, D, X, r) is given by

$$Y := Z - rD, \quad Z = X + (rD - X)^+. \quad (4.1)$$

In our setting, Z corresponds to this put option. In the following, we define $Z := X$ for a CF. We assume without loss of generality that $(rD - X)^+ \neq 0$ for a GF, otherwise $X \geq rD$ a.s. and such a GF may be considered as a CF. Contrary to the model for CFs proposed by Modigliani–Miller, it is more justified to suppose that the GFs benefit from the risk-free interest rate r^0 as they are guaranteed by the government. Actually, we only suppose in this section that a guaranteed firm (S_A, D_A, X_A, r_A) benefits from a lower interest rate $r_A \leq r_B$ than the classical firm (S_B, D_B, X_B, r_B) such that $X_A \equiv X_B$.

Since the Modigliani–Miller and the Markowitz approaches are equivalent by Theorem 3.2, we therefore use the Markowitz approach when introducing guaranteed firms. In this section, we show that government guarantees trigger arbitrage opportunities between firms that have a priori the same value and benefit from the same interest rate r . More precisely, an agent who switches his investment from a classical firm to a guaranteed firm of same profile obtains a larger income for less risk, i.e., a larger Sharpe ratio. Following the reasoning of Theorem 2.2, we immediately deduce the following statement.

Corollary 4.1. *Suppose that each agent measures his investment risk by an individual measure ρ satisfying condition (ρ) . Assume that, in the Modigliani–Miller setting, the market is composed of classical and guaranteed firms which may borrow and lend money at an interest rate r depending on their profile. Assume that the market does not admit any arbitrage opportunity of Markowitz type. Then, two firms A and B such that $Z_A \equiv Z_B$ have the same value. Moreover, if $\rho(X) = \text{var}(X)$, the Sharpe ratio of an investment in firm A and a bond of interest rate r_A is given by*

$$\Theta_A = \frac{\mathbf{E} Z_A - r_A V_A}{\sqrt{\text{var}(Z_A)}}. \quad (4.2)$$

Contrary to the case where the market is only composed of CFs, it may appear arbitrage opportunities of Markowitz type in presence of GFs. In-

deed, consider a CF A and a GF B such that X_A and X_B have the same distribution X and the same structure (S, D) . Suppose a moment that $\rho(X) = \text{var}(X)$. Therefore, if they have the same value as stated by the Modigliani–Miller theorem, by virtue of Lemma 6.2 in Appendix, the corresponding Sharpe ratios satisfy $\Theta_A < \Theta_B$, i.e., there exists an arbitrage opportunity of Markowitz type. Therefore, governments' guarantees favor arbitrage opportunities. Once the market's equilibrium is modified, we get the following results.

Theorem 4.1. *Suppose that, in the Modigliani–Miller setting, the market is composed of classical and guaranteed firms which may borrow and lend money at an interest rate r which depends on their profile.*

Assume that the market does not admit any arbitrage opportunity. Consider a classical firm A and a guaranteed firm B such that $X_A \equiv X_B$ and $S_B \neq 0$. Then, $V_A < V_B$ and when $V_A > D_B$, we have⁴⁾

$$V_B \leq \frac{1}{r_B} \left[\mathbf{E} X_A + \mathbf{E} (r_B D_B - X_B)^+ - \frac{S_B (\mathbf{E} X_A - r_A V_A) \sqrt{\rho \left(\frac{(X_B - r_B D_B)^+}{S_B} \right)}}{(V_A - D_B) \sqrt{\rho \left(\frac{X_A - r_A D_B}{V_A - D_B} \right)}} \right]. \quad (4.3)$$

P r o o f. Since A is a classical firm, its value is independent of its capital structure (see Theorem 2.1). Therefore, we may assume without loss of generality that A and B have the same debt $D_A = D_B = D$. The case where $S_A = 0$ is trivial since $V_A = D < V_B = D + S_B$ by assumption. So, we suppose that $S_A \neq 0$. The Sharpe ratios, when investing in A or B , are given by

$$\begin{aligned} \Theta_A &= \frac{\mathbf{E} X_A - r_A V_A}{S_A \sqrt{\rho \left(\frac{X_A - r_A D}{S_A} \right)}}, \\ \Theta_B &= \frac{\mathbf{E} X_B + \mathbf{E} (r_B D - X_B)^+ - r_B V_B}{S_B \sqrt{\rho \left(\frac{X_B + (r_B D - X_B)^+ - r_B D}{S_B} \right)}}, \end{aligned}$$

⁴⁾ If $V_A = D_B$, since V_A is independent of its capital structure, this means that $S_A = 0$ for the firm of same value V_A with $D = D_B$. In this case, clearly $V_A = 0$ is a trivial case.

where $X_B + (r_B D - X_B)^+ - r_B D = (X_B - r_B D)^+ \geq 0$. Suppose that $V_A \geq V_B$. Then, since $D = D_A = D_B$, we get that $S_B \leq S_A$. Since $\mathbf{E}(r_B D - X_B)^+ > 0$ by assumption and $r_B \leq r_A$, we deduce that

$$\Theta_B > \frac{\mathbf{E} X_A - r_A V_A}{S_A \sqrt{\rho \left(\frac{(X_B - r_B D)^+}{S_B} \right)}}. \quad (4.4)$$

Since $S_B \leq S_A$ and $r_B \leq r_A$, we get that

$$\frac{(X_B - r_B D)^+}{S_B} \geq \frac{(X_B - r_B D)^+}{S_A} \geq \frac{(X_B - r_A D)^+}{S_A}.$$

Recall that $\mathbf{E} X_A \geq r_A V_A$, hence $\mathbf{E}(X_B - r_A D) \geq 0$ and, by condition (i) and the inequalities above, we get that

$$\rho \left(\frac{(X_B - r_B D)^+}{S_B} \right) \leq \rho \left(\frac{X_A - r_A D}{S_A} \right).$$

We conclude from (4.4) that $\Theta_B > \Theta_A$, i.e., there exists an arbitrage opportunity of Markowitz type between two firms of distinct values. Therefore, $V_A < V_B$. This strict inequality implies that $\Theta_A \leq \Theta_B$. Indeed, otherwise there would be an arbitrage opportunity of Markowitz type. The conclusion follows. Theorem 4.1 is proved.

As it is expected, this result above shows that the value of a GF B is strictly greater than the value of a CF A of same profile when the new market's equilibrium is reached even if they benefit from the same interest rate. We may also compare the values of two GFs.

Theorem 4.2. *Suppose that, in the Modigliani–Miller setting, the market is composed of classical and guaranteed firms which may borrow and lend money at an interest rate r which depends on their profile. Assume that the market does not admit any arbitrage opportunity of Markowitz type. Consider two GFs (S_A, D_A, X_A, r_A) and (S_B, D_B, X_B, r_B) such that $X_A \equiv X_B$. Suppose that condition (iii) (a) holds with $c(k) = k^2$. Then, $D_A < D_B$ implies that*

$$V_A < V_B \leq \frac{1}{r} \left[\mathbf{E} Z_B - \frac{\sqrt{\rho(X - r D_B)^+}}{\sqrt{\rho(X - r D_A)^+}} (\mathbf{E} Z_A - r V_A) \right]. \quad (4.5)$$

P r o o f. Observe that if an agent invests the capital αS_B in A or αS_A in B, the proportions are given by α and $\tilde{\alpha} = \alpha S_A / S_B$. In this case, by Lemma 3.1, the Sharpe ratios do not depend on α or $\tilde{\alpha}$ and Θ_A is given by

$$\Theta_A = \frac{\mathbf{E} X_A - r_A V_A}{\sqrt{\rho((X_A - r_A D_A)^+)}}.$$

The expression for Θ_B is similar. If we suppose that $V_A \geq V_B$, when $D_A < D_B$, then $\Theta_A < \Theta_B$, i.e., there exists an arbitrage opportunity of Markowitz type hence a contradiction. We deduce that $V_A < V_B$. At last, since there is no arbitrage opportunity of Markowitz type, the strict inequality $V_A < V_B$ implies that $\Theta_A \leq \Theta_B$ and the conclusion follows. Theorem 4.2 is proved.

This last result shows that GFs' values depend on their capital structures contrary to CFs' values. Moreover, government guarantees favor leverage for GFs.

Policy implications. The theoretical result of this research contradicts the shared belief on the relevance of the Modigliani–Miller theorem in the banking sector as suggested by the bulk of the academic literature. Policy implications can be derived from this paper. The first implication concerns the Government interventions. Indeed, there are abundant examples of interventions that create distortions to the Modigliani–Miller theorem such as tax shield, deposit insurances or implicit guarantees. For example, the Basel Committee asks for more equity to banks whereas the Governments subsidize debt. Another example stems from the Governments willingness to reduce risk-taking in the banking sector, while offering an implicit guarantee to banks, which creates an incentive to excessive leverage. As shown in this extended version of the Modigliani–Miller theorem, a bank cannot be longer considered as a classical firm given the safety net offered by the governments. A comprehensive policy needs to reconcile the Governments interventions with the Basel regulator requirements given that the first creates various incentives to debt while the last asks for more equity arguing of the Modigliani–Miller theorem.

5. Conclusion. This paper shows that the Modigliani–Miller theorem may be reformulated under a generalized Markowitz approach. Under this new perspective, we show that government guarantees modify market's equilibrium as they provoke arbitrage opportunities by increasing the Sharpe ratio of the guaranteed firms. Consequently, these guaranteed firms (e.g. banks) have strictly greater values than classical firms of same profiles and debt is optimal for banks.

6. Appendix.

Lemma 6.1. *Consider a random variable such that $\mathbf{E}X \geq 0$. Then, $\text{var}(X) \geq \text{var}(X^+)$.*

P r o o f. Since $X^2 \geq (X^+)^2$, we have $\mathbf{E}X^2 \geq \mathbf{E}(X^+)^2$. Moreover, $X \leq X^+$ implies that $0 \leq \mathbf{E}X \leq \mathbf{E}X^+$ and $0 \leq (\mathbf{E}X)^2 \leq (\mathbf{E}X^+)^2$. The conclusion follows. Lemma 6.1 is proved.

Lemma 6.2. *Consider the Sharpe ratios given by*

$$\Theta_A^r = \frac{\mathbf{E}X - rV}{\sqrt{\rho(X)}}, \quad \Theta_B^r = \frac{\mathbf{E}X + \mathbf{E}(rD - X)^+ - rV}{\sqrt{\rho(X + (rD - X)^+)}}. \quad (6.1)$$

Then, $\Theta_A^r < \Theta_B^r$.

P r o o f. Observe that $\rho(X) = \rho(X - rD)$ if condition (iii) (a) holds and

$$\rho(X + (rD - X)^+) = \rho(X + (rD - X)^+ - rD) = \rho((X - rD)^+).$$

Moreover, $\mathbf{E}(X - rD) \geq r(V - D) \geq 0$ by assumption. Using condition (i), we deduce that

$$\rho(X + (rD - X)^+) \leq \rho(X).$$

In the case where condition (iii) (b) holds, this inequality is satisfied by assumption. It follows that $\Theta_A^r < \Theta_B^r$. Lemma 6.2 is proved.

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